REAL ZEROS OF RANDOM ALGEBRAIC POLYNOMIALS WITH BINOMIAL ELEMENTS

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This paper provides an asymptotic estimate for the expected number of real zeros of a random algebraic polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$. The coefficients a_j ($j = 0,1,2,\ldots,n-1$) are assumed to be independent normal random variables with mean zero. For integers m and $k = O(\log n)^2$ the variances of the coefficients are assumed to have nonidentical value $\operatorname{var}(a_j) = \binom{k-1}{j-ik}$, where $n = k \cdot m$ and $i = 0,1,2,\ldots,m-1$. Previous results are mainly for identically distributed coefficients or when $\operatorname{var}(a_j) = \binom{n}{j}$. We show that the latter is a special case of our general theorem.

1. Introduction

Let $(\Omega, \Pr, \mathscr{A})$ be a fixed probability space and

$$P_n(x) = \sum_{j=0}^{n-1} a_j x^j, \tag{1.1}$$

where for $\omega \in \Omega$, $a_j(\omega) \equiv a_j$, $j = 0,1,2,\ldots,n-1$, is a sequence of random variables defined on Ω. Denote $N_n(a,b)$ as the number of real zeros of $P_n(x)$ in the interval (a,b). One way to determine the mathematical behavior of $P_n(x)$ is by looking at $EN_n(a,b)$, its expected number of real zeros. This has been achieved for many forms of $P_n(x)$ and varieties of assumptions for the distributions of a_j's. However, the majority of results assume identical distributions for the coefficients, albeit, EN_n is now known for most classes of distributions, see for example Ibragimov and Maslova [9] or Farahmand [7] and its references. Recently, in an interesting work Edelman and Kostlan [6] introduced a case of nonidentical coefficients in which $\operatorname{var}(a_j) = \binom{n}{j}$, $j = 0,1,2,\ldots,n-1$. Indeed, the above

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case is also motivated by several physical applications which are studied, for example, in Ramponi [11], Bleher and Di [2, 3], Bogomolny et al. [4, 5], and Aldous and Fyodorov [1]. There have been further works to study the mathematical behavior of EN_n of which the most recent is Farahmand and Nezakati [8].

The special feature of choosing the above assumptions for the $\operatorname{var}(a_j)$'s, indeed beside physical applications mentioned above, is the significant increase to $EN_n(-\infty,\infty)$. In fact this expected number increases from $(2/\pi)\log n$ for the case of identical normal standard to $\sqrt{n}$ for the case when $a_j(\omega)$ is normal with mean zero and variance $\binom{n}{j}$. This assumption for the variances of the coefficients introduces a new class of polynomials in which there are more zeros than algebraic polynomials with identical variances and less than random trigonometric polynomials defined as $\sum_{j=0}^{n-1} a_j \cos j\theta$. The expected number of real zeros of the latter polynomial is $EN_n(0,2\pi) \sim n/\sqrt{3}$. Indeed, these results show that the oscillatory behavior of the above random polynomials has the least number of oscillations for the algebraic with identical coefficients case and most for the trigonometric case. Our case of random algebraic polynomial with nonidentical coefficients has a number of oscillations between these two extreme cases. Therefore, it is natural to ask whether or not for other cases of nonidentical variances with binomial elements, the latter increases in EN_n remain stainable. To this end we prove the following theorem.

Theorem 1.1. *For random algebraic polynomial $P_n(x)$, let n be separated into two multipliers such that $n = k \cdot m$, where $k = f(n)$ is an integer and increasing function of n, such that $f(n) = O(\log n)^2$. The random variables a_j, $j = 0,1,2,\dots,n-1$ are normally distributed with means zero and* $\operatorname{var}(a_j) = \binom{k-1}{j-ik}$, $j = ik, ik+1,\dots,(i+1)k-1$, $i = 0,1,\dots,m-1$. *Then for sufficiently large n, the expected number of real zeros of $P_n(x)$ is*

$$EN_n(-\infty,\infty) \sim \sqrt{k-1}. \tag{1.2}$$

2. Proof of theorem

For proof of the theorem we use an approach based on Kac's [10] or Rice's [12] results. In this case for

$$\begin{aligned} A^2 &= \operatorname{var}(P_n(x)), \qquad B^2 = \operatorname{var}(P_n'(x)), \\ C &= \operatorname{cov}(P_n(x), P_n'(x)), \qquad \Delta^2 = A^2B^2 - C^2, \end{aligned} \tag{2.1}$$

