EXISTENCE OF SOLUTIONS TO SOBOLEV-TYPE PARTIAL NEUTRAL DIFFERENTIAL EQUATIONS

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This work is concerned with a nonlocal partial neutral differential equation of Sobolev type. Specifically, existence of the solutions to the abstract formulations of such type of problems in a Banach space is established. The results are obtained by using Schauder’s fixed point theorem. Finally, an example is provided to illustrate the applications of the abstract results.

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1. Introduction

Let $X$ and $Y$ be two real Banach spaces. We consider the following neutral differential equation of Sobolev type with a nonlocal history condition:

$$\frac{d}{dt} \left[ Bu(t) + f(t, u(t - \tau_1)) \right] + Au(t) = g(t, u(t), u(t - \tau_2)), \quad 0 < t \leq T,$$

$$h(u|[-\tau,0]) = \phi,$$

where $\tau = \max\{\tau_1, \tau_2\}$, $\tau_i > 0$, $i = 1, 2$, $T < \infty$, $\phi \in \mathcal{C}_0 := C([-\tau,0], X)$, and $B$ and $A$ are linear operators with the domains contained in $X$ and the ranges contained in $Y$. $f : [0, T] \times X \rightarrow Y$, $g : [0, T] \times X \times X \rightarrow Y$ are two given functions, and the map $h$ is defined from $\mathcal{C}_0$ into $\mathcal{C}_0$. Here $\mathcal{C}_t := C([-\tau, t]; X)$ for $t \in [0, T]$ is a Banach space of all continuous functions from $[-\tau, t]$ into $X$ endowed with the norm

$$\|\psi\|_t := \sup \{\|\psi(\theta)\|_X : \theta \in [-\tau, t] \}.$$  

The study of differential equations with nonlocal conditions is of significance due to its applications in problems in physics and other areas of applied mathematics. Byszewski [6] proved the existence of mild, strong, and classical solutions for the nonlocal Cauchy problem. Some more results on the existence, uniqueness, and stability of solutions are
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The theory of neutral differential equations has been extensively studied in the literature. Hernández [10] established the existence results for partial neutral functional differential equations with nonlocal conditions modeled as

\[
\frac{d}{dt}(u(t) + F(t, u_t)) = Au(t) + G(t, u_t), \quad 0 \leq t \leq T,
\]

\[\]

where \( A \) is the infinitesimal generator of an analytic semigroup \( T(t) \) on a Banach space. He made use of fixed point theorems and the results mentioned in Pazy’s [13]. For results on neutral partial differential equations with nonlocal and classical conditions, we refer to the papers of Hernández and Henríquez [11], Balachandran and Sakthivel [3], Fu and Ezzinbi [9], and references therein.

Our aim in this paper is to study the existence of a solution of partial neutral differential equation of form (1.1) by using Schauder’s fixed point theorem. For this purpose, we first prove the existence of a solution of (1.1) on \([-\tau, \tilde{T}]\) for some \( 0 < \tilde{T} \leq T \) and then prove that this solution can be extended to a solution of (1.1) either on \([-\tau, T]\) or on the maximal interval of existence \([-\tau, t_{\text{max}}]\), \( 0 < t_{\text{max}} \leq T \), and in the latter case we show that \( \lim_{t \to t_{\text{max}}^-} \| u(t) \|_X = \infty \). Finally, we present an example to show an application of the existence theorem.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary results which we require to establish the existence of a solution of (1.1).

Henceforth, \( (X, \| \cdot \|_X) \) and \( (Y, \| \cdot \|_Y) \) will denote two real Banach spaces. The space of continuous linear mapping from \( X \) into \( Y \) will be denoted by \( L(X, Y) \).

To prove our main theorem, we consider the following assumptions on the operators \( A : D(A) \subset X \to Y \) and \( B : D(B) \subset X \to Y \):

(i) \( A \) and \( B \) are closed, linear operators,
(ii) \( D(B) \subset D(A) \) and \( B \) is bijective,
(iii) \( B^{-1} : Y \to D(B) \) is compact.

