This paper is devoted to showing the null exact controllability for a class of parabolic equations with equivalued surface boundary condition. Our method is based on the duality argument and global Carleman-type estimate for a parabolic operator.

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1. Introduction

Let \( T > 0, \Omega \subset \mathbb{R}^n (n \in \mathbb{N}) \) be a given bounded domain, \( \partial \Omega = \Gamma_0 \cup \Gamma_1 (\Gamma_1 \neq \emptyset) \) (where \( \Gamma_0 \) is the interior boundary and \( \Gamma_1 \) the outer boundary), \( \Gamma_0 \cap \Gamma_1 = \emptyset \). For simplicity, we assume that \( \Gamma_0, \Gamma_1 \in C^\infty \) and \( \omega \neq \emptyset \) is a given subdomain of \( \Omega \). Denote the characteristic function of \( \omega \) by \( \chi_\omega \), and the unit outward normal vector of \( \Omega \) by \( (n_1, \ldots, n_n) \). Put \( Q = \Omega \times (0, T), Q_\omega = \omega \times (0, T) \), and \( \Sigma = \partial \Omega \times (0, T) \). Let \( a_{ij}(x) \in C^2(\Omega) \) satisfy \( a_{ij} = a_{ji} \), and for some \( \Lambda > 0 \), it holds that

\[
\sum_{i,j} a_{ij} \xi_i \xi_j \geq \Lambda |\xi|^2, \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n. \tag{1.1}
\]

Here and henceforth, we denote \( \sum_{i,j=1}^n \) simply by \( \sum_{i,j} \).

We consider the following controlled parabolic equation with equivalued surface boundary condition:

\[
\frac{\partial y}{\partial t} - \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial y}{\partial x_j} \right) = \chi_\omega(x)b \quad \text{in } Q, \\
y|_{\Gamma_1} = 0, \quad y|_{\Gamma_0} = m(t), \\
\int_{\Gamma_0} \frac{\partial y}{\partial n_A} \, ds = 0, \quad y(x, 0) = y_0(x) \quad \text{in } \Omega, \tag{1.2}
\]
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where \( y = y(x, t) \) is the state, \( b = b(x, t) \) is the control, \( m(t) \in L^2(0, T) \) is an unknown function which depends only on the time variable \( t \), \( y_0 \) is the initial state, and

\[
\frac{\partial y}{\partial n_A} = \sum_{i,j} a_{ij}(x) \frac{\partial y}{\partial x_j} n_i,
\]

(1.3)

In system (1.2), the state space is chosen as \( L^2(\Omega) \), and the control space is \( L^2(\omega \times (0, T)) \). Let

\[
Y \triangleq \left\{ y \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \mid y|_{\Gamma_1} = 0, y|_{\Gamma_0} = m(\cdot) \in L^2(0, T) \right\}.
\]

(1.4)

It can be shown that for any \( y_0 \in L^2(\Omega) \) and \( b \in L^2(\omega \times (0, T)) \), system (1.2) admits one and only one weak solution \( y \in Y \) (cf. [5, 6, 8]).

The null exact controllability problem of (1.2) is formulated as follows: for any given \( y_0 \in L^2(\Omega) \), find a control \( b(x, t) \in L^2(\omega \times (0, T)) \) (if possible) such that the weak solution \( y(\cdot) \in Y \) satisfies \( y(T) = 0 \).

There are many concrete physical backgrounds for problem (1.2), for example, the problem of resistivity well logging, the unstable temperature field around an underground electric cable, and so on (cf. [5, 6]).

In recent years, great progress has been made in the exact controllability problem of the linear and semilinear partial differential equations with Dirichlet or Neumann boundary condition, or other sorts of pointwise boundary value conditions ([1–4, 7, 9], and the references cited therein). However, to the author’s best knowledge, there is no reference devoted to the same problem for the parabolic equations but with a spatial nonlocal boundary condition. In this paper, we will show the null exact controllability for system (1.2). By duality, the problem is reduced to the obtention of an observability inequality for the corresponding adjoint equation, which in turn is derived by means of a global Carleman-type estimate. Our method is stimulated by that in [4].

The rest of this paper is organized as follows. In Section 2, we state some preliminary results and our main results. The final section, Section 3, is devoted to the proof of our main theorem.

