EXISTENCE OF SOLUTIONS TO SOBOLEV-TYPE PARTIAL NEUTRAL DIFFERENTIAL EQUATIONS

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Received 3 January 2005; Revised 30 March 2005; Accepted 31 March 2005

This work is concerned with a nonlocal partial neutral differential equation of Sobolev type. Specifically, existence of the solutions to the abstract formulations of such type of problems in a Banach space is established. The results are obtained by using Schauder's fixed point theorem. Finally, an example is provided to illustrate the applications of the abstract results.

1. Introduction

Let X and Y be two real Banach spaces. We consider the following neutral differential equation of Sobolev type with a nonlocal history condition:

$$\begin{gathered}\frac{d}{dt}\left[Bu(t)+f\left(t,u(t-\tau_1)\right)\right]+Au(t)=g\left(t,u(t),u(t-\tau_2)\right),\quad 0<t\le T,\\ h\left(u|_{[-\tau,0]}\right)=\phi,\end{gathered}\tag{1.1}$$

where $\tau=\max\{\tau_1,\tau_2\}$, $\tau_i>0$, $i=1,2$, $T<\infty$, $\phi\in\mathscr{C}_0:=C([-\tau,0],X)$, and B and A are linear operators with the domains contained in X and the ranges contained in Y. $f:[0,T]\times X\to Y$, $g:[0,T]\times X\times X\to Y$ are two given functions, and the map h is defined from $\mathscr{C}_0$ into $\mathscr{C}_0$. Here $\mathscr{C}_t:=C([-\tau,t];X)$ for $t\in[0,T]$ is a Banach space of all continuous functions from $[-\tau,t]$ into X endowed with the norm

$$\|\psi\|_t:=\sup\left\{\left\|\psi(\theta)\right\|_X:\theta\in[-\tau,t]\right\}.\tag{1.2}$$

The study of differential equations with nonlocal conditions is of significance due to its applications in problems in physics and other areas of applied mathematics. Byszewski [6] proved the existence of mild, strong, and classical solutions for the nonlocal Cauchy problem. Some more results on the existence, uniqueness, and stability of solutions are

Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2006, Article ID 16308, Pages 1–10
DOI 10.1155/JAMSA/2006/16308

given by Byszewski and Akca [7], Balachandran and Chandrasekaran [2], and Byszewski and Lakshmikantham [8]. Recently, Bahuguna [1] obtained existence and regularity results for functional differential equation with nonlocal condition using semigroup theory. Brill [5] and Showalter [14, 15] established the existence of solutions of semilinear evolution equations of Sobolev type in Banach spaces. On the other hand, Balachandran and Uchiyama [4] considered an integrodifferential equation of Sobolev type with a nonlocal condition and proved the existence of mild and strong solutions.

The theory of neutral differential equations has been extensively studied in the literature. Hernández [10] established the existence results for partial neutral functional differential equations with nonlocal conditions modeled as

$$\begin{gathered}\frac{d}{dt}\big(u(t)+F(t,u_t)\big)=Au(t)+G(t,u_t),\quad 0\le t\le T,\\ u_\sigma=\phi+q\big(x_{t_1},x_{t_2},\dots,x_{t_n}\big)\in\Omega,\end{gathered}\tag{1.3}$$

where A is the infinitesimal generator of an analytic semigroup $T(t)$ on a Banach space. He made use of fixed point theorems and the results mentioned in Pazy's [13]. For results on neutral partial differential equations with nonlocal and classical conditions, we refer to the papers of Hernández and Henríquez [11], Balachandran and Sakthivel [3], Fu and Ezzinbi [9], and references therein.

Our aim in this paper is to study the existence of a solution of partial neutral differential equation of form (1.1) by using Schauder's fixed point theorem. For this purpose, we first prove the existence of a solution of (1.1) on $[-\tau,\widetilde{T}]$ for some $0<\widetilde{T}\le T$ and then prove that this solution can be extended to a solution of (1.1) either on $[-\tau,T]$ or on the maximal interval of existence $[-\tau,t_{\max}]$, $0<t_{\max}\le T$, and in the latter case we show that $\lim_{t\to t_{\max}-}\|u(t)\|_X=\infty$. Finally, we present an example to show an application of the existence theorem.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary results which we require to establish the existence of a solution of (1.1).

