MEAN CONVERGENCE THEOREMS AND WEAK LAWS OF LARGE NUMBERS FOR DOUBLE ARRAYS OF RANDOM VARIABLES

LE VAN THANH

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For a double array of random variables $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$, mean convergence theorems and weak laws of large numbers are established. For the mean convergence results, conditions are provided under which $\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X_{ij} - EX_{ij}) \xrightarrow{L_r} 0$ $(0 < r \leqslant 2)$ where $\{a_{mnij};\ m,n,i,j \geqslant 1\}$ are constants, and $\{k_n, n \geqslant 1\}$ and $\{l_n, n \geqslant 1\}$ are sequences of positive integers. The weak law results provide conditions for $\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n} a_{mnij}(X_{ij} - EX_{ij}) \xrightarrow{P} 0$ to hold where $\{T_m,\ m \geqslant 1\}$ and $\{\tau_n,\ n \geqslant 1\}$ are sequences of positive integer-valued random variables. The sharpness of the results is illustrated by examples.

1. Introduction

Consider a double array $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{k_n,\ n \geqslant 1\}$ and $\{l_n,\ n \geqslant 1\}$ be sequences of positive integers, let $\{T_n,\ n \geqslant 1\}$ and $\{\tau_n,\ n \geqslant 1\}$ be sequences of positive integer-valued random variables (called *random indices*), and let $\{a_{mnij};\ m,n,i,j \geqslant 1\}$ be constants. In the current work, mean convergence theorems will be established for the weighted double sums $\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X_{ij} - EX_{ij})$, $m,n \geqslant 1$, and weak laws of large numbers will be established for the weighted double sums with random indices $\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n} a_{mnij}(X_{ij} - EX_{ij})$, $m,n \geqslant 1$.

Limit theorems for weighted sums (with or without random indices) for random variables (real-valued or Banach space-valued) are studied by many authors. The reader may refer to Wei and Taylor [14], Ordóñez Cabrera [8, 9], Adler et al. [1], or more recently, Rosalsky et al. [11], Ordóñez Cabrera and Volodin [10]. However, the same problems for double sums have not yet been studied.

The main results of the paper are Theorems 3.1 and 3.2. Theorem 3.1 provides conditions for $\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X_{ij} - EX_{ij}) \xrightarrow{L_r} 0$ $(0 < r \leqslant 2)$ as $\max\{m,n\} \to \infty$ to hold. In

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Theorem 3.2, conditions are provided under which $\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n} a_{mnij}(X_{ij}-EX_{ij}) \xrightarrow{P} 0$ as $\min\{m,n\}\to\infty$.

The plan of the paper is as follows. Notation, technical definitions, and the lemmas used in proving the main results are consolidated into Section 2. The main results and their corollaries are established in Section 3. Some interesting examples/counterexamples are presented in Section 4.

2. Preliminaries

Some definitions and preliminary results will be presented prior to establishing the main results.

A sequence of random variables $\{X_n,\ n\geqslant 1\}$ is said to be *stochastically dominated* by a random variable X if for some constant $D<\infty$,

$$P\{|X_n|>t\}\leqslant DP\{|DX|>t\},\quad t\geqslant 0,\ n\geqslant 1. \tag{2.1}$$

If the $\{X_n,\ n\geqslant 1\}$ are identically distributed, then (2.1) is automatic with $X=X_1$ and $D=1$. It follows from [14, Lemma 3] of Wei and Taylor that stochastic dominance can be accomplished by the sequence of random variables having a bounded absolute rth moment ($r>0$). Specifically, if $\sup_{n\geqslant 1}E|X_n|^r<\infty$ for some $r>0$, then there exists a random variable X with $E|X|^p<\infty$ for all $0<p<r$ such that (2.1) holds with $D=1$. (The proviso that $r>1$ in [14, Lemma 3] of Wei and Taylor is not needed as was pointed out by Adler et al. [2].)

A sequence of random variables $\{X_n,\ n\geqslant 1\}$ is said to be *Cesàro uniformly integrable* if

$$\lim_{a\to\infty}\sup_{n\geqslant 1}\frac{\sum_{j=1}^{k_n}E(|X_j|I(|X_j|>a))}{k_n}=0, \tag{2.2}$$

where $\{k_n,\ n\geqslant 1\}$ is a sequence of positive integers. This condition was introduced by Chandra [4]. For a sequence of random variables $\{X_n,\ n\geqslant 1\}$, it is immediate that $\{|X_n|^r,\ n\geqslant 1\}$ being Cesàro uniformly integrable for some $r>0$ is a weaker condition than the $\{|X_n|^r,\ n\geqslant 1\}$ being stochastically dominated by a random variable X with $E|X|^r<\infty$.

Relationships between Cesàro uniform integrability and the weak law of large numbers, the strong law of large numbers, and mean convergence for a sequence of random variables were studied by Chandra [4], Chandra and Goswami [5], and Bose and Chandra [3], respectively.

The notion of Cesàro uniform integrability was extended by Ordóñez Cabrera [9] who introduced the following condition for a sequence of random variables. Let $\{a_{nj},\ 1\leqslant j\leqslant k_n\}$ be an array of constants. A sequence of random variables $\{X_n,\ n\geqslant 1\}$ is said to be $\{a_{nj}\}$*-uniformly integrable* if

$$\lim_{a\to\infty}\sup_{n\geqslant 1}\sum_{j=1}^{k_n}|a_{nj}|E(|X_j|I(|X_j|>a))=0. \tag{2.3}$$

Of course, $\{a_{nj}\}$-uniform integrability reduces to Cesàro uniform integrability when $a_{nj} = 1/k_n$, $j \geqslant 1$. Ordóñez Cabrera [9] obtained mean convergence results for weighted sums of random variables under an $\{a_{nj}\}$-uniform integrability condition.

