Approaching Fixed Points of Non-Self Asymptotically Nonexpansive Mappings in Banach Spaces

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Suppose $K$ is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space $E$ with $P$ as a nonexpansive retraction. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $F(T)$ is nonempty, where $F(T)$ denotes the fixed points set of $T$. Let $\{\alpha_n\}, \{\alpha'_n\}$, and $\{\alpha''_n\}$ be real sequences in $(0, 1)$ and $\epsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \epsilon$ for all $n \in \mathbb{N}$ and some $\epsilon > 0$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by $x_1 \in K, z_n = P(\alpha''_n T(PT)^{n-1} x_n + (1 - \alpha''_n) x_n), y_n = P(\alpha'_n T(PT)^{n-1} z_n + (1 - \alpha'_n) x_n), x_{n+1} = P(\alpha_n T(PT)^{n-1} y_n + (1 - \alpha_n) x_n)$. (i) If the dual $E^*$ of $E$ has the Kadec-Klee property, then $\{x_n\}$ converges weakly to a fixed point $p \in F(T)$; (ii) if $T$ satisfies condition (A), then $\{x_n\}$ converges strongly to a fixed point $p \in F(T)$.

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1. Introduction and preliminaries

Let $E$ be a real Banach space, let $K$ be a nonempty subset of $X$ and $F(T)$ denote the set of fixed points of $T$. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad \forall x, y \in K. \quad (1.1)$$

This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4]. They proved that if $K$ is a nonempty bounded closed convex subset of a uniformly convex Banach space $E$, then every asymptotically nonexpansive self-mapping $T$ of $K$ has a fixed point. Moreover, the fixed points set $F(T)$ of $T$ is closed and convex.

Many authors have contributed their efforts to investigate the problem of finding a fixed point of asymptotically nonexpansive mapping. In [8, 9], Schu introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive...
self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space \( H \). More precisely, he proved the following theorems.

**Theorem 1.1** (see [8]). Let \( H \) be a Hilbert space, \( K \) a nonempty closed convex and bounded subset of \( H \), and let \( T : K \to K \) be a completely continuous asymptotically nonexpansive with sequence \( \{k_n\} \subset [1, \infty) \), \( k_n \to 1 \), and \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \). Let \( \{\alpha_n\}_{n=1}^{\infty} \) be a real sequence in \([0, 1]\) satisfying the condition \( \epsilon \leq \alpha_n \leq 1 - \epsilon \) for all \( n \geq 1 \) and for some \( \epsilon > 0 \). Then the sequence \( \{x_n\} \) generated from arbitrary \( x_1 \in K \) by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1,
\]

converges strongly to a fixed point of \( T \).

**Theorem 1.2** (see [9]). Let \( E \) be a uniformly convex Banach space satisfying Opial’s condition, \( K \) a nonempty closed convex and bounded subset of \( E \), and \( T : K \to K \) an asymptotically nonexpansive with sequence \( \{k_n\} \subset [1, \infty) \), \( k_n \to 1 \), and \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \). Let \( \{\alpha_n\}_{n=1}^{\infty} \) be a real sequence in \([0, 1]\) satisfying the condition \( 0 < a \leq \alpha_n \leq b < 1 \) for all \( n \geq 1 \) and some \( a, b \in (0, 1) \). Then the sequence \( \{x_n\} \) generated from arbitrary \( x_1 \in K \) by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1,
\]

converges weakly to a fixed point of \( T \).

Subsequently, Tan and Xu [13] first proved that Schu’s theorem remains true if the assumption that \( E \) satisfies Opial’s condition is replaced by the one that \( E \) has a Fréchet differential norm. Meantime, Tan and Xu [13] proved the weak convergence of the modified Ishikawa iterative scheme in a uniformly convex Banach space which either satisfies Opial’s condition or has a Fréchet differential norm. In [7], Rhoades extended [8, Theorem 1.1] to uniformly convex Banach space using a modified Ishikawa iteration method. In [6], Osilike and Aniagbosor proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on \( K \), provided that \( F(T) = \{x \in K : Tx = x\} \neq \emptyset \).

In [12], Tan and Xu introduced a modified Ishikawa process to approximate fixed points of nonexpansive mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space \( E \). More precisely, they proved the following theorem.