Henceforth, $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ will denote two real Banach spaces. The space of continuous linear mapping from X into Y will be denoted by $L(X,Y)$.

To prove our main theorem, we consider the following assumptions on the operators $A:D(A)\subset X\to Y$ and $B:D(B)\subset X\to Y$:

(i) A and B are closed, linear operators,

(ii) $D(B)\subset D(A)$ and B is bijective,

(iii) $B^{-1}:Y\to D(B)$ is compact.

From the above fact and the closed graph theorem, we get the boundedness of the linear operator $AB^{-1}:Y\to Y$. Consequently, AB^{-1} generates a uniformly continuous semigroup $e^{-tAB^{-1}}$, $t\ge 0$.

Further, we assume that

(iv) there exists a Lipschitz continuous function $\chi\in\mathscr{C}_0$ such that $h(\chi)=\phi$ on $[-\tau,0]$ and $\chi(0)\in D(B)$,

(v) $f : [0,T] \times X \to Y$ is a Lipschitz continuous function, that is,

$$\|f(t,u) - f(s,v)\|_Y \le L[|t-s| + \|u-v\|_X], \tag{2.1}$$

for all $t \in [0,T]$ and $u,v \in X$. $L > 0$ is a Lipschitz constant.

Next, we choose a $0 < \widetilde{T} \le T$ such that $\widetilde{T} = \min\{\tau_1, \tau_2\}$, then for $0 < t \le \widetilde{T}$, the functions $u(t-\tau_1)$ and $u(t-\tau_2)$ transform to known function χ.

Under the above assumptions, we prove the existence of a solution u of (1.1) in the sense that there exists a continuous function $u : [-\tau, \widetilde{T}] \to X$ such that $u(t) \in D(B)$ for all $t \in [0, \widetilde{T}]$, and in addition, $[Bu(t) + f(t, \chi(t-\tau_1))]$ is differentiable on $(0, \widetilde{T}]$ satisfying

$$\begin{gathered} \frac{d}{dt}[Bu(t) + f(t,\chi(t-\tau_1))] + Au(t) = g(t,u(t),\chi(t-\tau_2)), \quad 0 < t \le \widetilde{T}, \\ u = \chi, \quad \text{on } [-\tau, 0]. \end{gathered} \tag{2.2}$$

Note that $Bu(t)$ itself may not be differentiable on the interval of existence.

Now we mention a few results needed to establish our main result.

Proposition 2.1. *Suppose the assumptions (i)–(v) are satisfied. Let $F : [0,T] \to Y$ and $\widetilde{F} : [0,T] \to Y$ be Bochner integrable functions and let*

$$\begin{aligned} U(t) &= Bu(t) + H(t), \\ \widetilde{U}(t) &= B\widetilde{u}(t) + \widetilde{H}(t), \\ w &= \|AB^{-1}\|_Y. \end{aligned} \tag{2.3}$$

If u, $\widetilde{u}$ are solutions of

$$\begin{gathered} \frac{d}{dt}[Bu(t) + H(t)] + Au(t) = F(t), \\ u = \chi, \quad \text{on } [-\tau, 0]; \end{gathered} \tag{2.4}$$

$$\begin{gathered} \frac{d}{dt}[B\widetilde{u}(t) + \widetilde{H}(t)] + A\widetilde{u}(t) = \widetilde{F}(t), \\ \widetilde{u} = \widetilde{\chi}, \quad \text{on } [-\tau, 0], \end{gathered} \tag{2.5}$$

respectively, where

$$H(t) = f(t,\chi(t-\tau_1)), \qquad \widetilde{H}(t) = f(t,\widetilde{\chi}(t-\tau_1)), \quad \forall [0,T]. \tag{2.6}$$

Then, the following hold for $0 \le s \le t \le T$:

(I)

$$\begin{aligned} \|U(t) - \widetilde{U}(t)\|_Y \le{}& e^{w(t-s)}\|U(s) - \widetilde{U}(s)\|_Y + \int_s^t e^{w(t-\eta)}\|F(\eta) - \widetilde{F}(\eta)\|_Y d\eta \\ &+ \int_s^t e^{w(t-\eta)} w \|H(\eta) - \widetilde{H}(\eta)\|_Y d\eta, \end{aligned} \tag{2.7}$$

(II)

$$||U(t)-U(s)||_Y \le \int_s^t e^{w(t-\eta)}||F(\eta)-Au(s)||_Y d\eta + \int_s^t e^{w(t-\eta)} w||H(\eta)-H(s)||_Y d\eta, \tag{2.8}$$

(III)

$$\begin{aligned} ||Au(t)-A\chi(0)||_Y \le w\Bigg[&\int_0^t e^{w(t-\eta)}||F(\eta)-A\chi(0)||_Y d\eta \\ &+ \int_0^t e^{w(t-\eta)} w||H(\eta)-H(0)||_Y d\eta + ||H(t)-H(0)||_Y \Bigg]. \end{aligned} \tag{2.9}$$

Proof. Let Y^* be the dual space of Y and let the duality map $\mathcal{J} : Y \to 2^{Y^*}$ be given by

$$\mathcal{J}(y) = \left\{ y^* \in Y^* : \langle y, y^* \rangle = \|y\|_Y^2 = ||y^*||_{Y^*}^2 \right\}. \tag{2.10}$$

Here, 2^{Y^*} denotes the power set of Y^*, $\|\cdot\|_Y$ and $\|\cdot\|_{Y^*}$ are the norms of Y and Y^* respectively, and $\langle y, y^* \rangle$ is the value of $y^* \in Y^*$ at $y \in Y$.

Define a function $\langle \cdot, \cdot \rangle : Y \times Y \to \mathbb{R}$ by

$$\langle x, y \rangle := \inf \{ (x, y^*) : y^* \in F(y) \}. \tag{2.11}$$

(I) Since $\|U - \widetilde{U}\|_Y : [0,T] \to \mathbb{R}$ is a differentiable function for a.e. $\eta \in (0,T)$, we may apply the result of Kato [12, Lemma 1.3]. Thus, we have

$$\begin{aligned} &||U(\eta)-\widetilde{U}(\eta)||_Y \left(\frac{d}{d\eta}\right) ||U(\eta)-\widetilde{U}(\eta)||_Y \\ &\quad = \langle F(\eta) - Au(\eta) - \widetilde{F}(\eta) + A\widetilde{u}(\eta), U(\eta) - \widetilde{U}(\eta) \rangle \\ &\quad \le ||F(\eta)-\widetilde{F}(\eta)||_Y ||U(\eta)-\widetilde{U}(\eta)||_Y + ||Au(\eta)-A\widetilde{u}(\eta)||_Y ||U(\eta)-\widetilde{U}(\eta)||_Y. \end{aligned} \tag{2.12}$$

It implies

$$\begin{aligned} \frac{d}{d\eta}||U(\eta)-\widetilde{U}(\eta)||_Y &\le ||F(\eta)-\widetilde{F}(\eta)||_Y + w||Bu(\eta)-B\widetilde{u}(\eta)||_Y \\ &\le ||F(\eta)-\widetilde{F}(\eta)||_Y + w\Big[||U(\eta)-\widetilde{U}(\eta)||_Y + ||H(\eta)-\widetilde{H}(\eta)||_Y \Big], \end{aligned} \tag{2.13}$$

which gives

$$\begin{aligned} ||U(t)-\widetilde{U}(t)||_Y \le e^{w(t-s)}||U(s)-\widetilde{U}(s)||_Y &+ \int_s^t e^{w(t-\eta)}||F(\eta)-\widetilde{F}(\eta)||_Y d\eta \\ &+ \int_s^t e^{w(t-\eta)} w||H(\eta)-\widetilde{H}(\eta)||_Y d\eta. \end{aligned} \tag{2.14}$$

(II) If we take

$$\widetilde{u}(t) \equiv u(s), \quad \widetilde{H}(t) \equiv H(s), \quad \text{for } t \in [0,T], \tag{2.15}$$

then the estimate (II) is followed by (I).