For $a, b \in \mathbb{R}$, $\min\{a,b\}$ and $\max\{a,b\}$ will be denoted, respectively, by $a \wedge b$ and $a \vee b$. Throughout this paper, the symbol C will denote a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

The first lemma is due to von Bahr and Esseen [13].

Lemma 2.1. *Let $\{X_i,\ 1 \leqslant i \leqslant n\}$ be random variables such that $E\{X_{k+1} \mid S_k\} = 0$ for $0 \leqslant k \leqslant n-1$, where $S_0 = 0$ and $S_k = \sum_{i=1}^{k} X_i$ for $1 \leqslant k \leqslant n$. Then*

$$E\left|S_n\right|^p \leqslant 2\sum_{i=1}^{n} E|X_i|^p, \quad \forall 1 \leqslant p \leqslant 2. \tag{2.4}$$

Note that Lemma 2.1 holds when $\{X_i,\ 1 \leqslant i \leqslant n\}$ are independent random variables with $EX_i = 0$ for $1 \leqslant i \leqslant n$.

Lemma 2.2. *If $\{X_{kl}, \mathcal{F}_l,\ l \geqslant 1\}$, $k = 1, \ldots, m$ are nonnegative submartingales, then $\{\max_{1 \leqslant k \leqslant m} X_{kl}, \mathcal{F}_l,\ l \geqslant 1\}$ is a nonnegative submartingale.*

Proof. For $L > l \geqslant 1$,

$$E\left(\max_{1 \leqslant k \leqslant m} X_{kL} \mid \mathcal{F}_l\right) \geqslant \max_{1 \leqslant k \leqslant m} E(X_{kL} \mid \mathcal{F}_l) \geqslant \max_{1 \leqslant k \leqslant m} X_{kl}. \tag{2.5}$$

□

Lemma 2.3. *Let $\{X_{ij},\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ be a collection of mn independent random variables. If $EX_{ij} = 0$ for all $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, then*

$$E\left(\max_{1 \leqslant k \leqslant m, 1 \leqslant l \leqslant n} \left|S_{kl}\right|^p\right) \leqslant C\sum_{i=1}^{m}\sum_{j=1}^{n} E\left|X_{ij}\right|^p, \quad \forall 0 < p \leqslant 2, \tag{2.6}$$

where $S_{kl} = \sum_{i=1}^{k}\sum_{j=1}^{l} X_{ij}$, the constant C is independent of m and n. In the case $0 < p \leqslant 1$, the independence hypothesis and the hypothesis that $EX_{ij} = 0$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ are superfluous.

Proof. If $E|X_{ij}|^p = \infty$ for some $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, then (2.6) is immediate. Thus, we can assume that $E|X_{ij}|^p < \infty$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$.

Suppose, initially, that $1 < p \leqslant 2$ and $m \wedge n \geqslant 2$. Set

$$\begin{gathered} Y_l = \max_{1 \leqslant k \leqslant m} \left|S_{kl}\right|, \\ \mathcal{F}_l = \sigma(X_{ij},\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant l), \quad 1 \leqslant l \leqslant n. \end{gathered} \tag{2.7}$$

Now for each $1 \leqslant k \leqslant m$ and $2 \leqslant l \leqslant n$,

$$\begin{aligned} E(S_{kl} \mid \mathcal{F}_{l-1}) &= E(S_{k,l-1} + X_{1l} + \cdots + X_{kl} \mid \mathcal{F}_{l-1}) \\ &= E(S_{k,l-1} \mid \mathcal{F}_{l-1}) + E(X_{1l} \mid \mathcal{F}_{l-1}) + \cdots + E(X_{kl} \mid \mathcal{F}_{l-1}) \\ &= S_{k,l-1} \quad a.s., \end{aligned} \tag{2.8}$$

and so $\{S_{kl}, \mathcal{F}_l,\ 1 \leqslant l \leqslant n\}$ is a martingale for each $k = 1,\dots,m$. Since the function $|\cdot|$ is convex, it follows from Chow and Teicher [6, Lemma 7.4.1, page 244] that $\{|S_{kl}|, \mathcal{F}_l,\ 1 \leqslant l \leqslant n\}$ is a nonnegative submartingale for each $k = 1,\dots,m$. Then by Lemma 2.2, $\{Y_l, \mathcal{F}_l,\ 1 \leqslant l \leqslant n\}$ is a nonnegative submartingale and so by Doob's inequality (see, e.g., Chow and Teicher [6, page 255]),

$$E\left(\max_{1\leqslant k\leqslant m, 1\leqslant l\leqslant n} |S_{kl}|^p\right) = E\left(\max_{1\leqslant l\leqslant n} Y_l\right)^p \leqslant \left(\frac{p}{p-1}\right)^p EY_n^p. \tag{2.9}$$

Set $\mathcal{G}_k = \sigma(X_{ij},\ 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n)$, $1 \leqslant k \leqslant m$. We also have that $\{|S_{kn}|, \mathcal{G}_k,\ 1 \leqslant k \leqslant m\}$ is a submartingale and so by another application of Doob's inequality,

$$\begin{aligned} EY_n^p &= E\left(\max_{1\leqslant k\leqslant m} |S_{kn}|\right)^p \\ &\leqslant \left(\frac{p}{p-1}\right)^p E|S_{mn}|^p \leqslant 2\left(\frac{p}{p-1}\right)^p \sum_{i=1}^{m}\sum_{j=1}^{n} E|X_{ij}|^p \quad \text{(by Lemma 2.1).} \end{aligned} \tag{2.10}$$

The conclusion (2.6) follows immediately from (2.9) and (2.10).