**Theorem 1.3** (see [12]). Let \( E \) be a uniformly convex Banach space which satisfies Opial’s condition or has a Fréchet differentiable norm and \( C \) a nonempty closed convex bounded subset of \( E \), \( T : C \to C \) a nonexpansive mapping and let \( \{\alpha_n\}, \{\beta_n\} \) be real sequences in \([0, 1]\) such that \( \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty \) and \( \sum_{n=1}^{\infty} \beta_n (1 - \alpha_n) = \infty \). Then the sequence \( \{x_n\} \) generated from arbitrary \( x_1 \in C \) by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n], \quad n \geq 1,
\]

converges weakly to a fixed point of \( T \).

In the above results, \( T \) remains a self-mapping of a nonempty closed convex subset \( K \) of a uniformly convex Banach space if, however, the domain \( K \) of \( T \) is a proper subset
of $E$ (and this is the case in several applications), and $T$ maps $K$ into $E$, then iteration processes of Mann and Ishikawa studied by these authors may fail to be well defined.

Chidume [1] studied the iteration scheme defined by

$$x_{n+1} = P\left( (1 - \alpha_n)x_n + \alpha_n T PT \right)^n x_n, \quad n \geq 1,$$

in the framework of uniformly convex Banach space, where $K$ is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space $E$ with $P$ as a nonexpansive retraction. $T : K \rightarrow E$ is an asymptotically nonexpansive nonself map with sequence $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$. $\{\alpha_n\}_{n=1}^\infty$ is a real sequence in $[0, 1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1 - \epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. They proved strong and weak convergence theorems for asymptotically nonexpansive nonself maps.

Recently, Shahzad [11] studied the sequence $\{x_n\}$ defined by

$$x_{n+1} = P\left( (1 - \alpha_n)x_n + \alpha_n T PT \left( (1 - \beta_n)x_n + \beta_n Tx_n \right) \right), \quad n \geq 1,$$

where $K$ is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space $E$ with $P$ as a nonexpansive retraction. He proved weak and strong convergence theorems for nonself nonexpansive mappings in Banach spaces.

Motivated by Chidume [1], Shahzad [11], and some others, the purpose of this paper is to construct an iterative scheme for approximating a fixed point of asymptotically nonexpansive nonself maps (when such a fixed point exists) and to prove some strong and weak convergence theorems for such maps.

Let $K$ be a nonempty closed convex subset of a real uniformly convex Banach space $E$. In this paper, the following iteration scheme is studied:

$$x_1 \in K,$$

$$z_n = P\left( \alpha_n' T PT \right)^n x_n + (1 - \alpha_n') x_n,$$

$$y_n = P\left( \alpha_n' T PT \right)^n z_n + (1 - \alpha_n') x_n,$$

$$x_{n+1} = P\left( \alpha_n T PT \right)^n y_n + (1 - \alpha_n) x_n,$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, and $\{\alpha_n''\}$ are real sequences in $(0, 1)$.

Our theorems improve and generalize some previous results. Our weak convergence result applies not only to $L^p$-spaces with $1 < p < \infty$ but also to other spaces which do not satisfy Opial’s condition or have a Fréchet differentiable norm. More precisely, we prove weak convergence of the modified Noor-type iteration process (Noor-type iteration process was introduced by Xu and Noor [14]) in a uniformly convex Banach space whose dual has the Kadec-Klee property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a Fréchet differentiable norm nor Opial’s property; however their dual does have the Kadec-Klee property (see, e.g., [3, 5]).

Let $E$ be a real Banach space. A subset $K$ of $E$ is said to be a retract of $E$ if there exists a continuous map $P : E \rightarrow E$ such that $Px = x$ for all $x \in K$. A map $P : E \rightarrow E$ is said to be a retraction if $P^2 = P$. It follows that if a map $P$ is a retraction, then $Py = y$ for all $y$ in the range of $P$. A set $K$ is optimal if each point outside $K$ can be moved to be closer to all
points of $K$. It is well known (see, e.g., [2]) that

(i) if $E$ is a separable, strictly convex, smooth, reflexive Banach space, and if $K \subset E$

is an optimal set with interior, then $K$ is a nonexpansive retract of $E$;

(ii) a subset of $l^p$, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, op-
timal sets are closed and convex. However, every closed convex subset of a Hilbert space
is optimal and also a nonexpansive retract.

