A REMARK ON RECENT RESULTS FOR FINDING ZEROES OF ACCRETIVE OPERATORS

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By means of the Yosida approximate of an accretive operator, we extended two recent results by Chidume and Chidume and Zegeye (2003) to set-valued operators, and we made the connection with two recent convergence results obtained by Benavides et al. for a relaxed version of the so-called proximal point algorithm.

1. Introduction and preliminaries

In this paper we deal with methods for finding zeroes of set-valued operators A in Banach spaces. The first aim of this note is to show that the Yosida regularization may be combined with the schemes in [1, 2] keeping the strong convergence properties of the iterates and extending to set-valued operators two recent results by Chidume [1] and Chidume and Zegeye [2]. The second goal of the note is to make the connection with the iterative method studied in Benavides et al. [3]. This permits to obtain two convergence results under weaker conditions on the underlying operator.

Let X be a real Banach space, a (possibly multivalued) operator A with domain $D(A)$ and range $R(A)$ in X is called accretive if, for each $x_i \in D(A)$ and $y_i \in A(x_i)$ $(i = 1,2)$, there is $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$, where J stands for the normalized duality map on X, namely,

$$J(x) = \left\{x^* \in X^* : \langle x, x^*\rangle = |x|^2 = \left|x^*\right|^2\right\}, \quad x \in X. \tag{1.1}$$

An accretive operator A in X is said to be m-accretive if $R(I + \lambda A) = X$ for all $\lambda > 0$. A is said to be ϕ-strongly accretive, if there is a strictly increasing function $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ which satisfies $\phi(0) = 0$ and such that for each $x_i \in D(A)$ and $y_i \in A(x_i)$ $(i = 1,2)$, there is $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq \phi(|x_1 - x_2|)|x_1 - x_2|$, and it is strongly accretive

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if for each $x_i \in D(A)$ and $y_i \in A(x_i)$ $(i = 1,2)$, there is $j \in J(x_1 - x_2)$ and a constant $\alpha > 0$ such that $\langle y_1 - y_2, j\rangle \geq \alpha|x_1 - x_2|^2$.

Throughout the paper we always assume that A is an m-accretive operator such that $0 \in R(A)$ or in other words $S := A^{-1}(0) \neq \emptyset$ and we work with the normalized duality map for sake of simplicity and for a clearer presentation of our results.

It is well known that for each $x \in D(A)$ and $\lambda > 0$ there is a unique $z \in X$ such that $x \in (I + \lambda A)z$. The single-valued operator $J_\lambda^A := (I + \lambda A)^{-1}$ is called the resolvent of A of parameter λ. It is a nonexpansive mapping from $\overline{D(A)}$ to $\overline{D(A)}$ and is related with its Yosida approximate, $A_\lambda(x) := (x - J_\lambda^A(x))/\lambda$, by the relation

$$A_\lambda(x) \in A\left(J_\lambda^A(x)\right) \quad \forall x \in D(A). \tag{1.2}$$

Furthermore, it is clear that $x \in S := A^{-1}(0) \Leftrightarrow x = J_\lambda^A(x) \Leftrightarrow 0 = A_\lambda(x)$.

Let us finally recall that the inverse A^{-1} of A is the operator defined by $x \in A^{-1}(y) \Leftrightarrow y \in A(x)$.

In what follows, we will focus our attention on the classical problem of finding a zero of a maximal monotone operator A on a real Banach space X, namely,

$$\text{find } x \in X \text{ such that } A(x) \ni 0. \tag{1.3}$$

In [1], to solve (1.3) in the case where A is Lipschitz, Chidume considered a method which generates the next iterates x_{n+1} by

$$x_{n+1} = x_n - \mu A(x_n), \tag{1.4}$$

where x_n is the current iterate and $\mu := \alpha/(1 + L(3 + L - \alpha))$, where L and α are, respectively, the Lipschitz and the strong accretivity constants of A. He obtained the following result.

Proposition 1.1. *Let X be a real Banach space, and let $A : X \to X$ be a Lipschitz and strongly accretive map with Lipschitz constant $L > 0$ and strong accretivity constant $\alpha \in (0,1)$. For any arbitrary $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (1.4) strongly converges to the solution x^* of (1.3) with*

$$\left|x_{n+1} - x^*\right| \leq \delta^n \left|x_1 - x^*\right|, \tag{1.5}$$

where

$$\delta = 1 - \frac{\alpha\mu}{2} \in (0,1). \tag{1.6}$$

In [2], Chidume and Zegeye consider the case where the parameter μ is variable, namely,

$$x_{n+1} = x_n - \mu_n A(x_n), \tag{1.7}$$

and establish the following result.