Next, if $1 < p \leqslant 2$ and $m \wedge n = 1$, then (2.6) follows as in the $m \wedge n \geqslant 2$ case, *mutatis mutandis*.

Finally, if $0 < p \leqslant 1$, then

$$\begin{aligned} E\left(\max_{1\leqslant k\leqslant m, 1\leqslant l\leqslant n} |S_{kl}|^p\right) &\leqslant E\left(\max_{1\leqslant k\leqslant m, 1\leqslant l\leqslant n} \sum_{i=1}^{k}\sum_{j=1}^{l} |X_{ij}|^p\right) \\ &= E\left(\sum_{i=1}^{m}\sum_{j=1}^{n} |X_{ij}|^p\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} E|X_{ij}|^p, \end{aligned} \tag{2.11}$$

again establishing (2.6). □

3. Main results

With the preliminaries accounted for, Theorem 3.1 may now be established.

Theorem 3.1. *Let $0 < r \leqslant p \leqslant 2$, let $\{k_n,\ n \geqslant 1\}$ and $\{l_n,\ n \geqslant 1\}$ be sequences of positive integers, and let $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ be a double array of random variables such that $\{|X_{mn}|^r,\ m \geqslant 1,\ n \geqslant 1\}$ is $\{|a_{mnij}|^r\}$-uniformly integrable in the sense that*

$$\lim_{a\to\infty} \sup_{m\geqslant 1, n\geqslant 1} \sum_{i=1}^{k_m}\sum_{j=1}^{l_n} |a_{mnij}|^r E(|X_{ij}|^r I(|X_{ij}| > a)) = 0, \tag{3.1}$$

where $\{a_{mnij};\ m,n,i,j \geqslant 1\}$ *are constants satisfying*

$$\lim_{m\vee n\to\infty} \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} |a_{mnij}|^p = 0. \tag{3.2}$$

Suppose that the random variables $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ *are independent if* $1 < p \leqslant 2$. *Then*

$$\lim_{m\vee n\to\infty} E\left| \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} a_{mnij}(X_{ij} - c_{ij}) \right|^r = 0, \tag{3.3}$$

where $c_{ij} = 0$ *if* $0 < p \leqslant 1$ *and* $c_{ij} = EX_{ij}$ *if* $1 < p \leqslant 2$.

Proof. For arbitrary $\epsilon > 0$, there exists $M > 0$ such that

$$\sum_{i=1}^{k_m} \sum_{j=1}^{l_n} |a_{mnij}|^r E(|X_{ij}|^r I(|X_{ij}| > M)) < \epsilon, \quad m \geqslant 1,\ n \geqslant 1. \tag{3.4}$$

Set

$$\begin{aligned} X'_{mn} &= X_{mn} I(|X_{mn}| \leqslant M), \quad m \geqslant 1,\ n \geqslant 1, \\ X''_{mn} &= X_{mn} I(|X_{mn}| > M), \quad m \geqslant 1,\ n \geqslant 1. \end{aligned} \tag{3.5}$$

For $m \geqslant 1,\ n \geqslant 1$

$$\sum_{i=1}^{k_m} \sum_{j=1}^{l_n} |a_{mnij}|^r E|X''_{ij} - EX''_{ij}|^r \leqslant 2\sum_{i=1}^{k_m} \sum_{j=1}^{l_n} |a_{mnij}|^r E|X''_{ij}|^r < 2\epsilon. \tag{3.6}$$

If $0 < p \leqslant 1$, then

$$\begin{aligned} &E\left| \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} a_{mnij} X_{ij} \right|^r \\ &\quad \leqslant E\left| \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} a_{mnij} X'_{ij} \right|^r + \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} E|a_{mnij} X''_{ij}|^r \\ &\quad \leqslant \left(E\left| \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} a_{mnij} X'_{ij} \right|^p \right)^{r/p} + \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} |a_{mnij}|^r E|X''_{ij}|^r \quad \text{(by the Jensen inequality)} \\ &\quad \leqslant \left(\sum_{i=1}^{k_m} \sum_{j=1}^{l_n} |a_{mnij}|^p E|X'_{ij}|^p \right)^{r/p} + \epsilon \ \text{(by (3.4))} \\ &\quad \leqslant M^r \left(\sum_{i=1}^{k_m} \sum_{j=1}^{l_n} |a_{mnij}|^p \right)^{r/p} + \epsilon. \end{aligned} \tag{3.7}$$

The conclusion (3.3) follows from (3.2) and(3.7).