A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be demiclosed at $p$ if
whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and
$\{Tx_n\}$ converges strongly to $p$, then $Tx^* = p$.

A Banach space $E$ is said to have the Kadec-Klee property if for every sequence $\{x_n\}$ in
$E$, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ strongly together imply $\|x_n - x\| \to 0$.

Recall that the mapping $T : K \to E$ with $F(T) \neq \emptyset$, where $K$ is a subset of $E$, is said
to satisfy [10, condition (A)] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with
$f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in K$,

$$\|x - Tx\| \geq f(d(x,F(T))),$$

where $d(x,F(T)) = \inf \{\|x - p\| : p \in F(T)\}$.

In order to prove our main results, we will make use of the following lemmas.

**Lemma 1.4** (Schu [9]). Suppose that $E$ is a uniformly convex Banach space and $0 < p \leq
\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \to \infty} r_n = 0$.

**Lemma 1.5** (demiclosed principle for nonselfmap [1]). Let $E$ be a uniformly convex Ban-
ach space, $K$ a nonempty closed convex subset of $E$. Let $T : K \to E$ be an asymptotically
nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed
with respect to zero.

**Lemma 1.6** (see [3]). Let $E$ be a real reflexive Banach space such that its dual $E^*$ has the
Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in $E$ and $x^* = w^*(x_n)$; here $w^*(x_n)$
denotes the weak $w$-limit set of $\{x_n\}$. Suppose $\lim_{n \to \infty} \|tx_n + (1 - t)x^* - y^*\|$ exists for all
$t \in [0, 1]$. Then $x^* = y^*$.

**Lemma 1.7** (Tan and Xu [12]). Let $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ be three nonnegative sequences
satisfying the following conditions:

$$r_{n+1} \leq (1 + s_n)r_n + t_n \quad \forall n \geq 1.$$  \hspace{1cm} (1.10)

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \to \infty} r_n$ exists.
2. Main results

Definition 2.1 (see [1]). Let $E$ be a real-normed linear space, $K$ a nonempty subset of $E$. Let $P : E \to K$ be the nonexpansive retraction of $E$ onto $K$. A map $T : K \to E$ is said to be an asymptotically nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$ such that the following inequality holds:

$$\| T(PT)^{n-1}x - T(PT)^{n-1}y \| \leq k_n \| x - y \| \quad \forall x, y \in K, \ n \geq 1. \quad (2.1)$$

$T$ is called uniformly $L$-Lipschitzian if there exists $L > 0$ such that

$$\| T(PT)^{n-1}x - T(PT)^{n-1}y \| \leq L \| x - y \| \quad \forall x, y \in K, \ n \geq 1. \quad (2.2)$$

Lemma 2.2. Let $E$ be a uniformly convex Banach space and $K$ a nonempty closed convex subset which is also a nonexpansive retract of $E$. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7). Then $\lim_{n \to \infty} \| x_n - p \|$ exists, for any $p \in F(T)$, where $F(T)$ denotes the nonempty fixed points set of $T$.

Proof. For any given $p \in F(T)$, it follows from (1.7) that

$$\| z_n - p \| = \| P(\alpha''_n T(PT)^{n-1}x_n + (1 - \alpha''_n)x_n) - p \|
\leq \| \alpha''_n T(PT)^{n-1}x_n + (1 - \alpha''_n)x_n - p \|
\leq \alpha''_n \| T(PT)^{n-1}x_n - p \| + (1 - \alpha''_n) \| x_n - p \| \quad (2.3)
\leq \alpha''_n k_n \| x_n - p \| + (1 - \alpha''_n) \| x_n - p \|
\leq k_n \| x_n - p \|.$$

That is,

$$\| z_n - p \| \leq k_n \| x_n - p \|. \quad (2.4)$$

From (1.7) and (2.4) we get

$$\| y_n - p \| = \| P(\alpha'_n T(PT)^{n-1}z_n + (1 - \alpha'_n)x_n) - p \|
\leq \| \alpha'_n T(PT)^{n-1}z_n + (1 - \alpha'_n)x_n - p \|
\leq \alpha'_n \| T(PT)^{n-1}z_n - p \| + (1 - \alpha'_n) \| x_n - p \| \quad (2.5)
\leq \alpha'_n k_n \| z_n - p \| + (1 - \alpha'_n) \| x_n - p \|
\leq k_n^2 \| x_n - p \|.$$