From the above fact and the closed graph theorem, we get the boundedness of the linear operator \( AB^{-1} : Y \to Y \). Consequently, \( AB^{-1} \) generates a uniformly continuous semigroup \( e^{-tAB^{-1}}, t \geq 0 \).

Further, we assume that

(iv) there exists a Lipschitz continuous function \( \chi \in C_0 \) such that \( h(\chi) = \phi \) on \([-\tau, 0]\) and \( \chi(0) \in D(B) \),
(v) \( f : [0, T] \times X \to Y \) is a Lipschitz continuous function, that is,

\[
\| f(t, u) - f(s, v) \|_Y \leq L[|t - s| + \|u - v\|_X],
\]

for all \( t \in [0, T] \) and \( u, v \in X \). \( L > 0 \) is a Lipschitz constant.

Next, we choose a \( 0 < \tilde{T} \leq T \) such that \( \tilde{T} = \min\{\tau_1, \tau_2\} \), then for \( 0 < t \leq \tilde{T} \), the functions \( u(t - \tau_1) \) and \( u(t - \tau_2) \) transform to known function \( \chi \).

Under the above assumptions, we prove the existence of a solution \( u \) of (1.1) in the sense that there exists a continuous function \( u : [-\tau, \tilde{T}] \to X \) such that \( u(t) \in D(B) \) for all \( t \in [0, \tilde{T}] \), and in addition, \( [Bu(t) + f(t, \chi(t - \tau_1))] \) is differentiable on \( (0, \tilde{T}] \) satisfying

\[
\frac{d}{dt}[Bu(t) + f(t, \chi(t - \tau_1))] + Au(t) = g(t, u(t), \chi(t - \tau_2)), \quad 0 < t \leq \tilde{T},
\]

\[
u = \chi, \quad \text{on } [-\tau, 0].
\]

Note that \( Bu(t) \) itself may not be differentiable on the interval of existence.

Now we mention a few results needed to establish our main result.

**Proposition 2.1.** Suppose the assumptions (i)--(v) are satisfied. Let \( F : [0, T] \to Y \) and \( \tilde{F} : [0, T] \to Y \) be Bochner integrable functions and let

\[
U(t) = Bu(t) + H(t),
\]

\[
\tilde{U}(t) = B\tilde{u}(t) + \tilde{H}(t),
\]

\[
w = ||AB^{-1}||_Y.
\]

If \( u, \tilde{u} \) are solutions of

\[
\frac{d}{dt}[Bu(t) + H(t)] + Au(t) = F(t),
\]

\[
u = \chi, \quad \text{on } [-\tau, 0];
\]

\[
\frac{d}{dt}[B\tilde{u}(t) + \tilde{H}(t)] + A\tilde{u}(t) = \tilde{F}(t),
\]

\[
\tilde{u} = \tilde{\chi}, \quad \text{on } [-\tau, 0],
\]

respectively, where

\[
H(t) = f(t, \chi(t - \tau_1)), \quad \tilde{H}(t) = f(t, \tilde{\chi}(t - \tau_1)), \quad \forall [0, T].
\]

Then, the following hold for \( 0 \leq s \leq t \leq T \):

(1)

\[
\| U(t) - \tilde{U}(t) \|_Y \leq e^{w(t-s)}\| U(s) - \tilde{U}(s) \|_Y + \int_s^t e^{w(t-\eta)}\| F(\eta) - \tilde{F}(\eta) \|_Y d\eta
\]

\[
+ \int_s^t e^{w(t-\eta)}w\| H(\eta) - \tilde{H}(\eta) \|_Y d\eta,
\]

(2.7)
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(II)
\[ ||U(t) - U(s)||_Y \leq \int_s^t e^{w(t-\eta)}||F(\eta) - Au(s)||_Y d\eta + \int_s^t e^{w(t-\eta)}w||H(\eta) - H(s)||_Y d\eta, \]
\( (2.8) \)