If $1 < p \leqslant 2$ and $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ are independent, then

$$\begin{aligned}
&E\left|\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X_{ij}-EX_{ij})\right|^r \\
&\quad\leqslant 2E\left|\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X'_{ij}-EX'_{ij})\right|^r + 2E\left|\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X''_{ij}-EX''_{ij})\right|^r \\
&\quad\leqslant 2\left(E\left|\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X'_{ij}-EX'_{ij})\right|^p\right)^{r/p} \\
&\qquad + 4\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} |a_{mnij}|^r E|X''_{ij}-EX''_{ij}|^r \quad \text{(by the Jensen inequality and Lemma 2.1)} \\
&\quad\leqslant 4\left(\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} |a_{mnij}|^p E|X'_{ij}-EX'_{ij}|^p\right)^{r/p} + 8\epsilon \quad \text{(by Lemma 2.1 and (3.6))} \\
&\quad\leqslant 8M^r\left(\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} |a_{mnij}|^p\right)^{r/p} + 8\epsilon.
\end{aligned} \tag{3.8}$$

The conclusion (3.3) follows from (3.2) and (3.8). □

In the following theorem, we establish the weak law of large numbers with random indices for weighted double sums of random variables.

Theorem 3.2. *Let $0 < r \leqslant p \leqslant 2$, let $\{k_n,\ n \geqslant 1\}$ and $\{l_n,\ n \geqslant 1\}$ be sequences of positive integers, and let $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ be a double array of random variables such that $\{|X_{mn}|^r,\ m \geqslant 1,\ n \geqslant 1\}$ is $\{|a_{mnij}|^r\}$-uniformly integrable in the sense that (3.1) holds where $\{a_{mnij};\ m,n,i,j \geqslant 1\}$ are constants satisfying*

$$\lim_{m\wedge n\to\infty}\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} |a_{mnij}|^p = 0. \tag{3.9}$$

Suppose that the random variables $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ are independent if $1 < p \leqslant 2$. Let $\{T_n,\ n \geqslant 1\}$ and $\{\tau_n,\ n \geqslant 1\}$ be sequences of positive integer-valued random variables such that

$$\lim_{n\to\infty} P\{T_n > k_n\} = \lim_{n\to\infty} P\{\tau_n > l_n\} = 0. \tag{3.10}$$

Then

$$\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n} a_{mnij}(X_{ij}-c_{ij}) \xrightarrow{P} 0, \quad \text{as } m\wedge n \longrightarrow \infty, \tag{3.11}$$

where $c_{ij} = 0$ if $0 < p \leqslant 1$ and $c_{ij} = EX_{ij}$ if $1 < p \leqslant 2$.

Proof. For arbitrary $\epsilon > 0$, define M, X'_{mn} and X''_{mn}, $m \geqslant 1$, $n \geqslant 1$ as in the proof of Theorem 3.1. Note that

$$\begin{aligned} &P\left\{\left|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n} a_{mnij}(X_{ij}-c_{ij})\right| > \epsilon\right\} \\ &\quad\leqslant P\left\{\left[\left|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n} a_{mnij}(X_{ij}-c_{ij})\right| > \epsilon\right] \cap [T_m \leqslant k_m] \cap [\tau_n \leqslant l_n]\right\} \\ &\qquad + P\{T_m > k_m\} + P\{\tau_n > l_n\}. \end{aligned} \tag{3.12}$$

If $1 < p \leqslant 2$, then

$$\begin{aligned} &P\left\{\left[\left|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n} a_{mnij}(X_{ij}-EX_{ij})\right| > \epsilon\right] \cap [T_m \leqslant k_m] \cap [\tau_n \leqslant l_n]\right\} \\ &\quad\leqslant P\left\{\bigcup_{k=1}^{k_m}\bigcup_{l=1}^{l_n}\left[\left|\sum_{i=1}^{k}\sum_{j=1}^{l} a_{mnij}(X_{ij}-EX_{ij})\right| > \epsilon\right]\right\} \\ &\quad= P\left\{\max_{1\leqslant k\leqslant k_m}\max_{1\leqslant l\leqslant l_n}\left|\sum_{i=1}^{k}\sum_{j=1}^{l} a_{mnij}(X_{ij}-EX_{ij})\right| > \epsilon\right\} \\ &\quad\leqslant P\left\{\max_{1\leqslant k\leqslant k_m}\max_{1\leqslant l\leqslant l_n}\left|\sum_{i=1}^{k}\sum_{j=1}^{l} a_{mnij}(X'_{ij}-EX'_{ij})\right| > \frac{\epsilon}{2}\right\} \\ &\qquad + P\left\{\max_{1\leqslant k\leqslant k_m}\max_{1\leqslant l\leqslant l_n}\left|\sum_{i=1}^{k}\sum_{j=1}^{l} a_{mnij}(X''_{ij}-EX''_{ij})\right| > \frac{\epsilon}{2}\right\} \\ &\quad\leqslant \frac{4^p}{\epsilon^p}E\left(\max_{1\leqslant k\leqslant k_m}\max_{1\leqslant l\leqslant l_n}\left|\sum_{i=1}^{k}\sum_{j=1}^{l} a_{mnij}(X'_{ij}-EX'_{ij})\right|\right)^p \\ &\qquad + \frac{4^r}{\epsilon^r}E\left(\max_{1\leqslant k\leqslant k_m}\max_{1\leqslant l\leqslant l_n}\left|\sum_{i=1}^{k}\sum_{j=1}^{l} a_{mnij}(X''_{ij}-EX''_{ij})\right|\right)^r \quad \text{(by the Markov inequality)} \\ &\quad\leqslant C\sum_{i=1}^{k_m}\sum_{j=1}^{l_n}|a_{mnij}|^p E|X'_{ij}-EX'_{ij}|^p \\ &\qquad + C\sum_{i=1}^{k_m}\sum_{j=1}^{l_n}|a_{mnij}|^r E|X''_{ij}-EX''_{ij}|^r \quad \text{(by Lemma 2.3)} \\ &\quad\leqslant C\sum_{i=1}^{k_m}\sum_{j=1}^{l_n}|a_{mnij}|^p + C\sum_{i=1}^{k_m}\sum_{j=1}^{l_n}|a_{mnij}|^r E|X''_{ij}|^r \leqslant C\sum_{i=1}^{k_m}\sum_{j=1}^{l_n}|a_{mnij}|^p + C\epsilon. \end{aligned} \tag{3.13}$$

Combining (3.9), (3.10), (3.12), and (3.13) we get (3.11).