Proposition 1.2. *Let X be a real normed linear space, and let $A : X \to X$ be a uniformly continuous ϕ-strongly accretive mapping. Then there exists $\gamma_0 > 0$ such that if the parameters*

$\mu_n \in [0,\gamma_0]$ for all $n \in \mathbb{N}$ satisfy the following conditions:

$$\lim_{n\to+\infty} \mu_n = 0, \qquad \sum_n \mu_n = \infty, \tag{1.8}$$

then, for any arbitrary $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (1.7) strongly converges to x^* solution of (1.3).

However, the Lipschitz and the strong accretivity conditions were rather stringent and they exclude important applications. In a recent paper [3], Benavides et al. [3] consider the following iterative scheme:

$$x_{n+1} = \beta_n x_n + (1-\beta_n) J_{\lambda_n}^A(x_n), \tag{1.9}$$

and prove the following results under m-accretivity of the underlying operator.

Proposition 1.3. *Let X be a uniformly convex Banach space with a Fréchet diffrentiable norm. Assume that*

$$\lim_{n\to+\infty} \beta_n = 0, \qquad \lim_{n\to+\infty} \lambda_n = \infty. \tag{1.10}$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (1.9) weakly converges to a solution of (1.3).

Proposition 1.4. *Let X be a uniformly convex Banach space with either a Fréchet diffrentiable norm or satisfies Opial's property. Assume for some $\varepsilon > 0$ that*

$$\varepsilon \le \beta_n \le 1-\varepsilon, \qquad \lambda_n \ge \varepsilon \ \forall n \in \mathbb{N}. \tag{1.11}$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (1.9) weakly converges to a solution of (1.3).

Our analysis is based on the observation that the solution set of (1.3) coincides with that of the problem

$$\text{find } x \in X \text{ such that } A_\lambda(x) = 0, \tag{1.12}$$

where A_λ is the Yosida approximate of A with parameter $\lambda > 0$.

We will apply the previous methods to A_λ, and show that with a judicious choice of the regularization parameter λ. We will first improve the results by Chidume [1] and Chidume and Zegeye [2]. Second, we will be in position to apply the results by Benavides et al. [3] and derive two new results under weaker conditions on the involving operator. The main interest is that the mapping A_λ is always Lipschitz continuous even when A is not and is strongly accretive if A is strongly accretive. For the simplicity of the exposition and a unified presentation of our results, we work in a uniformly convex Banach space, but Theorems 2.3 and 2.5 still hold true in a real Banach space and in a real normed linear space, respectively, by using the *subdifferential inequality*

$$|x+y|^2 \le |x|^2 + 2\langle y, j(x+y)\rangle \quad \forall j(x+y) \in J(x+y) \tag{1.13}$$

instead of Lemma 2.1.

2. Convergence results

To begin with, let us state the following inequality which will be needed in the proof of the next lemma.

Lemma 2.1 [5]. *Assume X is a uniformly convex Banach space. Then, there is a constant $c > 0$ such that*

$$|x+y|^2 \geq |x|^2 + 2\langle y, j(x)\rangle + c|y|^2 \quad \forall x, y \in X, \tag{2.1}$$

where $j(x) \in J(x)$.

Throughout, X is a uniformly convex Banach space.

Lemma 2.2. *If A is strongly accretive with constant α, then A_λ is strongly accretive with constant $\alpha_\lambda = \alpha/(1+2\alpha\lambda)$. Moreover, A_λ is $1/(\lambda\sqrt{c})$-Lipschitz continuous.*

Proof. Since for all $x, y \in X$, we have

$$x - y = J_\lambda^A(x) - J_\lambda^A(y) + \lambda\big(A_\lambda(x) - A_\lambda(y)\big). \tag{2.2}$$

By applying Lemma 2.1 and taking into account the fact that A is α-strongly accretive, we get

$$\begin{aligned} |x-y|^2 &\geq \left|J_\lambda^A(x) - J_\lambda^A(y)\right|^2 + c\lambda^2\left|A_\lambda(x) - A_\lambda(y)\right|^2 \\ &\quad + 2\lambda\langle A_\lambda(x) - A_\lambda(y), j\big(J_\lambda^A(x) - J_\lambda^A(y)\big)\rangle \\ &\geq \left|J_\lambda^A(x) - J_\lambda^A(y)\right|^2 + c\lambda^2\left|A_\lambda(x) - A_\lambda(y)\right|^2 + 2\lambda\alpha\left|J_\lambda^A(x) - J_\lambda^A(y)\right|^2. \end{aligned} \tag{2.3}$$

From which we derive

$$\left|A_\lambda(x) - A_\lambda(y)\right| \leq \frac{1}{\lambda\sqrt{c}}\left|x-y\right|, \tag{2.4}$$

$$\left|J_\lambda^A(x) - J_\lambda^A(y)\right|^2 \leq \frac{1}{1+2\alpha\lambda}\left|x-y\right|^2. \tag{2.5}$$