the expected number of real zeros is given by the Kac-Rice formula as

$$EN_n(a,b) = \frac{1}{\pi}\int_a^b \frac{\Delta}{A^2}\,dx. \tag{2.2}$$

In order to use (2.2) to obtain $EN_n(-\infty,\infty)$ we note that changing x to $1/x$ and x to $-x$ leaves the distribution of the coefficients of $P_n(x)$ in (1.1) invariant. Hence the expected number of real zeros in the interval $(0,1)$ is asymptotically the same as in $(1,\infty)$, $(-1,0)$, and $(-\infty,-1)$. Therefore it suffices to give the result for $EN_n(0,1)$ only. To this end, we present our calculations for any integer k. From the assumptions of Theorem 1.1 for the

distributions of the coefficients of $P_n(x)$, from (2.1) we can easily show that

$$A^2 = \sum_{j=0}^{n-1} \operatorname{var}(a_j) x^{2j}, \qquad B^2 = \sum_{j=0}^{n-1} \operatorname{var}(a_j) j^2 x^{2j-2}, \qquad C = \sum_{j=0}^{n-1} \operatorname{var}(a_j) j x^{2j-1}. \tag{2.3}$$

Now similar to the method of Sambandham [13], let

$$H(x,y) = \sum_{j=0}^{n-1} \operatorname{var}(a_j) x^j y^j, \tag{2.4}$$

then we can obtain

$$\begin{aligned} H(x,y) &= \binom{k-1}{0} + \binom{k-1}{1} xy + \cdots + \binom{k-1}{k-1} (xy)^{k-1} \\ &\quad + (xy)^k \left[\binom{k-1}{0} + \binom{k-1}{1} xy + \cdots + \binom{k-1}{k-1} (xy)^{k-1} \right] + \cdots \\ &\quad + (xy)^{(m-1)k} \left[\binom{k-1}{0} + \binom{k-1}{1} xy + \cdots + \binom{k-1}{k-1} (xy)^{k-1} \right] \\ &= (1+x^2)^{k-1} \left[1 + (xy)^k + (xy)^{2k} + \cdots + (xy)^{(m-1)k} \right] \\ &= \frac{1-x^n y^n}{1-x^k y^k} (1+xy)^{k-1}. \end{aligned} \tag{2.5}$$

Therefore from (2.3)-(2.4) we can easily see that

$$A^2 = H(x,x), \qquad B^2 = \left[\frac{\partial^2 H(x,y)}{\partial x \partial y} \right]_{y=x}, \qquad C = \left[\frac{\partial H(x,y)}{\partial x} \right]_{y=x}. \tag{2.6}$$

Therefore it is an easy exercise to obtain the value of A^2, B^2, and C given in (2.3) as

$$\begin{aligned} A^2 &= \frac{1-x^{2n}}{1-x^{2k}} (1+x^2)^{k-1}, \\ B^2 &= \left[\frac{k^2 x^{4k-2} + (n-k)^2 x^{2n+4k-2} + (2n^2 - 2nk - k^2) x^{2n+2k-2}}{(1-x^{2k})^3} \right. \\ &\quad \left. + \frac{k^2 x^{2k-2} - n^2 x^{2n-2}}{(1-x^{2k})^3} \right] (1+x^2)^{k-1} + 2(k-1) \left[\frac{(n-k) x^{2n+2k} + k x^{2k} - n x^{2n}}{(1-x^2 k)^2} \right] (1+x^2)^{k-2} \\ &\quad + \left[\frac{(k-1)(1+kx^2-x^2)(1-x^{2n})}{1-x^{2k}} \right] (1+x^2)^{k-3}, \\ C &= \left[\frac{(n-k) x^{2n+2k-1} + k x^{2k-1} - n x^{2n-1}}{(1-x^{2k})^2} \right] (1+x^2)^{k-1} + \left[\frac{(k-1) x (1-x^{2n})}{1-x^{2k}} \right] (1+x^2)^{k-2}. \end{aligned} \tag{2.7}$$

Also from (2.1) and (2.7), for any integer k, we obtain

$$\frac{\Delta^2}{A^4} = \frac{k^2x^{2k-2} - n^2x^{2n-2} + 2(n^2-k^2)x^{2n+2k-2} + (n^2+2k^2-4nk)x^{2n+4k-2}}{(1-x^{2k})^2(1-x^{2n})^2} + \frac{k^2x^{4n+2k-2} - 2(n-k)^2x^{4n+4k-2}}{(1-x^{2k})^2(1-x^{2n})^2} + \frac{k-1}{(1+x^2)^2}. \tag{2.8}$$

In order to continue the proof of Theorem 1.1, we first consider the interval $(0, 1-\eta)$, where for

$$a = 1 - \frac{\log\log k^{10}}{\log k}, \quad \text{we let } \eta = k^{-a}. \tag{2.9}$$