2. Main result

Throughout this paper, \( C \) denotes a positive constant depending only on \( \lambda, \mu, \Omega, T, \) and \( \omega \), which may change from line to line.

To begin with, we fix \( \omega_0 \) to be a nonempty open subset of \( \Omega \) such that \( \overline{\omega_0} \subset \omega \). Let \( \psi \in C^\infty(\overline{\Omega}) \) satisfy \( \psi > 0 \) in \( \Omega \), \( \psi = 0 \) on \( \partial \Omega \), and \( |\nabla \psi(x)| > 0 \) for all \( x \in \Omega_0 = \Omega \setminus \omega_0 \). The existence of function \( \psi \) was proved in [7]. In this paper, we further assume that the following technical condition holds:

\[
\left( \frac{\partial \psi}{\partial n} \right)^2 \sum_{i,j} a_{ij} n_i n_j |_{\Gamma_0} = \text{Const}.
\]

(2.1)

This technical condition admits several interesting cases such as \( \Omega = \{ x \in \mathbb{R}^n \mid r < |x| < R \} \)
for some $0 < r < R < \infty$, $a_{ij}(x) = \delta_i^j$, and $\partial\psi/\partial n = \text{Const}$ on $\Gamma_0$. In this case, one may choose $\psi(x) = (|x|^2 - r^2)(R^2 - |x|^2)$.

The main result in this paper is stated as follows.

**Theorem 2.1.** Under the assumption (2.1), system (1.2) is null exact controllable.

By means of the usual duality argument (see, e.g., [3, 9]), the proof of Theorem 2.1 is easily reduced to the obtention of an observability inequality for the following adjoint system of system (1.2):

$$-Lu \equiv -\frac{\partial u}{\partial t} - \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = 0 \quad \text{in } Q,$$

$$u|_{\Gamma_1} = 0, \quad u|_{\Gamma_0} = c(t), \quad \int_{\Gamma_0} \frac{\partial u}{\partial n} d\sigma = 0, \quad u(x, T) = u(T) \quad \text{in } \Omega,$$

(2.2)

where $u(T) \in L^2(\Omega)$, and similarly to system (1.2), $c(\cdot) \in L^2(0, T)$ is an unknown function. More precisely, we need to show the following.

**Theorem 2.2.** Under the assumption (2.1), there is a constant $C > 0$ such that solutions $u \in Y$ of system (2.2) satisfy

$$\|u(0)\|_{L^2(\Omega)} \leq C \|u(x, t)\|_{L^2(\omega \times (0, T))}.$$

(2.3)

**Remark 2.3.** It would be quite interesting to drop the technical condition (2.1). But this is by now an unsolved problem.

**3. Proof of the main theorem**

It suffices to prove Theorem 2.2. To this end, for any given parameters $\lambda$ and $\mu$, we set

$$\alpha(x, t) = (t(T - t))^{-1} (e^{\mu\psi(x)} - e^{2\mu\|\psi\|_{C(\bar{\Omega})}}),$$

$$\phi(x, t) = (t(T - t))^{-1} e^{\mu\psi(x)}, \quad \theta(x, t) = e^{\lambda\alpha(x, t)}.$$

(3.1)

Clearly, Theorem 2.2 is an easy consequence of the following global Carleman-type estimate for solutions of (2.2).

**Theorem 3.1.** Let (2.1) hold. Then there exist a constant $\mu_1$ and a function $\lambda_1 : \mathbb{R}^+ \rightarrow (1, \infty)$ such that for any $\mu > \mu_1$ and $\lambda > \lambda_1(\mu)$, solutions $u \in Y$ of system (2.2) satisfy

$$\lambda^3 \int_Q \theta^2 \phi^3 u^2 dx \, dt + \lambda \int_Q \theta^2 \phi |\nabla u|^2 dx \, dt + \lambda^{-1} \int_Q \theta^2 \phi^{-1} [u_t^2 + (\Delta u)^2] dx \, dt$$

$$\leq C \lambda^3 \int_{Q_0} \theta^2 \phi^3 u^2 dx \, dt.$$

(3.2)