(III) By using the fact $\|Ax\|_Y \le w\|Bx\|_Y$ for every $x \in D(B)$ in (II), we obtain

$$\begin{aligned} \|Au(t) - A\chi(0)\|_Y &\le w\|Bu(t) - B\chi(0)\|_Y \\ &\le w\Bigg[\int_0^t e^{w(t-\eta)}\|F(\eta) - A\chi(0)\|_Y d\eta \\ &\quad + \int_0^t e^{w(t-\eta)} w\|H(\eta) - H(0)\|_Y d\eta + \|H(t) - H(0)\|_Y\Bigg]. \end{aligned} \tag{2.16}$$

□

Corollary 2.2. *Let $F : [0,T] \to Y$ be a continuous function and let u be a solution of (2.4). Also, suppose that the assumptions (i)–(v) are satisfied. Then, there exist constants $\alpha = \alpha(T)$ and $\beta = \beta(T)$ such that*

$$\|u(t) - u(s)\|_X \le \alpha\left[\sup_{0\le\eta\le T}\|F(\eta) - A\chi(0)\|_Y + \beta\right](t-s), \quad \textit{for } 0 \le s < t \le T. \tag{2.17}$$

Proof. Let us take b as the norm of $B^{-1} \in L(Y,X)$.

Using Proposition 2.1(II) and (III), we obtain

$$\begin{aligned} \|u(t) - u(s)\|_X &\le b\|Bu(t) - Bu(s)\|_Y \\ &\le b\Bigg[\int_s^t e^{w(t-\eta)}\|F(\eta) - Au(s)\|_Y d\eta \\ &\quad + \int_s^t e^{w(t-\eta)} w\|f(\eta,\chi(\eta-\tau_1)) - f(s,\chi(s-\tau_1))\|_Y d\eta \\ &\quad + \|f(t,\chi(t-\tau_1)) - f(s,\chi(s-\tau_1))\|_Y\Bigg] \\ &\le b\Bigg[\sup_{0\le\eta\le T}\|F(\eta) - A\chi(0)\|_Y\left(\frac{e^{wt} - e^{ws}}{w}\right) \\ &\quad + c\left(e^{w(t-s)} - 1\right)\left(\frac{e^{wt} - 1}{w}\right) + c\left(\frac{e^{w(t-s)} - 1}{w}\right)\Bigg] \\ &\le b\left[\sup_{0\le\eta\le T}\|F(\eta) - A\chi(0)\|_Y + ce^{wT}\right]e^{wT}|t-s|. \end{aligned} \tag{2.18}$$

Set $\alpha = be^{wT}$ and $\beta = ce^{wT}$. Thus,

$$\|u(t) - u(s)\|_X \le \alpha\left[\sup_{0\le\eta\le T}\|F(\eta) - A\chi(0)\|_Y + \beta\right]|t-s|. \tag{2.19}$$

□

3. Existence of solutions

We start by establishing a result of existence of a solution to the following abstract Cauchy problem:

$$\begin{gathered}\frac{d}{dt}\left[Bu(t)+f\left(t,\chi(t-\tau_1)\right)\right]+Au(t)=F(t), \quad 0<t\leq \widetilde{T},\\ u=\chi, \quad \text{on } [-\tau,0].\end{gathered} \tag{3.1}$$

Proposition 3.1. *If assumptions (i)–(v) are satisfied and $F:[0,\widetilde{T}]\to Y$ is a continuous function, then the Cauchy problem (3.1) has a unique solution.*

Proof. Since $AB^{-1}:Y\to Y$ is a bonded linear operator, it follows that the problem

$$\begin{gathered}\frac{d}{dt}w(t)+AB^{-1}w(t)=G(t), \quad t\in(0,\widetilde{T}],\\ w(t)=\widetilde{\chi}(t), \quad t\in[-\tau,0],\end{gathered} \tag{3.2}$$

where

$$G(t)=F(t)+AB^{-1}f\left(t,\chi(t-\tau_1)\right), \quad \forall t\in[0,\widetilde{T}], \tag{3.3}$$

has a solution $w:[-\tau,\widetilde{T}]\to Y$, given by

$$w(t)=e^{-tAB^{-1}}\widetilde{\chi}(0)+\int_0^t e^{-(t-s)AB^{-1}}G(s)ds, \quad -\tau\leq t\leq \widetilde{T}. \tag{3.4}$$

