STABILITY OF INVARIANT SETS OF ITÔ STOCHASTIC DIFFERENTIAL EQUATIONS WITH MARKOVIAN SWITCHING

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Consider the nonlinear Itô stochastic differential equations with Markovian switching, some sufficient conditions for the invariance, stochastic stability, stochastic asymptotic stability, and instability of invariant sets of the equations are derived.

1. Introduction

Invariant sets of dynamic systems play an important role in many situations when the dynamic behavior is constrained in some way. Knowing that a set in the state space of a system is invariant means that we have bounds on the behavior. We can verify that pre-specified bounds which originate from, for example, safety restrictions, physical constraints, or state-feedback magnitude bounds are not invalidated.

There is significant literature devoted to the invariant sets of ordinary differential equations, functional differential equations, and stochastic differential equations, and we here mention [2, 4, 15–18].

Recently, much work has been done on stochastic differential equations with Markovian switching [1, 3, 5–14, 19, 20]. In particular, we here highlight Mao's significant contribution [6, 11, 12]. However, to the best of the author's knowledge to date, the problem of the invariant sets of equations of this kind, has not been investigated yet.

The aim of the present paper is to study the invariant sets of nonlinear Itô stochastic differential equations with Markovian switching. Similar to the result of [18], which investigates the usual stochastic differential equations, some sufficient conditions for the invariance and stochastic stability of invariant sets of equations of this kind are derived. At the same time, we establish some conditions for stochastic asymptotic stability and instability of the invariant sets, which are not discussed in [18] even in the case of equations without Markovian switching.

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2. Stochastic differential equations with Markovian switching

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a complete probability space with a filtration satisfying the usual conditions, that is, the filtration is continuous on the right and $\mathcal{F}_0$ contains all P-zero sets. Let $w(t) = (w_1(t), w_2(t), \ldots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ is the Euclidean norm in R^n, that is, $|x| = \sqrt{x^T x}$ $(x \in R^n)$.

Let $\{r(t),\ t \in R_+ = [0,+\infty]\}$ be a right-continuous Markov chain on the probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ taking values in a finite state space $S = \{1,2,\ldots,N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\delta + o(\delta), & \text{if } i = j, \end{cases} \tag{2.1}$$

where $\delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+, and $r(t)$ is ergodic.

Consider the Itô stochastic differential equations with Markovian switching:

$$dx(t) = f(t,x(t),r(t))dt + g(t,x(t),r(t))dw(t), \tag{2.2}$$

where $t \geq 0$, $f : R_+ \times R^n \times S \to R^n$, $g : R_+ \times R^n \times S \to R^{n\times m}$, and the initial condition is $x(t_0) = x_0 \in R^n$, $r(t_0) = r_0 \in S$, $t_0 \geq 0$.

In this paper we always assume that both f and g satisfy the local Lipschitz condition and the linear growth condition. Hence it is known from [6] that (2.2) has a unique continuous bounded solution $x(t) = x(t,t_0,x_0)$ on $t \geq t_0$.

Denote by $C^{2,1}(R_+ \times R^n \times S; R_+)$ the family of all nonnegative functions $V(t,x,i)$ on $R_+ \times R^n \times S$ which are continuously twice differentiable with respect to x and once differentiable with respect to t. For any $(t,x,i) \in R_+ \times R^n \times S$, we define an operator $\mathcal{L}$ by

$$\begin{aligned} \mathcal{L}V(t,x,i) = {} & \sum_{j=1}^{N} \gamma_{ij} V(t,x,j) + V_t(t,x,i) + V_x(t,x,i) f(t,x,i) \\ & + \frac{1}{2}\operatorname{trace}\left[g^T(t,x,i) V_{xx}(t,x,i) g(t,x,i)\right], \end{aligned} \tag{2.3}$$

where

$$\begin{gathered} V_t(t,x,i) = \frac{\partial V(t,x,i)}{\partial t}, \qquad V_x(t,x,i) = \left(\frac{\partial V(t,x,i)}{\partial x_1}, \ldots, \frac{\partial V(t,x,i)}{\partial x_n}\right), \\ V_{xx}(t,x,i) = \left(\frac{\partial^2 V(t,x,i)}{\partial x_i \partial x_j}\right)_{n\times n}. \end{gathered} \tag{2.4}$$