(III)
\[ ||Au(t) - A\chi(0)||_Y \leq w\left[ \int_0^t e^{w(t-\eta)}||F(\eta) - A\chi(0)||_Y d\eta + \int_0^t e^{w(t-\eta)}w||H(\eta) - H(0)||_Y d\eta + ||H(t) - H(0)||_Y \right]. \]
\( (2.9) \)

Proof. Let \( Y^* \) be the dual space of \( Y \) and let the duality map \( \mathcal{J} : Y \rightarrow 2^{Y^*} \) be given by
\[ \mathcal{J}(y) = \left\{ y^* \in Y^* : \langle y, y^* \rangle = ||y||_Y^2 = ||y^*||_{Y^*}^2 \right\}. \]
\( (2.10) \)

Here, \( 2^{Y^*} \) denotes the power set of \( Y^* \), \( || \cdot ||_Y \) and \( || \cdot ||_{Y^*} \) are the norms of \( Y \) and \( Y^* \) respectively, and \( \langle y, y^* \rangle \) is the value of \( y^* \in Y^* \) at \( y \in Y \).

Define a function \( \langle \cdot, \cdot \rangle : Y \times Y \rightarrow \mathbb{R} \) by
\[ \langle x, y \rangle := \inf \left\{ (x, y^*) : y^* \in F(y) \right\}. \]
\( (2.11) \)

(I) Since \( ||U - \tilde{U}||_Y : [0, T] \rightarrow \mathbb{R} \) is a differentiable function for a.e. \( \eta \in (0, T) \), we may apply the result of Kato [12, Lemma 1.3]. Thus, we have
\[ ||U(\eta) - \tilde{U}(\eta)||_Y \left( \frac{d}{d\eta} \right)||U(\eta) - \tilde{U}(\eta)||_Y \\
= \langle F(\eta) - Au(\eta) - \tilde{F}(\eta) + A\tilde{\eta}(\eta), U(\eta) - \tilde{U}(\eta) \rangle \\
\leq ||F(\eta) - \tilde{F}(\eta)||_Y ||U(\eta) - \tilde{U}(\eta)||_Y + ||Au(\eta) - A\tilde{\eta}(\eta)||_Y ||U(\eta) - \tilde{U}(\eta)||_Y. \]
\( (2.12) \)

It implies
\[ \frac{d}{d\eta} ||U(\eta) - \tilde{U}(\eta)||_Y \leq ||F(\eta) - \tilde{F}(\eta)||_Y + w||Bu(\eta) - B\tilde{\eta}(\eta)||_Y \\
\leq ||F(\eta) - \tilde{F}(\eta)||_Y + w\left[ ||U(\eta) - \tilde{U}(\eta)||_Y + ||H(\eta) - \tilde{H}(\eta)||_Y \right], \]
\( (2.13) \)

which gives
\[ ||U(t) - \tilde{U}(t)||_Y \leq e^{w(t-s)}||U(s) - \tilde{U}(s)||_Y + \int_s^t e^{w(t-\eta)}||F(\eta) - \tilde{F}(\eta)||_Y d\eta \\
+ \int_s^t e^{w(t-\eta)}w||H(\eta) - \tilde{H}(\eta)||_Y d\eta. \]
\( (2.14) \)
(II) If we take
\[ \tilde{u}(t) \equiv u(s), \quad \tilde{H}(t) \equiv H(s), \quad \text{for } t \in [0, T], \quad (2.15) \]
then the estimate (II) is followed by (I).

(III) By using the fact \( \|Ax\|_Y \leq w\|Bx\|_Y \) for every \( x \in D(B) \) in (II), we obtain
\[
\|Au(t) - A\chi(0)\|_Y \leq w\|Bu(t) - B\chi(0)\|_Y \\
\leq w\left[ \int_0^t e^{w(t-\eta)}\|F(\eta) - A\chi(0)\|_Y d\eta \\
+ \int_0^t e^{w(t-\eta)}w\|H(\eta) - H(0)\|_Y d\eta + \|H(t) - H(0)\|_Y \right].
\quad (2.16)
\]