If $0 < p \leqslant 1$, then (3.11) follows as in the $1 < p \leqslant 2$ case, *mutatis mutandis*. □

In (3.2), if the limit is taken as $m \wedge n \to \infty$, then we have the following result. This result should be compared with Example 4.3 below. Theorem 3.3 will only be stated as its proof is similar to that of Theorem 3.1.

Theorem 3.3. *Let $0 < r \leqslant p \leqslant 2$, let $\{k_n,\ n \geqslant 1\}$ and $\{l_n,\ n \geqslant 1\}$ be sequences of positive integers, and let $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ be a double array of random variables such that $\{|X_{mn}|^r,\ m \geqslant 1,\ n \geqslant 1\}$ is $\{|a_{mnij}|^r\}$-uniformly integrable in the sense that (3.1) holds, where $\{a_{mnij};\ m,n,i,j \geqslant 1\}$ are constants satisfying (3.9). Suppose that the random variables $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ are independent if $1 < p \leqslant 2$. Then*

$$\lim_{m \wedge n \to \infty} E \left| \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} a_{mnij} (X_{ij} - c_{ij}) \right|^r = 0, \tag{3.14}$$

where $c_{ij} = 0$ if $0 < p \leqslant 1$ and $c_{ij} = EX_{ij}$ if $1 < p \leqslant 2$.

Corollary 3.4. *Let $0 < r < 2$, let $\{k_n,\ n \geqslant 1\}$ and $\{l_n,\ n \geqslant 1\}$ be sequences of positive integers with $k_m l_n \to \infty$ as $m \wedge n \to \infty$, and let $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ be a double array of random variables such that $\{|X_{mn}|^r,\ m \geqslant 1,\ n \geqslant 1\}$ is Cesàro uniformly integrable in the sense that*

$$\lim_{a \to \infty} \sup_{m \geqslant 1, n \geqslant 1} \frac{\sum_{i=1}^{k_m} \sum_{j=1}^{l_n} E(|X_{ij}|^r I(|X_{ij}| > a))}{k_m l_n} = 0. \tag{3.15}$$

Let $\{a_{mnij};\ m,n,i,j \geqslant 1\}$ be constants satisfying

$$\max_{1 \leqslant i \leqslant k_m} \max_{1 \leqslant j \leqslant l_n} |a_{mnij}| \leqslant C \frac{1}{(k_m l_n)^{1/r}}, \quad m \geqslant 1,\ n \geqslant 1. \tag{3.16}$$

Suppose that the random variables $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ are independent if $1 \leqslant r < 2$. Let $\{T_n,\ n \geqslant 1\}$ and $\{\tau_n,\ n \geqslant 1\}$ be sequences of positive integer-valued random variables satisfying (3.10). Then

$$\sum_{i=1}^{T_m} \sum_{j=1}^{\tau_n} a_{mnij} (X_{ij} - c_{ij}) \xrightarrow{P} 0, \quad \text{as } m \wedge n \longrightarrow \infty, \tag{3.17}$$

where $c_{ij} = 0$ if $0 < r < 1$ and $c_{ij} = EX_{ij}$ if $1 \leqslant r < 2$.

Proof. Note that (3.15) and (3.16) ensure that the double array $\{|X_{mn}|^r,\ m \geqslant 1,\ n \geqslant 1\}$ is $\{|a_{mnij}|^r\}$-uniformly integrable. We choose $p = 1$ when $0 < r < 1$, and $p = 2$ when $1 \leqslant r < 2$. Then (3.9) is an immediate consequence of (3.16), $0 < r < p$, and $k_m l_n \to \infty$ as $m \wedge n \to \infty$. Corollary 3.4 follows immediately from Theorem 3.2. □

The following corollary reduces to a result of Thanh [12] when the array $\{|X_{mn}|^r,\ m \geqslant 1,\ n \geqslant 1\}$ is uniformly integrable, $a_{mnij} = 1/(mn)^{1/r}$, $m \geqslant 1$, $n \geqslant 1$, $i \geqslant 1$, $j \geqslant 1$.

Corollary 3.5. *Let $0 < r < 2$, let $\{k_n,\ n \geqslant 1\}$ and $\{l_n,\ n \geqslant 1\}$ be sequences of positive integers with $k_m l_n \to \infty$ as $m \vee n \to \infty$, and let $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ be a double array of random variables such that $\{|X_{mn}|^r,\ m \geqslant 1,\ n \geqslant 1\}$ is Cesàro uniformly integrable in*

the sense that (3.15) holds. Let $\{a_{mnij};\ m,n,i,j \geqslant 1\}$ *be constants satisfying (3.16). Suppose that the random variables* $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ *are pairwise independent if* $r = 1$ *and the random variables* $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ *are independent if* $1 < r < 2$. *Then*

$$\lim_{m\vee n\to\infty} E\left|\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X_{ij}-c_{ij})\right|^r = 0, \tag{3.18}$$

where $c_{ij} = 0$ *if* $0 < r < 1$ *and* $c_{ij} = EX_{ij}$ *if* $1 \leqslant r < 2$.