The generalized Itô formula reads as follows: if $V \in C^{2,1}([-\tau,+\infty) \times R^n \times S; R_+)$, then

$$EV(t+h, x(t+h), r(t+h)) = EV(t,x(t),r(t)) + E\int_t^{t+h} \mathcal{L}V(s,x(s),r(s))ds. \tag{2.5}$$

3. Main results

Denote by Q a certain Borel set in $\{t \geq 0\} \times R^n \times S$. Let $Q_t^{r(t)}$ be a set in R^n, here $Q_t^{r(t)} = \{x(t) : (t, x(t), r(t)) \in Q\}$ and $Q_t^{r(t)}$ is nonempty for $t \geq 0$.

Definition 3.1. A set Q is called invariant for (2.2) if for $(t_0, x_0, r_0) \in Q$,

$$P\{(t, x(t), r(t)) \in Q,\ \forall t \geq t_0\} = 1. \tag{3.1}$$

Remark 3.2. Definition 3.1 is equivalent to the condition

$$P\{(t, x(t), r(t)) \in Q\} = 1, \quad \forall t \geq t_0. \tag{3.2}$$

Definition 3.3. A set Q is called stochastically stable if for any $\varepsilon > 0$ the following holds:

$$\lim_{\rho(x_0, Q_{t_0}^{r_0}) \to 0} P\left\{\sup_{t \geq t_0} \rho\left(x(t), Q_t^{r(t)}\right) > \varepsilon\right\} = 0, \tag{3.3}$$

where $\rho(x, A)$ denotes the distance between a point x and a set A.

Definition 3.4. A set Q is called stochastically asymptotically stable if it is stochastically stable and moreover

$$\lim_{\rho(x_0, Q_{t_0}^{r_0}) \to 0} P\left\{\lim_{t \to \infty} \rho\left(x(t), Q_t^{r(t)}\right) = 0\right\} = 1. \tag{3.4}$$

Definition 3.5. A set Q is called unstable if there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that, for any $\delta > 0$, there exist x_0 and t^* such that for $\rho(x_0, Q_{t_0}^{r_0}) < \delta$ the following holds:

$$P\left\{\sup_{t^* \geq t \geq t_0} \rho\left(x, Q_t^{r(t)}\right) > \varepsilon_1\right\} \geq \varepsilon_2. \tag{3.5}$$

Denote by Γ the set of its zeros of the function V in $\{t \geq 0\} \times R^n \times S$, that is, $\Gamma = \{(t, x, i) : V(t, x, i) = 0\}$. Let Γ_t^i denote the set of $x \in R^n$ such that $V(t, x, i) = 0$ for fixed $t \geq 0$ and $i \in S$.

Theorem 3.6. *If*

$$\mathcal{L}V(t, x, i) \leq 0, \tag{3.6}$$

then the set Γ is a positive invariant set for (2.2). In addition, if

$$\inf_{\rho(x, \Gamma_t^i) > \delta} V(t, x, i) = V_\delta > 0 \tag{3.7}$$

for any $\delta > 0$, then the set Γ is stochastically stable.

Proof. Without loss of generality, we assume that $t_0 = 0$, that is, $x(0,x_0) = x_0 \in \Gamma_0^i$. Applying (2.5) we have

$$EV(t,x(t),i) = EV(0,x_0,i) + E\int_0^t \mathcal{L}V(s,x(s),i)\,ds. \tag{3.8}$$

From (3.6), we get

$$EV(t,x,i) \le 0, \tag{3.9}$$

that is,

$$V(t,x,i) = 0 \quad \text{a.e.} \tag{3.10}$$

Thus Γ is invariant.

