We introduce a generalized notion of invariance for differential inclusions, using a proximal aiming condition in terms of proximal normals. A set of sufficient conditions for the weak and strong invariance in the generalized sense are presented.

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1. Introduction

The existence of solutions and flow invariance for differential inclusions are considered in [1] by using a generalized concept of solutions, namely, the Euler solutions of differential equations, without any continuity assumptions. This is done by utilizing a proximal aiming condition in terms of proximal normals. In a recent paper [2], we generalized the concept of proximal normal in the spirit of [3], and then, employing a generalized proximal aiming condition, we proved the existence and flow invariance results for solutions of differential inclusions.

Here in this paper, we consider a generalized notion of invariance, retaining the original notion of proximal normals as in [1], and study the corresponding results for differential inclusions. This generalized notion of flow invariance is useful in studying the solution sets of fuzzy differential equations, which will be considered in a separate paper.

2. Preliminaries

Consider the Cauchy problem

\[ x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t_0 \geq 0, \]  

(2.1)

where \( f: [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) is any function.
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Let \( \pi = \{ t_0, t_1, \ldots, t_N = T \} \) be a partition of \([t_0, T]\). On the interval \([t_0, t_1]\), we consider the differential equation with constant right-hand side

\[
x'(t) = f(t, x(t)), \quad x(t_0) = x_0,
\]

which has a unique solution, \( x(t) \) on \([t_0, t_1]\). Let \( x_1 = x(t_1) \). Next, consider, on the interval \([t_1, t_2]\), the IVP

\[
x'(t) = f(t, x(t)), \quad x(t_1) = x_1.
\]

We take \( x_2 = x(t_2) = x(t_2, t_1, x_1) \) as the next node and proceeding in this manner until we get an arc \( x_\pi = x_\pi(t) \) defined on all of \([t_0, T]\). The notation \( x_\pi \) is employed to emphasize the role played by the particular partition \( \pi \) in defining \( x_\pi \) which is the Euler Polygonal arc corresponding to the partition \( \pi \). The diameter \( \mu_\pi \) of the partition \( \pi \) is given by

\[
\mu_\pi := \max \{ \tau_i - \tau_{i-1} : 1 \leq i \leq N \}.
\]

By an Euler solution to the IVP (2.1), we mean any arc \( x(t) \) which is the uniform limit of the Euler polygonal arcs \( x_\pi \), corresponding to some sequence of partitions \( \pi_j \) such that the diameters \( \mu_\pi \to 0 \) as \( j \to \infty \). Clearly, this Euler arc satisfies the initial condition \( x(t_0) = x_0 \) and the corresponding number \( N_j \) of the partition points in \( \pi_j \) tends to infinity.

The following theorem, concerning the existence of Euler solutions for (2.1), is proved in [2].

**Theorem 2.1.** Assume that

1. \( f : [t_0, T] \times \mathbb{R}^n \to \mathbb{R} \) and \( \| f(t, x) \| \leq g(t, \| x \|), (t, x) \in [t_0, T] \times \mathbb{R}^n, \) where \( g : [t_0, T] \times \mathbb{R}_+ \to \mathbb{R}_+ \) is a continuous function, nondecreasing in \((t, u)\);

2. the maximal solution \( r(t) = r(t, t_0, u_0) \) of the scalar differential equation

\[
u' = g(t, u), \quad u(t_0) = u_0 \geq 0,
\]

exists on \([t_0, T]\).

Then, there exists an Euler solution \( x(t) = x(t, t_0, x_0) \) of the IVP (2.1) on \([t_0, T]\) which satisfies a Lipschitz condition and any Euler solution of (2.1) has an estimate

\[
\| x(t) - x_0 \| \leq r(t, t_0, \| x_0 \|) - \| x_0 \|, \quad t \in [t_0, T].
\]

**Remark 2.2.** We can extend the notion of Euler solution of (2.1) on the interval \([t_0, T]\) to \([t_0, \infty)\) provided we define \( f \) and \( g \) on \([t_0, \infty)\) instead of \([t_0, T]\), assume that the maximal solution on \( r(t) \) exists on \([t_0, \infty)\), and show that Euler solution exists on every \([t_0, T]\), \( T \in (t_0, \infty)\).