The rest of this section is devoted to prove Theorem 3.1. For this, we need the following pointwise estimate for parabolic operator $Lu$, which is a special case of [4, Lemma 3.1].
Lemma 3.2. Let \( u, \alpha \in C^2(\mathbb{R}^{n+1}) \) and \( \lambda > 0 \) be given. Put \( \theta = e^{\lambda u} \) and \( v = \theta u \). Then it holds that

\[
\theta^2 |Lu|^2 \geq \left[ -\lambda \alpha_t v^2 - \sum_{i,j} (a_{ij} v_x v_{x_j} + \lambda (a_{ij}) v_x^2 + \lambda a_{ij} \alpha x_{x_j} v^2 - \lambda^2 a_{ij} \alpha \alpha x_{x_j} v^2) \right]_t \\
+ \sum_j \left[ 2 \sum_i \left( a_{ij} v_x v_t + \lambda^2 a_{ij} \alpha x_t v_t \right) \\
+ 2 \lambda \sum_i \left( a_{ij} a_{\ell m} \alpha x_{x_j} v_{x_{x_m}} + a_{ij} (a_{\ell m} \alpha x_{x_{x_{x_m}}})_x v^2 - 2 a_{ij} a_{\ell m} \alpha x_t v_x v_{x_{x_m}} \\
- 2 a_{ij} a_{\ell m} \alpha x_{x_j} v_{x_{x_m}} + \lambda a_{ij} a_{\ell m} \alpha x_{x_j} x_{x_{x_m}} v^2 \\
+ \lambda a_{ij} (a_{\ell m})_{x_{x_m}} \alpha x_{x_j} x_{x_{x_m}} v^2 - \lambda^2 a_{ij} a_{\ell m} \alpha x_{x_j} x_{x_{x_m}} v^2 \right) \\
+ \left[ \lambda \alpha_{tt} - \lambda \sum_{i,j} \left[ ((a_{ij})_x)_x + (a_{ij} \alpha x_{x_j})_x - \lambda \alpha x_{x_j} \alpha x_{x_j} \right]_t \\
- 2 \lambda (a_{ij} \alpha x_{x_j} \alpha_t) + 4 \lambda a_{ij} \alpha x_{x_j} \alpha_t \right] \\
+ 2 \lambda \sum_{i,j,\ell,m} \left[ 2 \lambda \alpha_{ij} (a_{\ell m} \alpha x_{x_j} \alpha x_{x_{x_m}} - (a_{ij} (a_{\ell m} \alpha x_{x_{x_{x_m}}})_x)_x) - \lambda \alpha_{\ell m} \alpha x_{x_j} \alpha x_{x_{x_m}} \\
- 2 \lambda a_{ij} a_{\ell m} \alpha x_{x_j} \alpha x_{x_{x_m}} \\
+ \lambda^2 (a_{ij} a_{\ell m} \alpha x_{x_j} \alpha x_{x_{x_m}} x_{x_{x_m}}) - 2 \lambda^2 a_{ij} a_{\ell m} \alpha x_{x_j} \alpha x_{x_{x_m}} x_{x_{x_m}} \right] v^2 \\
+ 2 \lambda \sum_{i,j,\ell,m} \left[ 2 a_{ij} (a_{\ell m} \alpha x_{x_j}) x_{x_m} v_{x_{x_m}} - (a_{ij} a_{\ell m})_{x_{x_m}} \alpha x_{x_j} x_{x_{x_m}} v_{x_{x_m}} + a_{ij} a_{\ell m} \alpha x_{x_{x_{x_m}}} v_{x_{x_m}} v_{x_{x_j}} \right] \right] \\
(3.3)
\]

Proof of Theorem 3.1. The main idea of our proof is to use the pointwise estimate in Lemma 3.2. The proof is divided into several steps.

Step 1. Recall that \( \theta = e^{\lambda u} \), \( v = \theta u \). We claim that

\[
\int_Q \theta^2 |Lu|^2 dx \, dt - \int_Q \nabla v \nabla \psi dx \, dt + C \lambda^3 \mu^3 \int_Q \phi^3 |\nabla v|^2 dx \, dt + C \lambda \mu^2 \int_Q \phi |\nabla \psi|^2 dx \, dt \\
\geq 2 \Lambda^2 \lambda^3 \mu^4 \int_Q \phi^3 |\nabla \psi|^4 dx \, dt + 2 \Lambda^2 \lambda \mu^2 \int_Q \phi |\nabla v|^2 |\nabla \psi|^2 dx \, dt, \tag{3.4}
\]

