EXISTENCE OF SOLUTIONS OF A SPECIAL CLASS OF FUZZY INTEGRAL EQUATIONS

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We prove an existence theorem for a special class of fuzzy integral equations involving fuzzy set-valued mappings. The results are obtained by using the contraction mapping principle.

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1. Introduction

Chandrasekhar [5] and Crum [6] considered the following integral equation:

$$H(t) = 1 + H(t) \int_0^1 \frac{t}{t+s} \psi(s) H(s) ds. \quad (1.1)$$

This equation arises in the study of radiation transfer in a semi-infinite atmosphere. The first rigorous proof of existence of solutions of (1.1) was given in [6]. By using operators on a Banach algebra and a fixed point theorem of Darbo for a set contraction map, Leggett [8] proved an existence theorem for an equation of the form

$$x = x_0 + xKx, \quad (1.2)$$

where K is a compact operator on the Banach algebra. His abstract theorems are applied to the integral equation of the form

$$x(t) = x_0(t) + x(t) \int_{\Omega} K(t,s) f(s, x(s)) ds, \quad t \in \Omega, \Omega \subset R^n. \quad (1.3)$$

2 Fuzzy integral equations

Cahlon and Eskin [4] considered the equation

$$H(t) = 1 + H(t) \int_0^1 \frac{t}{t+s} \psi(s) H(s) ds + \int_0^1 P(t, s, H(t), H(s)) ds. \quad (1.4)$$

This equation is a generalization of (1.3), where P is the perturbation of Chandrasekhar H -equation.

The problem of existence of solutions of fuzzy integral equations has been studied by many authors [1, 2, 9–11, 14–16]. Kaleva [7] and Seikkala [13] have discussed the existence of solutions of fuzzy differential equations. Subrahmanyam and Sudarsanam [14] studied existence results for fuzzy Volterra integral equation of the form

$$x(t) = \phi(t) + \int_0^t g(t, s, x(s)) ds, \quad (1.5)$$

where as Park et al. [11] proved the existence of solutions of fuzzy integral equation of the form

$$\phi(u) = w_0 + \int_{u_0}^u F(u, s, \phi(s)) ds \quad \phi(u_0) = w_0. \quad (1.6)$$

Balachandran and Dauer [2] established the local existence of solutions and approximate solutions of the perturbed fuzzy integral equation. Balachandran and Prakash [3] studied the existence of solutions of nonlinear fuzzy Volterra integral equations of the form

$$x(t) = \phi(t) + f\left(t, x(t), \int_0^t g(t, s, x(s)) ds\right). \quad (1.7)$$

In this paper we prove the existence of solutions of fuzzy integral equations of the form

$$x(t) = \phi(t) + x(t) \int_0^t k(t, s) f(s, x(s)) ds + \int_0^t g(t, s, x(s)) ds, \quad (1.8)$$

where $\phi : [0, T] \rightarrow E^n$, $k : [0, T] \times [0, T] \rightarrow R$, $f : [0, T] \times E^n \rightarrow E^n$, and $g : [0, T] \times [0, T] \times E^n \rightarrow E^n$ are continuous functions. This equation is a generalization of Chandrasekhar-type equation in fuzzy setting.

The outlay of the paper is as follows. In Section 2 we give some basic definitions for our study and in Section 3 we prove the main theorem on the existence of solutions of fuzzy integral equation (1.8). In Section 4 we state a theorem on the existence of solutions of a generalization of (1.8).

2. Preliminaries

Let $P_k(R^n)$ denote the family of all nonempty, compact, convex subsets of R^n . Addition and scalar multiplication in $P_k(R^n)$ are defined as usual. $\bar{U}$ denotes the closure of U , where U is contained in R^n . Let $I = [0, 1] \subseteq R$ be a compact interval and denote

$$E^n = \{u : R^n \rightarrow [0, 1] : u \text{ satisfies (i)–(iv) below}\}, \quad (2.1)$$

where

- (i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
- (ii) u is fuzzy convex,
- (iii) u is upper semicontinuous,
- (iv) $[u]^0 = \text{cl}\{x \in R^n : u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$ denote $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$. Then from (i)–(iv) it follows that the α -level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha \leq 1$.