3. Generalized flow invariance

Let \( S(t), \ t \in [0, \infty) \) be a family of nonempty closed subsets of \( \mathbb{R}^n \). Let \( x \in \mathbb{R}^n \) be such that \((t, x) \notin \{(t, s) : s \in S(t)\}\), for all \( t \geq 0 \). Suppose that, for \( t \geq 0 \), there exists an \( s_t \in S(t) \)
such that
\[ \| x - s_t \| = \| (t,x) - (t,s_t) \| = \inf \{ \| x - \tilde{s} \| : \tilde{s} \in S(t) \}. \] (3.1)

The set of all such \( s_t \in S(t) \), for each \( t \geq 0 \), is denoted by \( \text{proj}_{S(t)}(x) \). The vector \((t,x-s_t)\) determines a proximal normal direction to \((t,S(t))\) at \((t,s_t)\). We call any vector \( \eta_t \) of the form \((t,k(x-s_t))\), for any \( k \geq 0 \), a proximal normal (or P-normal) to \( S(t) \) at \( s_t \), at height \( t \). The set of all \( \eta_t \) obtained in this manner is called a proximal normal cone to \( S(t) \) at \( s_t \), at a height \( t \) and is denoted by \( N^p_{S(t)}(s_t) \). If \( s_t \in S(t) \) such that \( s_t \notin \text{proj}_{S(t)}(x) \) for all \((t,x) \notin \{(t,s) : s \in S(t)\}\), then we set \( N^p_{S(t)}(s_t) = \{0\} \). If \( s_t \notin S(t) \), then \( N^p_{S(t)} \) is not defined.

**Definition 3.1** (generalized flow invariance). The system \( \{(S(t),f) : t \geq t_0\} \) is said to be weakly invariant if for all \( x_0 \in S(t_0) \), there exists an Euler solution \( x(t) \) of (2.1) on \([t_0,\infty)\) such that \( x(t_0) = x_0 \) and \( x(t) \in S(t) \), \( t > t_0 \).

Note that this implies \((t,x(t)) \in (t,S(t))\), \( t \geq t_0 \). Also, if \( S(t) = S(t_0) \), for all \( t \geq t_0 \), then the above notion of weak invariance coincides with the one given in [1].

Throughout the rest of the paper, we make the following assumption.

**Assumption 3.2.** For all \( t > \tau \), \( t, \tau \in [t_0,\infty) \) and \( z \in \mathbb{R}^n \),
\[ d^2_{S(t)}(z) \leq d^2_{S(\tau)}(z) + (t-\tau)^2. \] (3.2)

We can now prove the following result which provides sufficient conditions in terms of the generalized proximal normal for weak invariance of \( \{(S(t),f) : t \geq t_0\} \).

**Theorem 3.3.** Let \( f \) and \( g \) satisfy the assumptions of Theorem 2.1 on \([t_0,\infty)\) and let \( x(t) \) be an Euler solution on \([t_0,\infty)\) of (2.1). Suppose that \( x(t) \) lies in an open set \( \Omega \subset \mathbb{R}^n \). Assume that for every \((t,z) \in [t_0,\infty) \times \omega \), there exists an \( s_t \in \text{proj}_{S(t)}(z) \) such that
\[ 2\langle f(t,z), (z-s_t) \rangle \leq q(t,d^2_{S(t)}(z)), \] (3.3)
where \( q \in C([t_0,\infty) \times \mathbb{R}^+,\mathbb{R}) \). Suppose also that the maximal solution \( r(t) = r(t,t_0,u_0) \) of the scalar differential equation \( u' = q(t,u), u(t_0) = u_0 \geq 0 \) exists on \([t_0,\infty)\). Then,
\[ d_{S(t)}(x(t)) \leq r(t,t_0,d^2_{S(t_0)}(x_0)). \] (3.4)

If, in addition, \( r(t,t_0,0) \equiv 0 \), then \((S(t),f), t \geq t_0 \), is weakly invariant.

**Proof.** Let \( x_\pi(t) \) be one polygonal arc in the sequence, converging uniformly to \( x \) as per the definition of Euler solution of (2.1). We denote, as before, its nodes at \( t_i \) by \( x_i \), \( i = 0,1,\ldots,N \), and hence \( x(t_0) = x_0 \). Let \( x_\pi(t) \) be in \( \Omega \) for all \( t_0 \leq t \leq T \), where \( T \in (t_0,\infty) \). Accordingly, there exists for each \( i \) an element \( s_{t_i} \in \text{proj}_{S(t_i)}(x_i) \) such that
\[ 2\langle f(t_i,x_i), x_i - s_{t_i} \rangle \leq q(t_i,\|x_i - s_{t_i}\|^2). \] (3.5)
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As in Theorem 2.1, letting \( \|x_\pi\| \leq k \), we find