Asymptotically nonexpansive mappings

That is,

\[ \| y_n - p \| \leq k_n^2 \| x_n - p \|. \]  \hfill (2.6)

Again, from (1.7) and (2.6) we have

\[ \| x_{n+1} - p \| = \| P(\alpha_n T(PT)^{n-1}y_n + (1 - \alpha_n)x_n) - p \| \\
= \| \alpha_n T(PT)^{n-1}y_n + (1 - \alpha_n)x_n - p \| \\
\leq \alpha_n \| T(PT)^{n-1}y_n - p \| + (1 - \alpha_n) \| x_n - p \| \\
\leq \alpha_n k_n \| y_n - p \| + (1 - \alpha_n) \| x_n - p \| \\
\leq \alpha_n k_n^2 \| x_n - p \| + (1 - \alpha_n) \| x_n - p \| \\
\leq k_n^2 \| x_n - p \|. \]  \hfill (2.7)

That is,

\[ \| x_{n+1} - p \| \leq (1 + (k_n^3 - 1)) \| x_n - p \|. \]  \hfill (2.8)

Note that \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \) is equivalent to \( \sum_{n=1}^{\infty} (k_n^3 - 1) < \infty \), therefore by Lemma 1.7, \( \lim_{n \to \infty} \| x_n - p \| \) exists for all \( p \in F(T) \). This completes the proof.

\textbf{Lemma 2.3.} Let \( E \) be a normed linear space, \( K \) a nonempty closed convex subset which is also a nonexpansive retract of \( E \), \( T : K \to E \) a uniformly \( L \)-Lipschitzian, starting from arbitrary \( x_1 \in K \), define the sequence \( \{ x_n \} \) by the recursion (1.7) and set \( C_n = \| x_n - T(PT)^{n-1}x_n \| \) for all \( n \geq 1 \). If \( \lim_{n \to \infty} C_n = 0 \), then \( \lim_{n \to \infty} \| x_n - Tx_n \| = 0 \).

\textbf{Proof.} It follows from (1.7) that

\[ \| x_{n+1} - x_n \| \leq \| \alpha_n T(PT)^{n-1}y_n + (1 - \alpha_n)x_n - x_n \| \\
\leq \| T(PT)^{n-1}y_n - x_n \| \\
\leq \| T(PT)^{n-1}x_n - x_n \| + \| T(PT)^{n-1}y_n - T(PT)^{n-1}x_n \| \\
\leq C_n + L \| y_n - x_n \| \\
\leq C_n + L \| \alpha_n T(PT)^{n-1}z_n + (1 - \alpha_n')x_n - x_n \| \\
\leq C_n + L \| T(PT)^{n-1}z_n - x_n \| \\
\leq C_n + L \| T(PT)^{n-1}z_n - x_n \| + L \| T(PT)^{n-1}z_n - T(PT)^{n-1}x_n \| \\
\leq C_n + LC_n + L^2 \| z_n - x_n \| \\
\leq C_n + LC_n + L^2 \| \alpha_n'' T(PT)^{n-1}x_n + (1 - \alpha_n')x_n - x_n \| \\
\leq C_n (1 + L + L^2), \]
\[ ||y_{n-1} - x_n|| \leq ||\alpha'_{n-1} T(PT)^{n-2}z_{n-1} + (1 - \alpha'_{n-1})x_{n-1} - x_n|| \]
\[ \leq ||T(PT)^{n-2}z_{n-1} - x_{n-1}|| + ||x_{n-1} - x_n|| \]
\[ \leq ||T(PT)^{n-2}x_{n-1} - x_{n-1}|| + ||T(PT)^{n-2}z_{n-1} - T(PT)^{n-2}x_{n-1}|| + ||x_{n-1} - x_n|| \]
\[ \leq C_{n-1} + LC_{n-1} + ||x_{n-1} - x_n||. \] (2.10)

Substituting (2.9) into (2.10) we obtain
\[ ||y_{n-1} - x_n|| \leq C_{n-1}(2 + 2L + L^2), \] (2.11)