If $g : R^n \times R^n \rightarrow R^n$ is a function, then using Zadeh's extension principle we can extend g to $E^n \times E^n \rightarrow E^n$ by the equation

$$\tilde{g}(u, v)(z) = \sup_{z=g(x,y)} \min \{u(x), v(y)\}. \quad (2.2)$$

It is well known that $[\tilde{g}(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha)$ for all $u, v \in E^n$, $0 \leq \alpha \leq 1$, and continuous function g . In addition the above equation gives $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$. The real numbers can be embedded in E^n by the rule $c \rightarrow \hat{c}(t)$, where

$$\hat{c}(t) = \begin{cases} 1 & \text{for } t = c, \\ 0 & \text{elsewhere.} \end{cases} \quad (2.3)$$

We can also generalize the multiplication by a real number and for any real number c we get $[cu]^\alpha = c[u]^\alpha$, where $0 \leq \alpha \leq 1$ and $u \in E^n$.

Let $D : E^n \times E^n \rightarrow R^+ \cup \{0\}$ be defined by $D(u, v) = \sup_{0 \leq \alpha \leq 1} H([u]^\alpha, [v]^\alpha)$, where H is the Hausdorff metric defined in $P_k(R^n)$. Then D is a metric on E^n . Further, (E^n, D) is a complete metric space [7, 12]. Also $D(u + w, v + w) = D(u, v)$ for every $u, v, w \in E^n$. Furthermore, $D(\lambda u, \lambda v) = |\lambda|D(u, v)$ for every $u, v \in E^n$ and $\lambda \in R$.

It can be proved straight away that $D(u + v, w + z) \leq D(u, w) + D(v, z)$ for u, v, w , and $z \in E^n$. (The proof is based on the fact $H(A_1 + A_2, B_1 + B_2) \leq H(A_1, B_1) + H(A_2, B_2)$, where H is the Hausdorff metric on $P_k(R^n)$ induced by the norm in R^n .)

Definition 2.1 [1]. Let I be $[0, 1]$ and for each t in I , let $F(t)$ be a nonempty subset of R^n . Let $\mathcal{F}$ be the set of all point-valued functions f from I to R^n such that f is integrable over I and $f(t) \in F(t)$ for all t in I . Then

$$\int_I F(t)dt = \left\{ \int_I f(t)dt : f \in \mathcal{F} \right\}. \quad (2.4)$$

Definition 2.2 [7]. A mapping $F : I \rightarrow E^n$ is strongly measurable if for all $\alpha \in [0, 1]$ the set-valued map $F_\alpha : I \rightarrow P_k(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $P_k(R^n)$ has the topology induced by the Hausdorff metric H .

Definition 2.3 [7]. A mapping $F : I \rightarrow E^n$ is said to be integrably bounded if there is an integrable function h such that $\|x\| \leq h(t)$ for every $x \in F_0(t)$.

4 Fuzzy integral equations

Definition 2.4 [12]. The integral of a fuzzy mapping $F : [0, 1] \rightarrow E^n$ is defined levelwise by

$$\begin{aligned} \left[\int_{[0,1]} F(t) dt \right]^\alpha &= \int_{[0,1]} F_\alpha(t) dt \\ &= \left\{ \int_{[0,1]} f(t) dt : f : [0, 1] \rightarrow R^n \text{ is a measurable selection for } F_\alpha \right\} \end{aligned} \quad (2.5)$$

for all $\alpha \in [0, 1]$.

It has been proved by Puri and Ralescu [12] that a strongly measurable and integrably bounded mapping $F : I \rightarrow E^n$ is integrable (i.e., $\int_I F(t) dt \in E^n$). The concept of a fuzzy integral generalizes the Aumann integral of a set-valued mapping. The following results are proved in [7].

THEOREM 2.5. *If $F : I \rightarrow E^n$ is continuous, then it is integrable.*

THEOREM 2.6. *Let $F, G : I \rightarrow E^n$ be integrable and $\lambda \in R$. Then*

- (i) $\int_I (F(t) + G(t)) dt = \int_I F(t) dt + \int_I G(t) dt$,
- (ii) $\int_I \lambda F(t) dt = \lambda \int_I F(t) dt$,
- (iii) $D(F, G)$ is integrable,
- (iv) $D(\int_I F(t) dt, \int_I G(t) dt) \leq \int_I D(F(t), G(t)) dt$.