**Corollary 2.2.** Let \( F : [0, T] \to Y \) be a continuous function and let \( u \) be a solution of (2.4). Also, suppose that the assumptions (i)–(v) are satisfied. Then, there exist constants \( \alpha = \alpha(T) \) and \( \beta = \beta(T) \) such that
\[
\|u(t) - u(s)\|_X \leq \alpha \left[ \sup_{0 \leq \eta \leq T} \|F(\eta) - A\chi(0)\|_Y + \beta \right] (t - s), \quad \text{for } 0 \leq s < t \leq T.
\quad (2.17)
\]

**Proof.** Let us take \( b \) as the norm of \( B^{-1} \in L(Y, X) \).

Using Proposition 2.1(II) and (III), we obtain
\[
\|u(t) - u(s)\|_X \leq b\|Bu(t) - Bu(s)\|_Y \\
\leq b\left[ \int_s^t e^{w(t-\eta)}\|F(\eta) - Au(s)\|_Y d\eta \\
+ \int_s^t e^{w(t-\eta)}w\|f(\eta, \chi(\eta - \tau_1)) - f(s, \chi(s - \tau_1))\|_Y d\eta \\
+ \|f(t, \chi(t - \tau_1)) - f(s, \chi(s - \tau_1))\|_Y \right] \\
\leq b\left[ \sup_{0 \leq \eta \leq T} \|F(\eta) - A\chi(0)\|_Y \left( \frac{e^{wt} - e^{ws}}{w} \right) \\
+ c(e^{w(t-s)} - 1) \left( \frac{e^{wt} - 1}{w} \right) + c\left( \frac{e^{w(t-s)} - 1}{w} \right) \right] \\
\leq b\left[ \sup_{0 \leq \eta \leq T} \|F(\eta) - A\chi(0)\|_Y + ce^{wT} \right] e^{wT}|t - s|.
\quad (2.18)
\]

Set \( \alpha = be^{wT} \) and \( \beta = ce^{wT} \). Thus,
\[
\|u(t) - u(s)\|_X \leq \alpha \left[ \sup_{0 \leq \eta \leq T} \|F(\eta) - A\chi(0)\|_Y + \beta \right] |t - s|. \quad (2.19)
\]

\[ \square \]
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3. Existence of solutions

We start by establishing a result of existence of a solution to the following abstract Cauchy problem:

\[
\frac{d}{dt} [Bu(t) + f(t, \chi(t - \tau_t))] + Au(t) = F(t), \quad 0 < t \leq \tilde{T},
\]

\[u = \chi, \quad \text{on } [-\tau, 0].\]  

\[u = \chi, \quad 0 < t \leq \tilde{T}, u = \chi, \quad \text{on } [-\tau, 0].\]  

(3.1)

**Proposition 3.1.** If assumptions (i)–(v) are satisfied and \(F : [0, \tilde{T}] \to Y\) is a continuous function, then the Cauchy problem (3.1) has a unique solution.

**Proof.** Since \(AB^{-1} : Y \to Y\) is a bonded linear operator, it follows that the problem

\[
\frac{d}{dt} w(t) + AB^{-1} w(t) = G(t), \quad t \in (0, \tilde{T}],
\]

\[w(t) = \tilde{\chi}(t), \quad t \in [-\tau, 0],\]

where

\[G(t) = F(t) + AB^{-1} f(t, \chi(t - \tau_1)), \quad \forall t \in [0, \tilde{T}],\]  

(3.2)

has a solution \(w : [-\tau, \tilde{T}] \to Y\), given by

\[w(t) = e^{-tAB^{-1}} \tilde{\chi}(0) + \int_{0}^{t} e^{-(t-s)AB^{-1}} G(s) ds, \quad -\tau \leq t \leq \tilde{T}.\]  

(3.4)

Consequently, a function \(u(t) = B^{-1}(w(t) - f(t, \chi(t - \tau_1)))\) is a solution of (3.1). Uniqueness of a solution is obtained by using Proposition 2.1(I).

Our main existence theorem is the following.