On the other hand, from (2.9) and (2.11) we have
\[ ||x_n - (PT)^{n-1}x_n|| \leq ||\alpha_n - 1 (PT)^{n-2}y_{n-1} + (1 - \alpha_n) x_{n-1} - T(PT)^{n-2}x_n|| \]
\[ \leq ||T(PT)^{n-2}y_{n-1} - T(PT)^{n-2}x_n|| + ||x_{n-1} - T(PT)^{n-2}x_n|| \]
\[ \leq L||y_{n-1} - x_n|| + ||x_{n-1} - T(PT)^{n-2}x_n|| \]
\[ + ||T(PT)^{n-2}x_{n-1} - T(PT)^{n-2}x_n|| \] (2.12)
\[ \leq L||y_{n-1} - x_n|| + C_{n-1} + L||x_{n-1} - x_n|| \]
\[ \leq LC_{n-1}(3 + 3L + 2L^2) + C_{n-1}. \]

It follows from (2.12) that
\[ ||x_n - Tx_n|| \leq ||x_n - T(PT)^{n-1}x_n|| + ||T(PT)^{n-1}x_n - Tx_n|| \]
\[ \leq C_n + L||(PT)^{n-1}x_n - x_n|| \] (2.13)
\[ \leq C_n + LC_{n-1}(1 + 3L + 3L^2 + 2L^3). \]

It follows from \( \lim_{n \to \infty} C_n = 0 \) that
\[ \lim_{n \to \infty} ||x_n - Tx_n|| = 0. \] (2.14)

This completes the proof. \( \square \)

**Theorem 2.4.** Let \( E \) be a uniformly convex Banach space and \( K \) a nonempty closed convex subset which is also a nonexpansive retract of \( E \). Let \( T : K \to E \) be an asymptotically nonexpansive mapping with \( \{k_n\} \subset [1, \infty) \) such that \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \) and \( F(T) \neq \emptyset \). Let \( \{\alpha_n\}, \{\alpha'_n\}, \) and \( \{\alpha''_n\} \) be real sequences in \((0, 1)\) and \( \epsilon \geq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \epsilon \) for all \( n \in \mathbb{N} \) and some \( \epsilon > 0 \), starting from arbitrary \( x_1 \in K \), define the sequence \( \{x_n\} \) by the recursion (1.7). Then \( \lim_{n \to \infty} x_n - Tx_n = 0 \).

**Proof.** Take \( p \in F(T) \), by Lemma 2.2 we know \( \lim_{n \to \infty} ||x_n - p|| \) exists. Let \( \lim_{n \to \infty} ||x_n - p|| = c \). If \( c = 0 \), then by the continuity of \( T \) the conclusion follows. Now suppose \( c > 0 \). We claim \( \lim_{n \to \infty} ||Tx_n - x_n|| = 0. \) Taking limsup on both sides in the inequality (2.4), we
Asymptotically nonexpansive mappings have
\[
\limsup_{n \to \infty} \|z_n - p\| \leq c. \tag{2.15}
\]
Similarly, taking limsup on both sides in the inequality (2.6), we have
\[
\limsup_{n \to \infty} \|y_n - p\| \leq c. \tag{2.16}
\]
Next, we consider
\[
\|T(PT)^{n-1}y_n - p\| \leq k_n\|y_n - p\|. \tag{2.17}
\]
Taking lim sup on both sides in the above inequality and using (2.16), we get
\[
\limsup_{n \to \infty} \|T(PT)^{n-1}y_n - p\| \leq c. \tag{2.18}
\]
Again, \(\lim_{n \to \infty} \|x_{n+1} - p\| = c\) means that
\[
\liminf_{n \to \infty} \|\alpha_n(T(PT)^{n-1}y_n - p) + (1 - \alpha_n)(x_n - p)\| \geq c. \tag{2.19}
\]
On the other hand, we have
\[
\|\alpha_n(T(PT)^{n-1}y_n - p) + (1 - \alpha_n)(x_n - p)\|
\leq \alpha_n\|T(PT)^{n-1}y_n - p\| + (1 - \alpha_n)\|x_n - p\|
\leq \alpha_n k_n\|y_n - p\| + (1 - \alpha_n)\|x_n - p\|
\leq \alpha_n k_n\|x_n - p\| + (1 - \alpha_n)\|x_n - p\|
\leq k_n\|x_n - p\|. \tag{2.20}
\]
Therefore, we obtain
\[
\limsup_{n \to \infty} \|\alpha_n(T(PT)^{n-1}y_n - p) + (1 - \alpha_n)(x_n - p)\| \leq c. \tag{2.21}
\]
Combining (2.19) and (2.21), we obtain
\[
\lim_{n \to \infty} \|\alpha_n(T(PT)^{n-1}y_n - p) + (1 - \alpha_n)(x_n - p)\| = c. \tag{2.22}
\]
Hence applying Lemma 1.4, we have
\[
\lim_{n \to \infty} \|T(PT)^{n-1}y_n - x_n\| = 0. \tag{2.23}
\]
Next, it follows from
\[
\|x_n - p\| \leq \|T(PT)^{n-1}y_n - x_n\| + \|T(PT)^{n-1}y_n - p\|
\leq \|T(PT)^{n-1}y_n - x_n\| + k_n\|y_n - p\|. \tag{2.24}
\]
that we have