Assume that ε_1 and ε_2 are arbitrary positive constants. Denote $\inf_{\rho(x,\Gamma_t^i)>\varepsilon_1} V(t,x,i) = V_{\varepsilon_1}$. Then, from (3.7) we know that $V_{\varepsilon_1} > 0$. It is easily seen that

$$P\left\{\sup_{t\ge t_0}\rho(x,\Gamma_t^i) > \varepsilon_1\right\} V_{\varepsilon_1} \le EV(t,x,i) \le V(t_0,x_0,i). \tag{3.11}$$

By the right-continuity of the function V, there exists $\delta > 0$ for $\varepsilon_2 > 0$ such that for $\rho(x_0,\Gamma_{t_0}^{r_0}) < \delta$, we get

$$V(t_0,x_0,i) \le V_{\varepsilon_1}\varepsilon_2. \tag{3.12}$$

By use of (3.11) and (3.12), we complete the proof. □

Remark 3.7. It is obvious that the condition (3.7) is automatically satisfied if the function V is independent on t.

Remark 3.8. If both of the coefficients of (2.2), f and g, are independent of the Markov chain $r(t)$, then Theorem 3.6 in this paper reduces to [18, Theorem 1].

Theorem 3.9. *Suppose that (3.7) and the following hold:*

$$\mathcal{L}V(t,x,i) \le -\varphi(V(t,x,i)), \tag{3.13}$$

where $\varphi(s)$, $s \ge 0$ is a continuous function, $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$, and $E\varphi(\tau) \ge \varphi(E\tau)$ for every nonnegative random variable τ. Then the set Γ is a stochastically asymptotically stable invariant set for (2.2).

Proof. Since the conditions of Theorem 3.6 are satisfied, Γ is a stable invariant set. Hence, for any sufficiently small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists $\delta > 0$ such that for $\rho(x_0,\Gamma_{t_0}^i) < \delta$, we have $P\{\sup_{t\ge t_0}\rho(x(t),\Gamma_t^i) > \varepsilon_1\} < \varepsilon_2$.

It is obviously seen from (2.5) and (3.16) that the function $EV(t,x(t),i)$ is nonnegative and nonincreasing. Let $\alpha = \lim_{t\to+\infty} EV(t,x(t),i)$. Hence, $\alpha \ge 0$. Assume that $\alpha > 0$. Then the inequality $\alpha \le EV(t,x(t),i) \le V_{\varepsilon_1}$ is true. Denote $c = \min_{\alpha\le s\le V_{\varepsilon_1}} \varphi(s)$. From (2.5) and (3.16), we get

$$EV(t,x(t),i) \le EV(0,x_0,i) - ct, \tag{3.14}$$

which, for sufficiently large t, contradicts the fact $\alpha > 0$. Hence, we have $\alpha = 0$. Thus, for $\rho(x_0, \Gamma^i_{t_0}) < \delta$, we get

$$\lim_{t\to\infty} \rho(x(t), \Gamma^i_t) = 0 \quad \text{a.e.} \tag{3.15}$$

The proof is complete. □

Theorem 3.10. *Suppose that (3.7) and the following hold:*

$$\mathscr{L}V(t,x,i) \geq \varphi(V(t,x,i)), \tag{3.16}$$

where $\varphi(s)$, $s \geq 0$ is a continuous function, $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$, and $E\varphi(\tau) \leq \varphi(E\tau)$ for every nonnegative random variable τ. Then the set Γ is an unstable set for (2.2).

Proof. Let $\delta > 0$ be a sufficiently small number. We choose x_0 such that $\rho(x_0, \Gamma^i_{t_0}) < \delta$ and assume that $V(0, x_0, i) = \alpha > 0$. To the contrary, we assume that $\rho(x(t), \Gamma^i_t) < \varepsilon$ for $t > 0$. From (2.5) and (3.16), it is seen that $EV(t,x,i) > 0$. Thus there exists an L such that $0 < \alpha \leq EV(t,x,i) \leq L$. Denote $c = \min_{\alpha\leq s\leq L} \varphi(s)$. Equations (3.16) and (2.5) yield $EV(t,x,i) \geq V(0,x_0,i) + ct$, which contradicts the fact that $EV(t,x,i)$ is bounded. Thus, the assumption $\rho(x(t), \Gamma^i_t) < \varepsilon$ for $t > 0$ is not true and the set Γ is unstable. □

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