**Theorem 3.2.** Let the assumptions (i)–(v) be satisfied and suppose that \(g\) is a continuous function from \([0, T] \times X \times X\) into \(Y\). Then, for any given \(\chi(0) \in D(B)\), there exists a solution of the abstract Cauchy problem (2.2) on the subinterval \([-\tau, \tilde{T}] \subset [-\tau, T]\).

**Proof.** Let \(0 < \tilde{T} \leq T\) and \(R > 0\) be real numbers and define the set

\[S_0 = S(A\chi(0), R, \tilde{T}) := \left\{ F \in C([0, \tilde{T}]; Y) : \sup_{0 \leq t \leq \tilde{T}} ||F(t) - A\chi(0)||_Y \leq R \right\}.\]  

(3.5)

It is easy to see that \(S_0\) is a bounded, closed, and convex subset of \(C([0, \tilde{T}]; Y)\).

Since all the hypotheses of Proposition 3.1 are verified, there exists a solution \(u_F, F \in S_0\), of the problem (3.1).

We define a mapping \(Z : S_0 \to C([0, \tilde{T}]; Y)\) by

\[Z(F)(t) := g(t, u_F(t), \chi(t - \tau_2)).\]  

(3.6)
The proof will be given in three steps.

**Step 1.** Z is continuous and maps \( S_0 \) into itself.

Let \( R > \max_{0 \leq t \leq T} \| g(t, \chi(0), \chi(0)) - A \chi(0) \|_Y \) be arbitrary. By the continuity of \( g \), we have \( \| g(t, x, \xi) - A \chi(0) \|_Y \leq R \), for all \( t \in [0, T] \), and \( x, \xi \in X \) provided \( \| (x, \xi) - (\chi(0), \chi(0)) \|_X \leq \delta \).

Suppose that \( \| B^{-1} \|_Y = b, c > 0 \) is a constant, and \( 0 < \tilde{T} \leq T \) is such that

\[
\frac{b}{w} \left[ e^{w \tilde{T}} - 1 \right] (R + c) \leq \delta. \tag{3.7}
\]

In view of these conditions and Proposition 2.1(II), we can easily show that \( Z \) maps \( S_0 \) into \( S_0 \). Now, we will show that \( Z \) is continuous. To this end, we introduce

\[
V = \{ u_F(t) : 0 \leq t \leq \tilde{T}, \ F \in S_0 \}, \tag{3.8}
\]

and using Proposition 2.1(II), we observe that \( B(V) \subset Y \) is a bounded set. Hence, \( V \) is relatively compact in \( X \). Therefore, \((t, v) \rightarrow g(t, v, \chi(t - \tau_2))\) is uniformly continuous on \([0, \tilde{T}] \times V\) and Proposition 2.1(II) implies that \( Z : S_0 \rightarrow C([0, \tilde{T}] ; Y) \) is continuous.

**Step 2.** \( Z(S_0) \) is equicontinuous.

Let \( F \in S_0 \) and \( t_1, t_2 \in [0, \tilde{T}] \). Then, if \( 0 < t_1 < t_2 \leq \tilde{T} \),

\[
\| Z(F)(t_1) - Z(F)(t_2) \|_Y = \| g(t_1, u_F(t_1), \chi(t_1 - \tau_2)) - g(t_2, u_F(t_2), \chi(t_2 - \tau_2)) \|_Y. \tag{3.9}
\]

The right-hand side tends to zero as \( t_2 - t_1 \rightarrow 0 \), since from Corollary 2.2 we obtain \( \| u_F(t_1) - u_F(t_2) \|_X \leq k \) and \( g \) is a uniformly continuous function. Thus, \( Z \) maps \( S_0 \) into an equicontinuous family of functions.

**Step 3.** \( Z \) maps \( S_0 \) into precompact set in \( Y \).

Let \( t \in [0, \tilde{T}] \) be fixed. Then, image of \( \{ t \} \times V \times \{ \chi \} \) under continuous function \( g \) is precompact in \( Y \). Therefore, the set \( \{ Z(F)(t) : F \in S_0 \} \) is precompact in \( Y \).