\[ c \leq \liminf_{n \to \infty} \| y_n - p \| \leq \limsup_{n \to \infty} \| y_n - p \| \leq c. \quad (2.25) \]

That is,

\[ \lim_{n \to \infty} \| y_n - p \| = c. \quad (2.26) \]

Again, \( \lim_{n \to \infty} \| y_n - p \| = c \) gives that

\[ \liminf_{n \to \infty} \| \alpha'_n (T(PT)^{n-1} z_n - p) + (1 - \alpha'_n) (x_n - p) \| \geq c. \quad (2.27) \]

Similarly,

\[
\| \alpha'_n (T(PT)^{n-1} z_n - p) + (1 - \alpha'_n) (x_n - p) \|
\leq \alpha'_n \| T(PT)^{n-1} z_n - p \| + (1 - \alpha'_n) \| x_n - p \|
\leq \alpha'_n k_n \| z_n - p \| + (1 - \alpha'_n) \| x_n - p \| \quad (2.28)
\leq \alpha'_n k_n^2 \| x_n - p \| + (1 - \alpha'_n) \| x_n - p \|
\leq k_n^2 \| x_n - p \|.
\]

Therefore, we have

\[ \limsup_{n \to \infty} \| \alpha'_n (T(PT)^{n-1} z_n - p) + (1 - \alpha'_n) (x_n - p) \| \leq c. \quad (2.29) \]

Combining (2.27) and (2.29) yields

\[ \lim_{n \to \infty} \| \alpha'_n (T(PT)^{n-1} z_n - p) + (1 - \alpha'_n) (x_n - p) \| = c. \quad (2.30) \]

On the other hand, we have

\[ \| (T(PT)^{n-1} z_n - p) \| \leq k_n \| z_n - p \|. \quad (2.31) \]

Taking \( \limsup \) on both sides in the above inequality and using (2.15), we have

\[ \limsup_{n \to \infty} \| (T(PT)^{n-1} z_n - p) \| \leq c. \quad (2.32) \]

Applying Lemma 1.4, it follows from (2.30) and (2.32) that

\[ \lim_{n \to \infty} \| (T(PT)^{n-1} z_n - x_n) \| = 0. \quad (2.33) \]
10 Asymptotically nonexpansive mappings

Notice that
\[
\|x_n - p\| \leq \|T(P^T)^n z_n - x_n\| + \|T(P^T)^n z_n - p\| \\
\leq \|T(P^T)^n z_n - x_n\| + k_n \|z_n - p\|
\]
which yields
\[
c \leq \liminf_{n \to \infty} \|z_n - p\| \leq \limsup_{n \to \infty} \|z_n - p\| \leq c.
\]
That is,
\[
\lim_{n \to \infty} \|z_n - p\| = c.
\]
Using the same method, we have
\[
\lim_{n \to \infty} \|\alpha_n' (T(P^T)^n x_n - p) + (1 - \alpha_n') (x_n - p)\| = c.
\]
Moreover,
\[
\|T(P^T)^n x_n - p\| \leq \|T(P^T)^n x_n - p\| \\
\leq k_n \|x_n - p\|
\]
which implies that
\[
\limsup_{n \to \infty} \|T(P^T)^n x_n - p\| \leq c.
\]
Lemma 1.4 combined with (2.37) and (2.39) yields
\[
\lim_{n \to \infty} \|T(P^T)^n x_n - x_n\| = 0.
\]
Since \(T\) is uniformly \(L\)-Lipschitzian for some \(L > 0\), it follows from Lemma 2.3 that
\[
\lim_{n \to \infty} \|x_n - T x_n\| = 0.
\]
This completes the proof. \(\square\)