\[
\begin{align*}
\left| d_{S(t)}^2(x_1) \right| & \leq d_{S(t)}^2(x_1) + (t_1 - t_0)^2 \\
& \leq \|x_1 - s_{t_0}\|^2 + (t_1 - t_0)^2 \\
& \leq (t_1 - t_0)^2 + \|x_1 - x_0\|^2 + \|x_0 - s_{t_0}\|^2 + 2 \langle x_1 - x_0, x_0 - s_{t_0} \rangle \\
& \leq (k^2 + 1) (t_1 - t_0)^2 + d_{S(t)}^2(x_0) + 2 \int_{t_0}^{t_1} \langle x'_{\pi}(t), x_0 - s_{t_0} \rangle dt \\
& = (k^2 + 1) (t_1 - t_0)^2 + d_{S(t)}^2(x_0) + 2 \int_{t_0}^{t_1} \langle f(t, x), x_0 - s_{t_0} \rangle dt \\
& \leq (k^2 + 1) (t_1 - t_0)^2 + d_{S(t)}^2(x_0) + q \left( t_0, d_{S(t)}^2(x_0) \right) (t_1 - t_0).
\end{align*}
\] (3.6)

Since similar estimates hold at any node, we have for \( i = 1, 2, \ldots, N \),

\[
\begin{align*}
\left| d_{S(t)}^2(x_i) \right| & \leq d_{S(t-1-i)}^2(x_{i-1}) + (k^2 + 1) (t_i - t_{i-1})^2 + q \left( t_{i-1}, d_{S(t-1-i)}^2(x_{i-1}) \right) (t_i - t_{i-1}).
\end{align*}
\] (3.7)

And therefore, it follows that

\[
\begin{align*}
\left| d_{S(t)}^2(x_i) \right| & \leq d_{S(t)}^2(x_0) + (k^2 + 1) \sum_{j=1}^{i} (t_j - t_{j-1})^2 + \sum_{j=1}^{i} q \left( t_{j-1}, d_{S(t-j-1)}^2(x_{j-1}) \right) (t_j - t_{j-1}) \\
& \leq d_{S(t)}^2(x_0) + (k^2 + 1) \mu_{\pi} \sum_{j=1}^{i} (t_j - t_{j-1})^2 + \sum_{j=1}^{i} q \left( t_{j-1}, d_{S(t-j-1)}^2(x_{j-1}) \right) (t_j - t_{j-1}) \\
& \leq d_{S(t)}^2(x_0) + (k^2 + 1) (T - t_0) \mu_{\pi} + \sum_{j=1}^{i} q \left( t_{j-1}, d_{S(t-j-1)}^2(x_{j-1}) \right) (t_j - t_{j-1}).
\end{align*}
\] (3.8)

We now consider the sequence \( x_{\pi_i}(t) \) of polygonal arcs converging to \( x(t) \). Since the last estimate is true at every node, \( \mu_{\pi_i} \to 0 \) as \( j \to \infty \), and the same \( k \) applies to each \( x_{\pi_i} \), we deduce in the limit the integral inequality

\[
\begin{align*}
\left| d_{S(t)}^2(x(t)) \right| & \leq d_{S(t)}^2(x_0) + \int_{t_0}^{t} q \left( \tau, d_{S(t)}^2(x(\tau)) \right) d\tau, \quad t_0 \leq t \leq T,
\end{align*}
\] (3.9)

which is the same as

\[
\begin{align*}
\left| d_{S(t)}^2(x(t)) \right| & \leq r \left( t, t_0, d_{S(t)}^2(x_0) \right).
\end{align*}
\] (3.10)

If \( r(t, t_0, 0) \equiv 0 \), then assuming \( x_0 \in S(t_0) \) implies \( x(t) \in S(t) \) for \( t \geq t_0 \) and therefore the system \( (S(t), f) \), \( t \geq t_0 \), is weakly invariant. The proof is complete.

\[
\Box
\]

4. Weak invariance for differential inclusions

Consider the IVP for the differential inclusion

\[
\begin{align*}
\dot{x} & \in F(t, x), \\
x(t_0) & = x_0,
\end{align*}
\] (4.1)
where $F$ satisfies the following hypotheses:
(a) $F$ is a nonempty convex set for each $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$;
(b) $F$ is upper semicontinuous;
(c) $v \in F(t,x)$ implies that $\|v\| \leq g(t,\|x\|)$, where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t,w)$ is nondecreasing in $w$, and the maximal solution $r(t) = r(t,t_0,w_0)$, of the scalar differential equation

$$
w' = g(t,w), \quad w(0) = w_0 \geq 0,
$$

exists on $[0,\infty)$.