3. Existence theorem

THEOREM 3.1. *Let a and b be positive numbers such that*

$$b = \max_{0 \leq t \leq a} D \left(\psi(t) \int_0^t k(t, s) f(s, \psi(s)) ds + \int_0^t g(t, s, \psi(s)) ds, \hat{0} \right). \quad (3.1)$$

Suppose that

- (i) $\phi : [0, a] \rightarrow E^n$ is continuous,
- (ii) $f : [0, T] \times E^n \rightarrow E^n$ and $k : [0, T] \times [0, T] \rightarrow R$ are continuous and there exists a constant $L > 0$ such that

$$D \left(x(t) \int_0^t k(t, s) f(s, x(s)) ds, y(t) \int_0^t k(t, s) f(s, y(s)) ds \right) \leq LD(x, y) \quad (3.2)$$

for $x, y \in E^n$,

- (iii) $g : U \rightarrow E^n$ is continuous, where $U = \{(t, s, x) : 0 \leq s \leq t \leq a, x \in E^n \text{ and } D(x, \phi(t)) \leq b\}$ and satisfies Lipschitz condition with respect to x on U , that is, there exists a constant $M > 0$ such that

$$D(g(t, s, x), g(t, s, y)) \leq MD(x, y) \quad \text{if } (t, s, x), (t, s, y) \in U. \quad (3.3)$$

If $c = (\alpha - L)/M$ for some fixed $\alpha \in (0, 1)$, then there is a unique solution of (1.7) on $[0, T]$, where $T = \min\{a, b, c\}$.

Proof. Let $\mathcal{C}$ be the space of continuous functions from $[0, T]$ into (E^n, D) with $H_1(\psi, \phi) \leq b$, that is, $\mathcal{C} = \{\psi : \psi : [0, T] \rightarrow E^n \text{ is continuous and } H_1(\psi, \phi) \leq b\}$, where $H_1(\psi, \phi) = \sup_{0 \leq t \leq T} D(\psi(t), \phi(t))$. Define an operator $A : \mathcal{C} \rightarrow \mathcal{C}$ by

$$A\psi(t) = \phi(t) + \psi(t) \int_0^t k(t, s) f(s, \psi(s)) ds + \int_0^t g(t, s, \psi(s)) ds. \quad (3.4)$$

To prove $A : \mathcal{C} \rightarrow \mathcal{C}$, we have to prove that $A\psi$ is continuous and $H_1(A\psi, \phi) \leq b$ whenever $\psi \in \mathcal{C}$. Consider

$$\begin{aligned} & D(A\psi(t+h), A\psi(t)) \\ &= D\left(\phi(t+h) + \psi(t+h) \int_0^{t+h} k(t+h, s) f(s, \psi(s)) ds + \int_0^{t+h} g(t+h, s, \psi(s)) ds, \phi(t)\right. \\ &\quad \left.+ \psi(t) \int_0^t k(t, s) f(s, \psi(s)) ds + \int_0^t g(t, s, \psi(s)) ds\right) \leq D(\phi(t+h), \phi(t)) \\ &\quad + D\left(\psi(t+h) \int_0^{t+h} k(t+h, s) f(s, \psi(s)) ds, \psi(t) \int_0^t k(t, s) f(s, \psi(s)) ds\right) \\ &\quad + D\left(\int_0^{t+h} g(t+h, s, \psi(s)) ds, \int_0^t g(t, s, \psi(s)) ds\right) \\ &\leq \frac{\epsilon}{3} + D\left(\psi(t+h) \int_0^{t+h} k(t+h, s) f(s, \psi(s)) ds, \psi(t) \int_0^t k(t, s) f(s, \psi(s)) ds\right) \\ &\quad + \int_0^t D(g(t+h, s, \psi(s)), g(t, s, \psi(s))) ds + \int_t^{t+h} D(g(t+h, s, \psi(s)), \hat{0}) ds. \end{aligned} \quad (3.5)$$

Clearly the right-hand side of (3.5) is less than ϵ as $h \rightarrow 0$. So $A\psi$ is continuous. Consider

$$\begin{aligned} H_1(A\psi, \phi) &= \sup_{0 \leq t \leq T} D(A\psi(t), \phi(t)) \\ &= \sup_{0 \leq t \leq T} D\left(\phi(t) + \psi(t) \int_0^t k(t, s) f(s, \psi(s)) ds + \int_0^t g(t, s, \psi(s)) ds, \phi(t)\right) \\ &= \sup_{0 \leq t \leq T} D\left(\psi(t) \int_0^t k(t, s) f(s, \psi(s)) ds + \int_0^t g(t, s, \psi(s)) ds, \hat{0}\right) \\ &\leq b. \end{aligned} \quad (3.6)$$

So $A\psi \in \mathcal{C}$ and A maps $\mathcal{C}$ into itself. We show that $\mathcal{C}$ is a closed subset of $C([0, T], E^n)$ a complete metric space with the metric H_1 (see [7]).