Proof. If $0 < r < 2$, $r \neq 1$, similar to the proof of Corollary 3.4, then the proof of Corollary 3.5 is an immediate consequence of Theorem 3.1. If $r = 1$ and $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ are pairwise independent, for arbitrary $\epsilon > 0$, then we define M, X'_{mn} and X''_{mn}, $m \geqslant 1$, $n \geqslant 1$ as in the proof of Theorem 3.1. Note that

$$\begin{aligned}
&E\left|\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X_{ij}-EX_{ij})\right| \\
&\quad\leqslant E\left|\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X'_{ij}-EX'_{ij})\right| + E\left|\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X''_{ij}-EX''_{ij})\right| \\
&\quad\leqslant \left(E\left(\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}(X'_{ij}-EX'_{ij})\right)^2\right)^{1/2} \\
&\qquad + \sum_{i=1}^{k_m}\sum_{j=1}^{l_n} |a_{mnij}|\,E|X''_{ij}-EX''_{ij}| \qquad \text{(by the Jensen inequality)} \\
&\quad\leqslant \left(\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} (a_{mnij})^2 E(X'_{ij}-EX'_{ij})^2\right)^{1/2} + 2\epsilon \\
&\quad\leqslant M\left(\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} (a_{mnij})^2\right)^{1/2} + 2\epsilon \leqslant C\left(\frac{1}{(k_m l_n)}\right)^{1/2} + 2\epsilon \qquad \text{(by (3.16))}.
\end{aligned} \tag{3.19}$$

The conclusion (3.18) follows from (3.19). □

4. Some interesting examples

Six illustrative examples will now be presented. The first example illustrates the essential role that condition (3.2) (condition (3.9)) plays in Theorem 3.1 (Theorem 3.3, resp.).

Example 4.1. Let $r = p = 2$ and let $\alpha \geqslant 1/2$, let $\{k_n,\ n \geqslant 1\}$ and $\{l_n,\ n \geqslant 1\}$ be sequences of positive integers with $\lim_{m\vee n\to\infty} k_m l_n = \infty$, and let $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ be a double array of independent identically distributed $N(0,1)$ random variables. For $m \geqslant 1$, $n \geqslant 1$, set

$$a_{mnij} = \frac{1}{(k_n l_n)^{\alpha}}, \quad i \geqslant 1,\ j \geqslant 1. \tag{4.1}$$

Then

$$\sum_{i=1}^{k_m}\sum_{j=1}^{l_n}|a_{mnij}|^p = \frac{1}{(k_n l_n)^{2\alpha-1}}, \quad m \geqslant 1,\ n \geqslant 1. \tag{4.2}$$

It is easy to see that (3.1) holds for $\alpha \geqslant 1/2$.

Now if $\alpha > 1/2$, then (3.2) and (3.9) hold. Thus by Theorem 3.1,

$$E\left|\sum_{i=1}^{k_n}\sum_{j=1}^{l_n} a_{mnij}X_{ij}\right|^2 \longrightarrow 0, \quad \text{as } m \vee n \longrightarrow \infty \tag{4.3}$$

(and so $E|\sum_{i=1}^{k_n}\sum_{j=1}^{l_n} a_{mnij}X_{ij}|^2 \to 0$ as $m \wedge n \to \infty$).

On the other hand, if $\alpha = 1/2$, then (3.2) and (3.9) fail. Moreover,

$$E\left|\sum_{i=1}^{k_n}\sum_{j=1}^{l_n} a_{mnij}X_{ij}\right|^r = \frac{1}{(k_m l_n)^{2\alpha}}E\left|\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} X_{ij}\right|^2 = E|X_{11}|^2\frac{1}{(k_m l_n)^{(2\alpha-1)}} = 1 \tag{4.4}$$

and so (3.3) and (3.14) also fail.

The next example shows that in Theorems 3.1 and 3.3 the condition (3.1) cannot be dispensed with.

Example 4.2. Let $0 < r < p \leqslant 2$, and let $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ be a double array of independent random variables with

$$X_{mn} \sim N\left(0, (mn)^{4/r}\right), \quad m \geqslant 1,\ n \geqslant 1. \tag{4.5}$$

Let $k_n = l_n = n$, $n \geqslant 1$, and let $a_{mnij} = 1/(mn)^{1/r}$, $m \geqslant 1$, $n \geqslant 1$, $i \geqslant 1$, $j \geqslant 1$. Then (3.2) and so (3.9) are automatic. It is easy to see that (3.1) fails. Moreover,

$$\begin{aligned} E\left|\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} a_{mnij}X_{ij}\right|^r &= \frac{1}{mn}E\left|\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}\right|^r \\ &= \frac{1}{mn}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}(ij)^{4/r}\right)^{r/2} E|X_{11}|^r > \frac{1}{mn}(mn)^2 E|X_{11}|^r \end{aligned} \tag{4.6}$$

and so (3.3) and (3.14) also fail.

The third example shows that Theorem 3.1 can fail if (3.2) is weakened to (3.9).