In this interval, for sufficiently large n, we can easily show that

$$x^k \le k^{-10}, \qquad x^n \le k^{-10m}. \tag{2.10}$$

Also note $a \to 1$ as $n \to \infty$. This is necessary to obtain the result later. Now from (2.8), we have

$$\frac{\Delta^2}{A^4} \sim \frac{k-1}{(1+x^2)^2}. \tag{2.11}$$

Therefore from (2.2), we can show that

$$EN_n(0, 1-\eta) \sim \frac{1}{\pi}\int_0^{1-\eta} \frac{\sqrt{k-1}}{1+x^2}dx = \frac{\sqrt{k-1}}{\pi}\arctan(1-\eta). \tag{2.12}$$

Hence for sufficiently large n,

$$EN_n(0, 1-\eta) \sim \frac{\sqrt{k-1}}{4}. \tag{2.13}$$

Now we assume $1-\eta \le x \le 1-\delta$, where for

$$b = 1 - \frac{\log\log k^{1/2}}{\log k}, \quad \text{we let } \delta = k^{-b}. \tag{2.14}$$

In this interval, for sufficiently large n, we can easily show that

$$x^{2k} \le k^{-1}, \qquad x^{2n} \le k^{-m}. \tag{2.15}$$

Now from (2.8), we have

$$\frac{\Delta}{A^2} \sim \sqrt{k^2x^{2k-2} + \frac{k-1}{(1+x^2)^2}} \le B\sqrt{k}, \tag{2.16}$$

where B is constant. Therefore, from (2.2), we can show that

$$EN_n(1-\eta, 1-\delta) = O(k^{-1/2}\log k). \tag{2.17}$$

When $1-\delta \le x \le 1-\epsilon$, where for

$$c = 1 - \frac{\log\log n^{10}}{\log n}, \quad \text{we let } \epsilon = n^{-c}. \tag{2.18}$$

In this interval, for sufficiently large n, we can easily show that

$$kx^{2k} \ge 1, \qquad x^n \le n^{-10}. \tag{2.19}$$

Now from (2.8), (2.14), and (2.18), we have

$$\frac{\Delta}{A^2} \sim \frac{kx^{k-1}}{1-x^{2k}}\sqrt{1+\frac{(k-1)\left(1-x^{2k}\right)^2}{k^2x^{2k-2}\left(1+x^2\right)^2}} \le C\left[\frac{kx^{k-1}}{1+x^k}+\frac{kx^{k-1}}{1-x^k}\right], \tag{2.20}$$

where C is constant. Therefore, from (2.2), we can show that

$$EN_n(1-\delta,1-\epsilon) \le \frac{C}{\pi}\int_{1-\delta}^{1-\epsilon}\left[\frac{kx^{k-1}}{1+x^k}+\frac{kx^{k-1}}{1-x^k}\right]dx = \frac{C}{\pi}\left[\log\frac{1+x^k}{1-x^k}\right]_{1-\delta}^{1-\epsilon}. \tag{2.21}$$

Therefore, for sufficiently large n,

$$EN_n(1-\delta,1-\epsilon) = O(\log n). \tag{2.22}$$

Finally, let $1-\epsilon \le x \le 1$, we know that always

$$\frac{\Delta}{A^2} \le n. \tag{2.23}$$

Therefore, from (2.2) and (2.18), and for sufficiently large n, we also have,

$$EN_n(1-\epsilon,1) = O(\log n). \tag{2.24}$$

Hence from (2.13), (2.17), (2.22), and (2.24), we have the proof of Theorem 1.1.

Example 2.1. In Theorem 1.1, if $k = n$, then our Theorem 1.1 result is similar to that obtained by Edelman and Kostlan [6].

Example 2.2. In random polynomials $P_n(x) = \sum_{j=0}^{n-1} a_j x^j$, where $n \in \{i^2;\ i = 1,2,\dots\}$, if $k = \sqrt{n}$ and conditions of Theorem 1.1 hold, then for sufficiently large n,

$$EN_n(-\infty,\infty) = \sqrt{\sqrt{n}-1}. \tag{2.25}$$

Example 2.3. In random polynomials $P_n(x) = \sum_{j=0}^{n-1} a_j x^j$, $n \in \{i^3;\ i = 1,2,\dots\}$, if $k = n^{1/3}$ and conditions of Theorem 1.1 hold, then for sufficiently large n,

$$EN_n(-\infty,\infty) = \sqrt{n^{1/3}-1}, \tag{2.26}$$

and, if $k = n^{2/3}$, then for sufficiently large n,

$$EN_n(-\infty,\infty) = \sqrt{n^{2/3}-1}. \tag{2.27}$$

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