6 Fuzzy integral equations

Let (ψ_n) be a sequence in $\mathcal{C}$ converging to ψ in $C([0, T], E^n)$. Consider

$$\begin{aligned}
 H_1(\psi, \phi) &= \sup_{0 \leq t \leq T} D(\psi(t), \phi(t)) \\
 &= \sup_{0 \leq t \leq T} \{D(\psi_n(t), \psi(t)) + D(\psi_n(t), \phi(t))\} \\
 &\leq H_1(\psi_n, \psi) + H_1(\psi_n, \phi) \\
 &\leq \epsilon + b
 \end{aligned} \tag{3.7}$$

for sufficiently large n and all positive ϵ . So $\psi \in \mathcal{C}$. This implies that $\mathcal{C}$ is a closed subset of $C([0, T], E^n)$. Therefore $\mathcal{C}$ is a complete metric space. We prove that A is a contraction mapping. For $\psi_1, \psi_2 \in \mathcal{C}$,

$$\begin{aligned}
 H_1(A\psi_1, A\psi_2) &= \sup_{0 \leq t \leq T} D(A\psi_1(t), A\psi_2(t)) \\
 &= \sup_{0 \leq t \leq T} D\left(\phi(t) + \psi_1(t) \int_0^t k(t, s) f(s, \psi_1(s)) ds + \int_0^t g(t, s, \psi_1(s)) ds, \phi(t) \right. \\
 &\quad \left. + \psi_2(t) \int_0^t k(t, s) f(s, \psi_2(s)) ds + \int_0^t g(t, s, \psi_2(s)) ds\right) \\
 &\leq \sup_{0 \leq t \leq T} \left\{ D\left(\psi_1(t) \int_0^t k(t, s) f(s, \psi_1(s)) ds, \psi_2(t) \int_0^t k(t, s) f(s, \psi_2(s)) ds\right) \right. \\
 &\quad \left. + \int_0^t D(g(t, s, \psi_1(s)), g(t, s, \psi_2(s))) ds \right\} \\
 &\leq (L + MT)H_1(\psi_1, \psi_2) \\
 &\leq (L + Mc)H_1(\psi_1, \psi_2) \\
 &\leq \alpha H_1(\psi_1, \psi_2) \quad \text{where } \alpha \in (0, 1).
 \end{aligned} \tag{3.8}$$

So $A : \mathcal{C} \rightarrow \mathcal{C}$ is a contraction map. Since $\mathcal{C}$ is a complete metric space and A is a contracting self-map on $\mathcal{C}$, it has a unique fixed point $x \in \mathcal{C}$. This fixed point is the required unique solution to (1.8). $\square$

4. General equations

As a generalization of (1.8) we consider the following fuzzy integral equation:

$$x(t) = \phi(t) + h(t, x(t)) \int_0^t k(t, s) f(s, x(s)) ds + \int_0^t g(t, s, x(s)) ds, \tag{4.1}$$

where $h : [0, T] \times E^n \rightarrow E^n$ is continuous and all other conditions are as before. Now we state without proof an existence theorem for (4.1).

THEOREM 4.1. Let a^* and b^* be positive numbers such that

$$b^* = \max_{0 \leq t \leq a^*} D \left(h(t, \psi(t)) \int_0^t k(t, s) f(s, \psi(s)) ds + \int_0^t g(t, s, \psi(s)) ds, \hat{0} \right). \quad (4.2)$$

Suppose that

- (i) $\phi : [0, a^*] \rightarrow E^n$ is continuous,
- (ii) $f, h : [0, T] \times E^n \rightarrow E^n$ and $k : [0, T] \times [0, T] \rightarrow R$ are continuous and there exists a constant $L^* > 0$ such that

$$D \left(h(t, x(t)) \int_0^t k(t, s) f(s, x(s)) ds, h(t, y(t)) \int_0^t k(t, s) f(s, y(s)) ds \right) \leq L^* D(x, y) \quad \text{for } x, y \in E^n, \quad (4.3)$$

- (iii) $g : U \rightarrow E^n$ is continuous where $U = \{(t, s, x) : 0 \leq s \leq t \leq a^*, x \in E^n \text{ and } D(x, \phi(t)) \leq b^*\}$ and satisfies Lipschitz condition with respect to x on U , that is, there exists a constant $M^* > 0$ such that

$$D(g(t, s, x), g(t, s, y)) \leq M^* D(x, y) \quad \text{if } (t, s, x), (t, s, y) \in U. \quad (4.4)$$

If $c^* = (\alpha - L^*)/M$ for some fixed $\alpha \in (0, 1)$, then there is a unique solution of (4.1) on $[0, T]$, where $T = \min\{a^*, b^*, c^*\}$.

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