Example 4.3. Let $r = p = 1$, and let $k_n = l_n = n$, $n \geqslant 1$. For $m \geqslant 1$, $n \geqslant 1$, set

$$a_{mnij} = \frac{1}{mn}\left(\frac{1}{m} + \frac{1}{n}\right), \quad i \geqslant 1,\ j \geqslant 1. \tag{4.7}$$

Let $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ be a double array of random variables with

$$P\{X_{mn} = 1\} = 1, \quad m \geqslant 1,\ n \geqslant 1. \tag{4.8}$$

Then (3.1) holds since $I(|X_{mn}| > 1) = 0$ a.s., $m \geqslant 1,\ n \geqslant 1$.

Now

$$\sum_{i=1}^{k_m} \sum_{j=1}^{l_n} |a_{mnij}|^p = \frac{1}{m} + \frac{1}{n}, \quad m \geqslant 1,\ n \geqslant 1, \tag{4.9}$$

so (3.9) holds but (3.2) fails. Thus (3.14) holds by Theorem 3.3. But (3.3) fails since

$$E\left|\sum_{i=1}^{k_m} \sum_{j=1}^{l_n} a_{mnij} X_{ij}\right|^r = E\left|\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{mn}\left(\frac{1}{m} + \frac{1}{n}\right) X_{ij}\right| = \frac{1}{m} + \frac{1}{n} \not\longrightarrow 0, \quad \text{as } m \vee n \longrightarrow \infty. \tag{4.10}$$

In Theorem 3.2, if (3.9) is replaced by (3.2), and (3.10) is replaced by

$$\lim_{m \vee n \to \infty} P\{[T_m > k_m] \cup [\tau_n > l_n]\} = 0, \tag{4.11}$$

then the weak law of large numbers

$$\sum_{i=1}^{T_m} \sum_{j=1}^{\tau_n} a_{mnij}(X_{ij} - c_{ij}) \xrightarrow{P} 0, \quad \text{as } m \vee n \longrightarrow \infty \tag{4.12}$$

is obtained by the same method. We see that condition (4.11) is not a good condition for this type of theorem because it will hold for only the most "uninteresting" sequences of random indices. However, in the following example we will show that (4.12) can fail if (4.11) is weakened to (3.10).

Example 4.4. Let $0 < r \leqslant p \leqslant 1$, and let $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ be a double array of random variables with

$$P\{X_{mn} = 1\} = 1, \quad m \geqslant 1,\ n \geqslant 1. \tag{4.13}$$

Suppose that $k_n = l_n = 2^n$, $n \geqslant 1$. For $m \geqslant 1$, $n \geqslant 1$, set

$$\begin{aligned} a_{mnij} &= \frac{1}{3^{(m+n)/r}} \quad \text{if } 1 \leqslant i \leqslant k_m,\ 1 \leqslant j \leqslant l_n, \\ a_{mnij} &= 1 \quad \text{if either } i > k_m \text{ or } j > l_n. \end{aligned} \tag{4.14}$$

Then (3.1) and (3.2) are automatic. Let $\{T_n,\ n \geqslant 1\}$ and $\{\tau_n,\ n \geqslant 1\}$ be sequences of identically distributed random variables such that

$$\{T_n,\ n \geqslant 1\} \text{ is independent of } \{\tau_n,\ n \geqslant 1\}, \tag{4.15}$$

$$\{T_n,\ n \geqslant 1\}, \{\tau_n,\ n \geqslant 1\} \text{ are independent of } \{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}, \tag{4.16}$$

$$P\{T_1 = 2^j\} = P\{\tau_1 = 2^j\} = 2^{-j}, \quad j \geqslant 1. \tag{4.17}$$

Now

$$\begin{aligned}
&P\{[T_m > k_m] \cup [\tau_n > l_n]\} \\
&\quad = P\{T_m > k_m\} + P\{\tau_n > l_n\} - P\{[T_m > k_m] \cap [\tau_n > l_n]\} \\
&\quad = \sum_{i=m+1}^{\infty} \frac{1}{2^i} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} - \left(\sum_{i=m+1}^{\infty} \frac{1}{2^i}\right)\left(\sum_{j=n+1}^{\infty} \frac{1}{2^j}\right) \quad \text{(by (4.15) and (4.17))},
\end{aligned} \tag{4.18}$$

so (3.10) holds. By Theorem 3.2, (3.11) also holds. Since

$$\begin{gathered}
\sum_{i=m+1}^{\infty} \frac{1}{2^i} < 1, \quad m \geqslant 1, \\
\left(\sum_{i=m+1}^{\infty} \frac{1}{2^i}\right)\left(\sum_{j=n+1}^{\infty} \frac{1}{2^j}\right) < \frac{1}{2}\left(\sum_{i=m+1}^{\infty} \frac{1}{2^i} + \sum_{j=n+1}^{\infty} \frac{1}{2^j}\right), \quad m \geqslant 1,\ n \geqslant 1.
\end{gathered} \tag{4.19}$$