As a consequence of Step 2 and Step 3 together with the Ascoli-Arzela theorem, we infer that \( Z(S_0) \) is relatively compact in \( Y \). Hence, by Schauder’s fixed point theorem, we deduce that the operator \( Z \) has a fixed point. This means that the problem (2.2) has a solution. \( \square \)

Next we will prove the following global existence result.

**Theorem 3.3.** Suppose that all the hypotheses of Theorem 3.2 are satisfied. Then, (2.2) has a solution either on \([−\tau, T]\) or on the maximal interval of existence \([−\tau, t_{\text{max}}]\), \( 0 < t_{\text{max}} \leq T \), and in the latter case, \( \lim_{t \rightarrow t_{\text{max}}} \| u(t) \|_X = \infty \).

**Proof.** Since all the assumptions of Theorem 3.2 are satisfied, there exists a solution \( u \) of (1.1) on \([−\tau, \tilde{T}]\).
Suppose $\tilde{T} < T$ and consider the problem

$$\frac{d}{dt}[Bw(t) + f(t + \tilde{T}, \tilde{\chi}(t - \tau_1))] + Aw(t) = g(t + \tilde{T}, \tilde{\chi}(t - \tau_2), w(t)), \quad 0 < t \leq T - \tilde{T},$$

$$w = \tilde{\chi}, \quad \text{on } [-\tau - \tilde{T}, 0],$$

(3.10)

where

$$\tilde{\chi}(t) = \begin{cases} \chi(t + \tilde{T}), & t \in [-\tau - \tilde{T}, -\tilde{T}], \\ u(t + \tilde{T}), & t \in [-\tilde{T}, 0]. \end{cases}$$

(3.11)

Since $\tilde{\chi}(0) = u(\tilde{T}) \in D(B)$, $f$ satisfies the assumption (v), and $g$ is a continuous function, we may proceed as before and prove the existence of a solution $w: [-\tau, T_1] \to X$, $0 < T_1 \leq T - \tilde{T}$, of the considered problem (3.10).

Then, the function $\tilde{u}: [-\tau, \tilde{T} + T_1] \to X$, given by

$$\tilde{u}(t) = \begin{cases} u(t), & t \in [-\tau, \tilde{T}], \\ w(t - T_0), & t \in [\tilde{T}, \tilde{T} + T_1], \end{cases}$$

(3.12)

is a solution of (2.2) on $[-\tau, \tilde{T} + T_1]$. Continuing this way, we may prove the existence either on the whole interval $[-\tau, T]$ or on the maximal interval of existence $[-\tau, t_{\max})$, $0 < t_{\max} \leq T$. In the latter case, if $\lim_{t \to -t_{\max}^-} \|u(t)\|_X < \infty$, then as $u(t) \in D(B)$ for $t \in [0, t_{\max})$, we have that $\lim_{t \to -t_{\max}^-} u(t)$ is in the closure of $D(B)$ in $X$, and if it is in $D(B)$, then proceeding as before, we may extend $u(t)$ beyond $t_{\max}$ but this will contradict the definition of the maximal interval of existence. Therefore, $\lim_{t \to -t_{\max}^-} \|u(t)\|_X = \infty$. \hfill \Box

4. An example

In this section, we consider an example to illustrate the abstract results.

Consider the following initial boundary value problem:

$$\frac{\partial}{\partial t} \left[ (1 - \Delta)w(t, x) + \int_{\Omega} k_1(t, x, y)b_1(w(t - \tau_1, y)) \, dy \right] - \Delta w(t, x)$$

$$= \int_{\Omega} k_2(t, x, y)b_2(w(t, y), w(t - \tau_2, y)) \, dy, \quad x \in \Omega, \; t \in (0, T],$$