We recall the notions of lower and upper Hamiltonians, which are functions from $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ as follows:

$$h(t,x,p) = \min_{v \in F(t,x)} \langle p,v \rangle, \quad H(t,x,p) = \max_{v \in F(t,x)} \langle p,v \rangle. \quad (4.3)$$

We are now in a position to discuss the existence and weak invariance of $(S(t), F)$.

**Theorem 4.1.** Assume that for each $t \geq t_0$ and every $s_t \in S(t)$,

$$h\left(t,s_t, \mathcal{N}_{S(t)}^P(s_t)\right) \leq 0. \quad (4.4)$$

Suppose further that $g(t,u)$ is subadditive in $u$, for each $t$. Then the system $(S(t), F)$, $t \geq t_0$, is weakly invariant.

**Proof.** For each $t \in [t_0,\infty)$ and $x \in \mathbb{R}^n$, choose $s_t = s_t(x) \in \text{proj}_{S(t)}(x)$, and let $v_t$ in $F(t,s_t)$, minimize over $F(t,s_t)$ the function $v_t \rightarrow \langle v_t, x - s_t \rangle$.

Set $f_p(t,x) = v_t$. Since $x - s_t \in \mathcal{N}_{S(t)}^P(s_t)$, we have $\langle f_p(t,x), x - s_t \rangle \leq 0$. This implies that the main assumption of Theorem 3.3 with $q(t,u) = 0$ is satisfied. If $s_0 \in S(t_0)$ is a given element, then for each $t \geq t_0$,

$$\|f_p(t,x)\| = \|v_t\| \leq g(t,\|s_t\|) = g(t,\|s_t - x + x\|)
\leq g(t,\|s_t - x\|) + g(t,\|x\|)
\leq g(t,\|s_0 - x\|) + g(t,\|t - t_0\|^2) + g(t,\|x\|)
\leq 2g(t,\|x\|) + g(t,\|s_0\|) + g(t,\|t - t_0\|^2) = \tilde{g}(t,\|x\|). \quad (4.5)$$

Clearly $\tilde{g}(t,u) \in C([t_0,T] \times \mathbb{R}_+, \mathbb{R}_+)$ and $\tilde{g}(t,u)$ is nondecreasing in $(t,u)$. Thus $f_p(t,x)$ satisfies the nonlinear growth condition required by Theorem 2.1. Thus, by Theorem 3.3, for any $x(0) = x_0$, we have $x(t) \in S(t)$, on $[t_0,\infty)$.

The proof will be complete if we show that $x(t)$ is a solution of (4.2). Since $f_p$ is not a selection of $F$, let us define another multifunction as follows:

$$
\text{for each } t \geq t_0, \quad F_{S(t)}(t,x) = \text{co} \{ F(t,s_t) : s_t \in \text{proj}_{S(t)}(x) \}. \quad (4.6)
$$

It can be verified that $f_p(t,x)$ is a selection for $F_{S(t)}(t,x)$, that $F_{S(t)}$ satisfies the hypothesis made at the beginning of this section, and that $F_{S(t)}(t,x) = F(t,x)$ for $x \in S(t)$. Since we know that an Euler solution $x(t)$ of any selection $f_p$ of $F_{S(t)}$ is also a solution of (4.2), it follows that $x'(t) \in F_{S(t)}(t,x(t))$ a.e. Since $F = F_{S(t)}$ on $S(t)$ and $x(t) \in S(t)$, $t \geq t_0$, it
follows that \( x(t) \) is a solution of (4.2), and therefore \((S(t), F)\) is weakly invariant. The proof is complete. □

5. Strong invariance

We begin with the following definition.

**Definition 5.1.** The multifunction \( F \) is said to be locally Lipschitz in \( x \), uniformly in \( t \), provided that for all \( t \in [t_0, \infty) \), each \( x \in \mathbb{R}^n \) admits a neighborhood \( U = U(x) \) and a positive constant \( K = K(x) \) such that

\[
x_1, x_2 \in U \implies F(t, x_2) \subseteq F(t, x_1) + K||x_1 - x_2||B,
\]

where \( B \) is the closed unit ball, centred at 0.

For the remainder of this section, we make the following assumption, which is stronger than Assumption 3.2.