Consequently, a function $u(t)=B^{-1}(w(t)-f(t,\chi(t-\tau_1)))$ is a solution of (3.1). Uniqueness of a solution is obtained by using Proposition 2.1(I). □

Our main existence theorem is the following.

Theorem 3.2. *Let the assumptions (i)–(v) be satisfied and suppose that g is a continuous function from $[0,T]\times X\times X$ into Y. Then, for any given $\chi(0)\in D(B)$, there exists a solution of the abstract Cauchy problem (2.2) on the subinterval $[-\tau,\widetilde{T}]\subset[-\tau,T]$.*

Proof. Let $0<\widetilde{T}\leq T$ and $R>0$ be real numbers and define the set

$$S_0=S(A\chi(0),R,\widetilde{T}):=\left\{F\in C([0,\widetilde{T}];Y):\sup_{0\leq t\leq\widetilde{T}}\|F(t)-A\chi(0)\|_Y\leq R\right\}. \tag{3.5}$$

It is easy to see that S_0 is a bounded, closed, and convex subset of $C([0,\widetilde{T}];Y)$.

Since all the hypotheses of Proposition 3.1 are verified, there exists a solution u_F, $F\in S_0$, of the problem (3.1).

We define a mapping $Z:S_0\to C([0,\widetilde{T}];Y)$ by

$$Z(F)(t):=g\left(t,u_F(t),\chi(t-\tau_2)\right). \tag{3.6}$$

The proof will be given in three steps.
Step 1. Z is continuous and maps S_0 into itself.

Let $R > \sup_{0\le t\le T} \|g(t,\chi(0),\chi(0)) - A\chi(0)\|_Y$ be arbitrary. By the continuity of g, we have $\|g(t,x,\xi) - A\chi(0)\|_Y \le R$, for all $t \in [0,T]$, and $x,\xi \in X$ provided $\|(x,\xi) - (\chi(0), \chi(0))\|_X \le \delta$.

Suppose that $\|B^{-1}\|_Y = b$, $c > 0$ is a constant, and $0 < \widetilde{T} \le T$ is such that

$$b\left[\frac{e^{w\widetilde{T}} - 1}{w}\right](R+c) \le \delta. \tag{3.7}$$

In view of these conditions and Proposition 2.1(II), we can easily show that Z maps S_0 into S_0. Now, we will show that Z is continuous. To this end, we introduce

$$V = \{u_F(t) : 0 \le t \le \widetilde{T},\ F \in S_0\}, \tag{3.8}$$

and using Proposition 2.1(II), we observe that $B(V) \subset Y$ is a bounded set. Hence, V is relatively compact in X. Therefore, $(t,v) \mapsto g(t,v,\chi(t-\tau_2))$ is uniformly continuous on $[0,\widetilde{T}] \times V$ and Proposition 2.1(I) implies that $Z : S_0 \to C([0,\widetilde{T}];Y)$ is continuous.
Step 2. $Z(S_0)$ is equicontinuous.

Let $F \in S_0$ and $t_1, t_2 \in [0,\widetilde{T}]$. Then, if $0 < t_1 < t_2 \le \widetilde{T}$,

$$\|Z(F)(t_1) - Z(F)(t_2)\|_Y = \|g(t_1,u_F(t_1),\chi(t_1-\tau_2)) - g(t_2,u_F(t_2),\chi(t_2-\tau_2))\|_Y. \tag{3.9}$$

The right-hand side tends to zero as $t_2 - t_1 \to 0$, since from Corollary 2.2 we obtain $\|u_F(t_1) - u_F(t_2)\|_X \le \alpha(R+\beta)|t_1 - t_2|$ and g is a uniformly continuous function. Thus, Z maps S_0 into an equicontinuous family of functions.
Step 3. Z maps S_0 into precompact set in Y.