$$w(t, x) = 0, \quad t \geq 0, \; x \in \partial \Omega,$n

$$h_0(w(t, x)) = \phi_0(x), \quad x \in \Omega, \; t \in [-\tau, 0],$$

(4.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial \Omega$, and $\Delta$ is the $N$-dimensional Laplacian. To represent problem (4.1) as the Cauchy problem (1.1), we take $X = L^p(\Omega)$, $Y = C(\bar{\Omega})$, and define $u(t)(x) = w(t, x)$, $B_1(u(t - \tau_1))(x) = b_1(w(t - \tau_1, x))$, and $B_2(u(t), u(t - \tau_2))(x) = b_2(w(t, x), w(t - \tau_1, x))$. The operators $A$ and $B$ are
Define
\[ f : \Omega \to \mathbb{R}, \quad \forall \Omega \in D(A), \]
\[ h : \Omega \to \mathbb{R}, \quad \forall \Omega \in D(B), \]
with the domains
\[ D(A) = D(B) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega), \quad p > n. \]

Then, \( A \) and \( B \) are closed, linear operators. Furthermore, \( A \) is an accerative operator and \( D(A) \) is compactly imbedded in \( Y \). Therefore, \( A \) and \( B \) verify the assumptions (i)--(iii).

We assume the following conditions.

(i) \( k_1(t,x,y) \) and \( k_2(t,x,y) \) are real-valued Lipschitz continuous and continuous functions, respectively, on \([0,T] \times \Omega \times \Omega\) satisfying
\[ \sup_{x \in \Omega} \int_{\Omega} \left( \int_{\Omega} |k_i(t,x,y)|^q dy \right)^{1/q} dx < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad i = 1,2. \]  

(ii) \( b_1 \) is a real-valued Lipschitz continuous function and \( b_2 \) is a real-valued continuous function.

Define \( f : [0,T] \times L^p(\Omega) \to C(\bar{\Omega}) \) and \( g : [0,T] \times L^p(\Omega) \times L^p(\Omega) \to C(\bar{\Omega}) \) by
\[ f(t,\xi)(x) = \int_{\Omega} k_1(t,x,y) B_1(\xi)(y) dy, \]
\[ g(t,v,\eta)(x) = \int_{\Omega} k_2(t,x,y) B_2(v,\eta)(y) dy, \]
respectively. Clearly, \( f \) is a Lipschitz continuous function and \( g \) is a continuous function.

For nonlocal history function \( h_0 \), we may have any of the following.

(I) \( h_0(\psi)(x) = \int_0^t k(s)\psi(s)(x)ds \), for \( x \in \Omega \) and \( \psi \in \mathcal{C}_0 \), where \( k \in L^1(\tau,0) \) with \( \kappa := \int_{\tau}^0 k(s)ds \neq 0 \).

(II) \( h_0(\psi)(x) = \sum_{i=1}^n c_i \psi(\theta_i)(x) \) for \( x \in \Omega \) and \( \psi \in \mathcal{C}_0 \), where \( -\tau = \theta_1 < \theta_2 < \cdots < \theta_n < 0 \), \( n \in \mathbb{N} \), \( c_i \geq 0 \), and \( C := \sum_{i=1}^n c_i \neq 0 \).

(III) \( h_0(\psi)(x) = \sum_{i=1}^n (c_i/\epsilon_i) \int_{\theta_i-\epsilon}^{\theta_i} \psi(s)(x)ds \) for \( x \in \Omega \) and \( \psi \in \mathcal{C}_0 \), where \( \theta_i \) and \( c_i \) are as in (II) and \( \epsilon_i > 0 \), for \( i = 1,2,\ldots,n \).

Define \( h(\psi)(t) = h_0(\psi) \), for \( t \in [-\tau,0] \). Let \( \phi(t) \equiv \phi_0 \) on \([-\tau,0]\), \( \phi_0 \in D(B) \). For (I), we may take \( \chi(t) = (1/\kappa)\phi_0 \), and for (II) as well as for (III), we may take \( \chi(t) = (1/C)\phi_0 \) on \([-\tau,0]\). Then, \( h \) satisfies (iv). Therefore, all the assumptions of Theorem 3.2 are satisfied and hence, the problem (4.1) admits a solution.

References


Sobolev-type neutral partial differential equations


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