**Lemma 2.5.** Let \(E\) be a uniformly convex Banach space and \(K\) a nonempty closed convex subset which is also a nonexpansive retract of \(E\). Let \(T : K \to E\) be an asymptotically nonexpansive mapping with \(F(T) \neq \emptyset\) and \(\{k_n\} \subset [1, \infty)\) such that \(\sum_{n=1}^\infty (k_n - 1) < \infty\). Let \(\{\alpha_n\}\), \(\{\alpha_n'\}\), and \(\{\alpha_n''\}\) be real sequences in \([0, 1]\) and \(\epsilon \leq \alpha_n, \alpha_n', \alpha_n'' \leq 1 - \epsilon\) for all \(n \in \mathbb{N}\) and some \(\epsilon > 0\). Starting from arbitrary \(x_1 \in K\), define the sequence \(\{x_n\}\) by the recursion (1.7). Then for all \(x^*, x^{**} \in F(T)\), the limit
\[
\lim_{n \to \infty} \|tx_n + (1 - t)x^* - x^{**}\| = 2(2.42)
\]
exists for all \(t \in [0, 1]\).
Proof. It follows from Lemma 2.2 that \( \{x_n\} \) is bounded, there exists \( \mathbb{R} > 0 \) such that \( \{x_n\} \subset C := B_R(0) \cap K \), where \( B_R(0) = \{x \in E; \|x\| \leq R\} \). Then \( C \) is a nonempty closed convex bounded subset of \( E \). We follow basically the idea of [12]. Let \( a_n(t) = \|tx_n + (1 - t)x^* - x^*\| \). Then \( \lim_{n \to \infty} a_n(0) = \|x^* - x^*\| \), and from Lemma 2.2, \( \lim_{n \to \infty} a_n(1) = \lim_{n \to \infty} \|x_n - x^*\| \) exists. Without loss of generality, we may assume that \( \lim_{n \to \infty} \|x_n - x^*\| = b > 0 \) and \( t \in (0, 1) \). Define \( T_n : C \to C \) by
\[
T_n x = P(\alpha_n T(PT)^{n-1} \{P(\alpha''_n T(PT)^{n-1}) (P(\alpha''_n T(PT)^{n-1}) + (1 - \alpha''_n))) + (1 - \alpha'_n) x\) + (1 - \alpha_n) x).
\]
(2.43)

Then
\[
\|T_n x - T_n y\| \leq \kappa^3_n \|x - y\|.
\]
(2.44)

Set \( S_{n,m} := T_{n+m-1} T_{n+m-2} \ldots T_n \), \( m \geq 1 \), and
\[
b_{n,m} = ||S_{n,m}(tx_n + (1 - t)x^*) - (tS_{n,m}x_n + (1 - t)x^*)||.
\]
(2.45)

Then
\[
||S_{n,m}x - S_{n,m}y|| \leq \prod_{j=n}^{n+m-1} k^3_j \|x - y\|,
\]
(2.46)

\( S_{n,m}x_n = x_{n+m} \) and \( S_{n,m}p = p \) for all \( p \in F(T) \). It follows from [3, Lemma 2.1 and Theorem 2.3] that \( \lim_{n \to \infty} b_{n,m} = 0 \). Note that
\[
a_{n+m}(t) = \|tx_{n+m} + (1 - t)x^* - x^*\|
\]
\[
\leq b_{n,m} + ||S_{n,m}(tx_n + (1 - t)x^*) - x^*||
\]
(2.47)

\[
\leq b_{n,m} + \prod_{j=n}^{n+m-1} k^3_j a_n(t).
\]