Let $t \in [0,\widetilde{T}]$ be fixed. Then, image of $\{t\} \times V \times \{\chi\}$ under continuous function g is precompact in Y. Therefore, the set $\{Z(F)(t) : F \in S_0\}$ is precompact in Y.

As a consequence of Step 2 and Step 3 together with the Ascoli-Arzela theorem, we infer that $Z(S_0)$ is relatively compact in Y. Hence, by Schauder's fixed point theorem, we deduce that the operator Z has a fixed point. This means that the problem (2.2) has a solution. □

Next we will prove the following global existence result.

Theorem 3.3. *Suppose that all the hypotheses of Theorem 3.2 are satisfied. Then, (2.2) has a solution either on* $[-\tau,T]$ *or on the maximal interval of existence* $[-\tau,t_{\max})$, $0 < t_{\max} \le T$, *and in the latter case,* $\lim_{t\to t_{\max}-} \|u(t)\|_X = \infty$.

Proof. Since all the assumptions of Theorem 3.2 are satisfied, there exists a solution u of (1.1) on $[-\tau,\widetilde{T}]$.

Suppose $\widetilde{T} < T$ and consider the problem

$$\begin{gathered}\frac{d}{dt}\left[Bw(t) + f\left(t+\widetilde{T},\widetilde{\chi}(t-\tau_1)\right)\right] + Aw(t) = g\left(t+\widetilde{T},\widetilde{\chi}(t-\tau_2),w(t)\right), \quad 0 < t \le T - \widetilde{T}, \\ w = \widetilde{\chi}, \quad \text{on } [-\tau - \widetilde{T}, 0],\end{gathered} \tag{3.10}$$

where

$$\widetilde{\chi}(t) = \begin{cases} \chi(t+\widetilde{T}), & t \in [-\tau - \widetilde{T}, -\widetilde{T}], \\ u(t+\widetilde{T}), & t \in [-\widetilde{T}, 0]. \end{cases} \tag{3.11}$$

Since $\widetilde{\chi}(0) = u(\widetilde{T}) \in D(B)$, f satisfies the assumption (v), and g is a continuous function, we may proceed as before and prove the existence of a solution $w : [-\tau, T_1] \to X$, $0 < T_1 \le T - \widetilde{T}$, of the considered problem (3.10).

Then, the function $\bar{u} : [-\tau, \widetilde{T} + T_1] \to X$, given by

$$\bar{u}(t) = \begin{cases} u(t), & t \in [-\tau, \widetilde{T}], \\ w(t - T_0), & t \in [\widetilde{T}, \widetilde{T} + T_1], \end{cases} \tag{3.12}$$

is a solution of (2.2) on $[-\tau, \widetilde{T} + T_1]$. Continuing this way, we may prove the existence either on the whole interval $[-\tau, T]$ or on the maximal interval of existence $[-\tau, t_{\max})$, $0 < t_{\max} \le T$. In the latter case, if $\lim_{t\to t_{\max}-} \|u(t)\|_X < \infty$, then as $u(t) \in D(B)$ for $t \in [0, t_{\max})$, we have that $\lim_{t\to t_{\max}-} u(t)$ is in the closure of $D(B)$ in X, and if it is in $D(B)$, then proceeding as before, we may extend $u(t)$ beyond $t_{\max}$, but this will contradict the definition of the maximal interval of existence. Therefore, $\lim_{t\to t_{\max}-} \|u(t)\|_X = \infty$. □

4. An example

In this section, we consider an example to illustrate the abstract results.