Now, applying again Lemma 2.1 with $J_\lambda^A(x) - J_\lambda^A(y) = x - y - \lambda(A_\lambda(x) - A_\lambda(y))$, we obtain

$$2\lambda\langle A_\lambda(x) - A_\lambda(y), x-y\rangle \geq |x-y|^2 - \left|J_\lambda^A(x) - J_\lambda^A(y)\right|^2 + c\lambda^2\left|A_\lambda(x) - A_\lambda(y)\right|^2, \tag{2.6}$$

which, in the light of (2.5), yields

$$\langle A_\lambda(x) - A_\lambda(y), x-y\rangle \geq \frac{\alpha}{1+2\alpha\lambda}|x-y|^2. \tag{2.7}$$

□

We are now able to give our convergence results without Lipschitz condition. First, we stress that the operator A and its Yosida regularization, A_λ, have the same zeroes. So, according to the fact that A_λ is Lipschitz even when A is not, we will use A_λ instead of A.

This leads to the following rule:

$$x_{n+1} = x_n - \mu A_\lambda(x_n). \tag{2.8}$$

Theorem 2.3. *Let $A : X \to 2^X$ be a strongly accretive map with strong accretivity constant α. Assume that for some $\lambda \in (0, 1/2)$ one has $\alpha \in (0, 1/(1-2\lambda))$. Then, for any arbitrary $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (2.4) strongly converges to x^*. Moreover,*

$$|x_{n+1} - x^*| \le \delta^n |x_1 - x^*| \quad \text{with } \delta = 1 - \frac{\alpha_\lambda \mu}{2}, \tag{2.9}$$

where L, α_λ stand for the Lipschitz and the strong accretivity constant of A_λ, and $\mu = \alpha_\lambda/(1 + L(3 + L - \alpha_\lambda))$.

Proof. Follows directly from Proposition 1.1 and Lemma 2.2. □

Remark 2.4. This result improves Proposition 1.1. Indeed, the assumption on the operator A is weaker and the bound on the strong accretivity constant α is better even if we work directly with A.

It is worth mentioning that the operator A_λ is in particular uniformly continuous, and it is easy to check that A_λ is ϕ-strongly accretive if A is so with a function ϕ satisfying $\phi(t) = (2 + r)t$, for all $t \in (0, +\infty)$ and for some $r > 0$. So, by replacing μ by μ_n in (2.8) and applying Proposition 1.2, we derive the following theorem.

Theorem 2.5. *Let $A : X \to 2^X$ be a ϕ-strongly accretive mapping. Then there exists $\gamma_0 > 0$ such that if the parameters $\mu_n \in [0, \gamma_0]$ for all $n \in \mathbb{N}$ satisfy the following conditions:*

$$\lim_{n\to+\infty} \mu_n = 0, \qquad \sum_n \mu_n = \infty, \tag{2.10}$$

then, for any arbitrary $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (2.8) (with $\mu := \mu_n$) strongly converges to x^*.

It is worth noticing that relation (2.8), with $\mu := \mu_n$, $\lambda := \lambda_n$, and $\beta_n := 1 - \mu_n/\lambda_n$, combined with the definition of the Yosida approximate, amounts to

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) J^A_{\lambda_n}(x_n). \tag{2.11}$$

This clearly paves the way to direct applications of both Propositions 1.3 and 1.4 and leads, without the strong accretivity assumption, to the next two convergence results.

Theorem 2.6. *Suppose that the norm of X is Fréchet diffrentiable and assume that*

$$\lim_{n\to+\infty} \lambda_n = +\infty, \qquad \lim_{n\to+\infty} \frac{\mu_n}{\lambda_n} = 1. \tag{2.12}$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (2.11) weakly converges to a solution of problem (1.3).

Theorem 2.7. *Suppose that X either has a Fréchet differentiable norm or satisfies Opial's property and assume for some $\varepsilon > 0$ that*

$$\varepsilon \leq \frac{\mu_n}{\lambda_n} \leq 1 - \varepsilon, \quad \lambda_n \geq \varepsilon \ \forall n \in \mathbb{N}. \tag{2.13}$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (2.11) weakly converges to a solution of problem (1.3).

3. Conclusion

The fact that the Yosida approximate, A_λ, is Lipschitz even when A is not makes it more attractive and is used here to improve two recent results by Chidume, and Chidume and Zegeye and in order to obtain two new ones. We would also like to emphasize that, in the particular case where $A = \partial f$, the subdifferential of a proper convex and lower semi-continuous function, the proposed method is reduced to that of Fukushima and Qi [4] whose implemented version converges globally and superlinearly for nonsmooth convex minimization problems.

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