Note that \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \) is equivalent to \( \sum_{n=1}^{\infty} (k_n^3 - 1) < \infty \). Since \( k_n \in [1, \infty) \), \( \lim_{n \to \infty} \prod_{j=n}^{\infty} k^3_j = 1 \). Therefore, we have
\[
\limsup_{n \to \infty} a_{n}(t) \leq \lim_{n,m \to \infty} b_{n,m} + \lim_{n \to \infty} a_n(t) = \liminf_{n \to \infty} a_n(t).
\]
(2.48)

That is,
\[
\lim_{n \to \infty} \|tx_n + (1 - t)x^* - x^*\|
\]
exists for all \( t \in [0,1] \). This completes the proof.

\[\square\]

**Theorem 2.6.** Let \( E \) be a uniformly convex Banach space such that its dual \( E^* \) has the Kadec-Klee property and \( K \) a nonempty closed convex subset which is also a nonexpansive retract of \( E \). Let \( T : K \to E \) be an asymptotically nonexpansive mapping with \( \{k_n\} \subset [1, \infty) \),
12 Asymptotically nonexpansive mappings

$k_n \to 1$ as $n \to \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\alpha'_n\}$, and $\{\alpha''_n\}$ be real sequences in $[0,1]$ and $\epsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \epsilon$ for all $n \in \mathbb{N}$ and some $\epsilon > 0$, starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7). Then $\{x_n\}$ converges weakly to some fixed point of $T$.

**Proof.** Since $\{x_n\}$ is bounded and $E$ is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $u \in K$. It follows from Theorem 2.4 that $\lim_{n \to \infty} \|Tx_{n_k} - x_{n_k}\| = 0$. By Lemma 1.5, we have $u = Tu$. Next we claim $\{x_n\}$ converges weakly to $u$. Suppose $\{x_{n_k}\}$ is another subsequence of $\{x_n\}$ converging to some $v \in K$. Then $u, v \in w_u(x_n) \cap F(T)$. Using Lemma 2.5 yields that the limit

$$\lim_{n \to \infty} \|tx_n + (1 - t)u - v\|$$

exists for all $t \in (0,1)$. By Lemma 1.6 we have that $u = v$. Then $\{x_n\}$ converges weakly to some fixed point of $T$. \qed

Next, we will prove a strong convergence theorem.

**Theorem 2.7.** Let $E$ be a uniformly convex Banach space and $K$ a nonempty closed convex subset which is also a nonexpansive retract of $E$. Let $T : K \to E$ be a nonexpansive mapping with $p \in F(T) := \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$, $\{\alpha'_n\}$, and $\{\alpha''_n\}$ be real sequences in $[0,1]$ and $\epsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \epsilon$ for all $n \in \mathbb{N}$ and some $\epsilon > 0$, starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7). Suppose $T$ satisfies condition (A). Then $\{x_n\}$ converges strongly to some fixed point of $T$.

**Proof.** By Lemma 2.2, $\lim_{n \to \infty} \|x_n - p\|$ exists for all $p \in F = F(T)$. Let $\lim_{n \to \infty} \|x_n - p\| = c$ for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. By Theorem 2.4, $\lim_{n \to \infty} \|Tx_n - x_n\| = 0$, and (2.8) gives

$$\inf_{p \in F} \|x_{n+1} - p\| \leq (1 + (k_n^3 - 1)) \inf_{p \in F} \|x_n - p\|.
$$

That is,

$$d(x_{n+1}, F) \leq (1 + (k_n^3 - 1)) d(x_n, F),$$

which gives that $\lim_{n \to \infty} d(x_n, F)$ exists by the virtue of Lemma 1.7. Now by condition (A), $\lim_{n \to \infty} f(d(x_n, F)) = 0$. Since $f$ is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n \to \infty} d(x_n, F) = 0$. Now we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and sequence $\{y_j\} \subset F$ such that $\|x_{n_k} - y_j\| < 2^{-j}$. Then following the method of the proof of Tan and Xu [12], we get that $\{y_j\}$ is a Cauchy sequence in $F$ and so it converges. Let $y_j \to y$. Since $F$ is closed, therefore $y \in F$ and then $x_{n_k} \to y$. As $\lim_{n \to \infty} \|x_n - p\|$ exists, $x_n \to y \in F = F(T)$ thereby completing the proof. \qed

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