Consider the following initial boundary value problem:

$$\begin{gathered}\frac{\partial}{\partial t}\left[(1-\Delta)w(t,x) + \int_\Omega k_1(t,x,y)b_1\left(w(t-\tau_1,y)\right)dy\right] - \Delta w(t,x) \\ = \int_\Omega k_2(t,x,y)b_2\left(w(t,y),w(t-\tau_2,y)\right)dy, \quad x \in \Omega,\ t \in (0,T], \\ w(t,x) = 0, \quad t \ge 0,\ x \in \partial\Omega, \\ h_0\left(w(t,x)\right) = \phi_0(x), \quad x \in \bar{\Omega},\ t \in [-\tau, 0],\end{gathered} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, and Δ is the N-dimensional Laplacian. To represent problem (4.1) as the Cauchy problem (1.1), we take $X = L^p(\Omega)$, $Y = C(\bar{\Omega})$, and define $u(t)(x) = w(t,x)$, $B_1(u(t-\tau_1))(x) = b_1(w(t-\tau_1,x))$, and $B_2(u(t),u(t-\tau_2))(x) = b_2(w(t,x),w(t-\tau_1,x))$. The operators A and B are

defined by

$$\begin{aligned} Az &= -\Delta z, \quad \forall z \in D(A), \\ B\widetilde{z} &= \widetilde{z} - \Delta\widetilde{z}, \quad \forall \widetilde{z} \in D(B), \end{aligned} \tag{4.2}$$

with the domains

$$D(A) = D(B) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega), \quad p > n. \tag{4.3}$$

Then, A and B are closed, linear operators. Furthermore, A is an accerative operator and $D(A)$ is compactly imbedded in Y. Therefore, A and B verify the assumptions (i)–(iii).

We assume the following conditions.

(i) $k_1(t,x,y)$ and $k_2(t,x,y)$ are real-valued Lipschitz continuous and continuous functions, respectively, on $[0,T] \times \Omega \times \Omega$ satisfying

$$\sup_{x\in\Omega} \int_\Omega \left(\int_\Omega |k_i(t,x,y)|^q dy \right)^{1/q} dx < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \ i = 1,2. \tag{4.4}$$

(ii) b_1 is a real-valued Lipschitz continuous function and b_2 is a real-valued continuous function.

Define $f : [0,T] \times L^p(\Omega) \to C(\bar{\Omega})$ and $g : [0,T] \times L^p(\Omega) \times L^p(\Omega) \to C(\bar{\Omega})$ by

$$\begin{aligned} f(t,\xi)(x) &= \int_\Omega k_1(t,x,y) B_1(\xi)(y) dy, \\ g(t,\nu,\eta)(x) &= \int_\Omega k_2(t,x,y) B_2(\nu,\eta)(y) dy, \end{aligned} \tag{4.5}$$

respectively. Clearly, f is a Lipschitz continuous function and g is a continuous function. For nonlocal history function h_0, we may have any of the following.

(I) $h_0(\psi)(x) = \int_{-\tau}^0 k(s)\psi(s)(x)ds$, for $x \in \Omega$ and $\psi \in \mathscr{C}_0$, where $k \in L^1(-\tau,0)$ with $\kappa := \int_{-\tau}^0 k(s)ds \neq 0$.

(II) $h_0(\psi)(x) = \sum_{i=1}^n c_i \psi(\theta_i)(x)$ for $x \in \Omega$ and $\psi \in \mathscr{C}_0$, where $-\tau \le \theta_1 \le \theta_2 < \cdots < \theta_n \le 0$, $n \in \mathbb{N}$, $c_i \ge 0$, and $C := \sum_{i=1}^n c_i \neq 0$.

(III) $h_0(\psi)(x) = \sum_{i=1}^n (c_i/\epsilon_i) \int_{\theta_i-\epsilon_i}^{\theta_i} \psi(s)(x)ds$ for $x \in \Omega$ and $\psi \in \mathscr{C}_0$, where θ_i and c_i are as in (II) and $\epsilon_i > 0$, for $i = 1,2,\ldots,n$.

Define $h(\psi)(t) = h_0(\psi)$, for $t \in [-\tau,0]$. Let $\phi(t) \equiv \phi_0$ on $[-\tau,0]$, $\phi_0 \in D(B)$. For (I), we may take $\chi(t) = (1/\kappa)\phi_0$, and for (II) as well as for (III), we may take $\chi(t) = (1/C)\phi_0$ on $[-\tau,0]$. Then, h satisfies (iv). Therefore, all the assumptions of Theorem 3.2 are satisfied and hence, the problem (4.1) admits a solution.

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