Thus

$$\begin{aligned}
&P\{[T_m > k_m] \cup [\tau_n > l_n]\} \\
&\quad = \sum_{i=m+1}^{\infty} \frac{1}{2^i} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} - \left(\sum_{i=m+1}^{\infty} \frac{1}{2^i}\right)\left(\sum_{j=n+1}^{\infty} \frac{1}{2^j}\right) > \frac{1}{2}\left(\sum_{i=m+1}^{\infty} \frac{1}{2^i} + \sum_{j=n+1}^{\infty} \frac{1}{2^j}\right),
\end{aligned} \tag{4.20}$$

and so (4.11) fails. Moreover,

$$\begin{aligned}
&P\left\{\left|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n} a_{mnij}X_{ij}\right| > \frac{1}{2}\right\} \\
&\quad \geqslant P\left\{\left[\left|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n} a_{mnij}X_{ij}\right| > \frac{1}{2}\right] \cap [(T_m > k_m) \cup (\tau_n > l_n)]\right\} \\
&\quad = P\{[T_m > k_m] \cup [\tau_n > l_n]\} \quad (\text{since } a_{mnij} = 1 \text{ if either } i > k_m \text{ or } j > l_n) \\
&\quad = P\{T_m > k_m\} + P\{\tau_n > l_n\} - P\{[T_m > k_m] \cap [\tau_n > l_n]\} \\
&\quad = \sum_{i=m+1}^{\infty} \frac{1}{2^i} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} - \left(\sum_{i=m+1}^{\infty} \frac{1}{2^i}\right)\left(\sum_{j=n+1}^{\infty} \frac{1}{2^j}\right) > \frac{1}{2}\left(\sum_{i=m+1}^{\infty} \frac{1}{2^i} + \sum_{j=n+1}^{\infty} \frac{1}{2^j}\right),
\end{aligned} \tag{4.21}$$

and so (4.12) fails.

Apropos of Theorem 3.2, the following example shows that its hypotheses do not guarantee that the convergence in mean of order r prevails in the conclusion (3.11).

Example 4.5. Let $r = p = 2$, and let $\{X_{mn}, m \geqslant 1, n \geqslant 1\}$ be a double array of independent identically distributed $N(0,1)$ random variables. Suppose that $k_n = l_n = 2^n$, $n \geqslant 1$, and let

$\{T_n,\ n \geqslant 1\}$ and $\{\tau_n,\ n \geqslant 1\}$ be sequences of identically distributed random variables satisfying (4.15), (4.16), and (4.17). For $m \geqslant 1,\ n \geqslant 1$, set

$$a_{mnij} = \frac{1}{2^{m+n}}, \quad i \geqslant 1,\ j \geqslant 1. \tag{4.22}$$

Then (3.1) and (3.9) are automatic. Moreover, (3.10) holds since

$$\lim_{n\to\infty} P\{T_n > k_n\} = \lim_{n\to\infty} P\{\tau_n > l_n\} = \lim_{n\to\infty} \sum_{j=n+1}^{\infty} \frac{1}{2^j} = 0. \tag{4.23}$$

Thus by Theorem 3.2,

$$\sum_{i=1}^{T_m} \sum_{j=1}^{\tau_n} a_{mnij} X_{ij} \xrightarrow{P} 0, \quad \text{as } m \wedge n \longrightarrow \infty. \tag{4.24}$$

However,

$$\begin{aligned}
E\left|\sum_{i=1}^{T_m} \sum_{j=1}^{\tau_n} a_{mnij} X_{ij}\right|^r &= \frac{1}{4^{m+n}} E\left[\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} X_{ij} I\left(\left[T_m = 2^k\right] \cap \left[\tau_n = 2^l\right]\right)\right)^2\right] \\
&= \frac{1}{4^{m+n}} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} E\left(\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} X_{ij} I\left(\left[T_m = 2^k\right] \cap \left[\tau_n = 2^l\right]\right)\right)^2 \\
&\quad \text{(by the Lebesgue monotone convergence theorem)} \\
&= \frac{1}{4^{m+n}} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} EX_{ij}^2 P\{T_m = 2^k\} P\{\tau_n = 2^l\}\right) \\
&= \frac{1}{4^{m+n}} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} \frac{1}{2^k 2^l} EX_{ij}^2 = \infty.
\end{aligned} \tag{4.25}$$

Thus the convergence in mean of order r does not prevail in (4.24).

Apropos of Theorem 3.3, the following example shows that its hypotheses do not guarantee that the a.s. convergence prevails in the conclusion (3.14).

Example 4.6. Let $0 < r < p = 2$, and let $\{X_{mn},\ m \geqslant 1,\ n \geqslant 1\}$ be a double array of independent identically distributed random variables with $E|X_{11}|^r < \infty$, and let $k_n = l_n = n,\ n \geqslant 1$. For $m \geqslant 1,\ n \geqslant 1$, set

$$a_{mnij} = \frac{1}{(mn)^{1/r}}, \quad i \geqslant 1,\ j \geqslant 1. \tag{4.26}$$

It is easy to check that all hypotheses of Theorem 3.3 are satisfied. Thus

$$E\left|\sum_{i=1}^{m}\sum_{j=1}^{n}a_{mnij}X_{ij}\right|^{r}=E\left|\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}}{(mn)^{1/r}}\right|^{r}\longrightarrow 0,\quad \text{as } m\wedge n\longrightarrow\infty. \tag{4.27}$$

However, according to a result of Gut [7],

$$\lim_{m\wedge n\to\infty}\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}}{(mn)^{1/r}}=0\quad \text{a.s.}, \tag{4.28}$$

if and only if $E(|X_{11}|^r\log^+|X_{11}|)<\infty$. Thus if $E(|X_{11}|^r\log^+|X_{11}|)=\infty$, then the a.s. convergence does not prevail in (4.27).

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Le Van Thanh: Department of Mathematics, Vinh University, Nghe An 42118, Vietnam
E-mail address: lvthanhvinh@yahoo.com