**Assumption 5.2.** For all \( t > \tau \), \( t, \tau \in [t_0, \infty) \), and \( z \in \mathbb{R}^n \),

\[
d_S(t)(z) \leq d_S(\tau)(z).
\]

**Theorem 5.3.** Let \((S(t), F)\) be weakly invariant and let \( F \) be locally Lipschitz in \( x \). Then there exists a feedback selection \( g_p \) for \( F \) under which \( S(t) \) is invariant.

**Proof.** Let \( f_p(t, x) \) be defined as in Theorem 4.1. Then, \( f_p(t, x) \) lies in \( F(s_t) \), where \( s_t \in \text{proj}_{S(t)}(x) \). Define, for each \( t \geq t_0 \), \( g_p(t, x) \) to be an element in \( F(T, x) \) closest to \( f_p(T, x) \) so that \( g_p \) is a selection for \( F \).

Now, suppose \( x_0 \in S(t_0) \) and \([t_0, T]\) is any interval. We will show that any Euler solution \( y(t) \) on \([t_0, T]\) from \( x_0 \) generated by \( g_p \) is such that \( y(t) \in S(t) \), \( t \in [t_0, T] \). We know there is a bound for \( y(t) \) on \([t_0, T]\) such that \( ||y(t) - x_0|| < M \). Let \( K \) be the Lipschitz constant for \( F \) on \( B[x_0, M_0] \).

If \( ||x - x_0|| < M \), then

\[
||s_t - x_0|| \leq ||s_t - x|| + ||x - x_0||
\]

\[
= d_S(t)(x) + ||x - x_0||
\]

\[
\leq d_S(t_0)(x_0) + |t - t_0| + ||x - x_0||
\]

\[
\leq 2||x - x_0|| + |T - t_0|
\]

\[
< M_0.
\]

Since \( \langle s_p(t, x), x - s_t \rangle \leq 0 \), we obtain the following estimate:

\[
\langle g_p(t, x), x - s_t \rangle = \langle f_p(t, x), x - s_t \rangle + \langle g_p(t, x) - f_p(t, x), x - s_t \rangle
\]

\[
\leq ||g_p(t, x) - f_p(t, x)||||x - s_t||^2
\]

\[
= Kd_S^2(t)(x).
\]
Thus, by [1, Exercise 2.2], and an application of Gronwall inequality, we get

\[ d_{S(t)}(y(t)) \leq d_{S(t_0)}(x_0) e^{Kt}, \quad t \in [t_0, T]. \]  

(5.5)

Since \( x_0 \in S(t_0) \), this implies that \( y(t) \in S(t), \ t \in [t_0, T], T \in (t_0, \infty) \).

We can now prove the strong invariance of the system \((S(t), F)\).

**Theorem 5.4.** Let \( F \) be locally Lipschitz and suppose that for each \( t \geq t_0 \) and every \( s_t \in S(t) \),

\[ H(t, x, N_{S(t)}(s_t)) \leq 0, \quad \forall S(t). \]  

(5.6)

Then, \((S(t), F), t \geq t_0,\) is strongly invariant.

**Proof.** Let \( y(t) \) be any solution for \( F \) on \([t_0, T]\) for each \( t \), with \( y(t_0) = x_0 \in S(0) \). As a consequence of Theorem 5.3, there exists an \( f \) such that \( y(t) \) is an Euler solution of the IVP \( x' = f(t, x), \ x(t_0) = x_0 \). As in Theorem 5.3, if \( M > 0 \) is such that all Euler solutions \( x(t) \) of this IVP satisfy \( \|x(t) - x_0\| < M \), then \( \|s_t - x\| \leq M_0 \), where \( s_t \in \text{proj}_{S(t)}(x) \). This means that \( s_t \in B(x_0, M_0) \).

Let \( K \) be the Lipschitz constant for \( F \) on \( B(x_0, M_0) \) and consider any \( x \in B(x_0, M_0) \) and \( s_t \in \text{proj}_{S(t)}(x) \). Then, \( x - s_t \in N^p_{S(t)}(s_t) \). Since \( f(t, x) \in F(t, x) \), there exists \( v \in F(t, s_t) \) so that

\[ \|v - f(t, x)\| \leq K\|s_t - x\| = Kd_{S(t)}(x). \]  

(5.7)

This leads us to

\[ \langle f(t, x), x - s_t \rangle \leq Kd^2_{S(t)}(x). \]  

(5.8)

Using an argument similar to Theorem 5.3, we conclude that \( y(t) \in S(t), \ t \in [t_0, T], \) since \( x_0 \in S(t_0) \). Since \( T \in (t_0, \infty) \), we have that \((S(t), F), t \geq t_0,\) is strongly invariant and the proof is complete. \( \square \)

**References**


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