where \( V = (V_1, V_2, \ldots, V_n) \), and

\[
V_j = -2 \sum_i (a_{ij} v_x v_t + \lambda^2 a_{ij} \alpha x_t v_t) \\
+ 2 \lambda \sum_{i,\ell,m} \left( a_{ij} a_{\ell m} \alpha x_{x_j} v_{x_{x_m}} - 2 a_{ij} a_{\ell m} \alpha x_{x_j} v_{x_{x_m}} + \lambda a_{ij} (a_{\ell m})_{x_{x_m}} \alpha x_{x_j} x_{x_{x_m}} v^2 \\
- \lambda^2 a_{ij} a_{\ell m} \alpha x_{x_j} x_{x_{x_m}} v^2 + \lambda a_{ij} a_{\ell m} \alpha x_{x_j} x_{x_{x_m}} v^2 \\
- 2 a_{ij} a_{\ell m} \alpha x_{x_{x_m}} v_{x_{x_j}} + a_{ij} (a_{\ell m} \alpha x_{x_{x_m}})_{x_{x_j}} v^2 \right). \tag{3.5}
\]
By the definition of $\alpha$ and $\phi$ in (3.1), it is easy to see that

$$
\int_Q \left[ -\lambda \alpha_t v^2 + \sum_{i,j} (a_{ij} v_x v_{x_j} + \lambda (a_{ij})_{x_j} \alpha_{x_j} v^2 - \lambda^2 a_{ij} \alpha_{x_j}^2 + \lambda a_{ij} \alpha_{x_j} v^2) \right] dt = 0. \quad (3.6)
$$

Let us estimate the last three “energy” terms of first order in the right-hand side of (3.3). First,

$$
\left| \int_Q 4 \lambda \sum_{i,j,m} a_{ij} (a_{\ell m} \alpha_{x_j})_x v_x v_{x_m} dx dt \right| = \left| \int_Q 4 \lambda \sum_{i,j,m} (a_{ij} (a_{\ell m})_x \mu \psi_x \psi_x v_x v_{x_m} + a_{ij} a_{\ell m} (\mu \psi_{x_j} \psi_x \phi + \mu^2 \psi_{x_j} \psi_x \phi) v_x v_{x_m}) \right| dx dt \leq C \lambda \mu \int_Q |\nabla v|^2 dx dt + C \lambda \mu^2 \int_Q |\nabla v|^2 dx dt,
$$

(3.7)

where $C$ is a positive constant for $\lambda$ large enough.

Next, it is easy to see that

$$
\left| - \int_Q 2 \lambda \sum_{i,j,m} (a_{ij} a_{\ell m})_x \alpha_{x_j} v_x v_{x_j} dx dt \right| \leq C \lambda \mu \int_Q |\nabla v|^2 dx dt,
$$

$$
\int_Q 2 \lambda \sum_{i,j,m} a_{ij} a_{\ell m} \alpha_{x_{x_m}} v_x v_{x_j} dx dt = \int_Q 2 \lambda \sum_{i,j,m} (\mu a_{ij} a_{\ell m} \psi_{x_{x_m}} \psi_x v_x v_{x_j} + \mu^2 a_{ij} a_{\ell m} \psi_x \psi_x \psi_{x_m} \psi_x v_{x_j}) dx dt \geq -C \lambda \mu \int_Q |\nabla v|^2 dx dt + 2 \lambda \mu^2 \Lambda^2 \int_Q |\nabla v|^2 |\nabla v|^2 dx dt.
$$

(3.8)

It remains to deal with the “energy” term of zero order, that is, $\int_Q \{ \cdots \} v^2 dx dt$, in the right-hand side of (3.3). Similarly to [4], we have

$$
-2 \int_\Omega \lambda^3 \sum_{i,j,m} (2 a_{ij} a_{\ell m} \alpha_{x_i} \alpha_{x_j} \alpha_{x_{x_m}} v^2 - (a_{ij} a_{\ell m} \alpha_{x_i} \alpha_{x_j} \alpha_{x_{x_m}}) v^2) dx dt
$$

$$
= - \int_\Omega 2 \lambda^3 \mu^3 \sum_{i,j,m} (2 a_{ij} a_{\ell m} \psi_x \psi_x \psi_{x_{x_m}} \psi_{x_{x_m}} \phi^3 v^2 + 2 \mu a_{ij} a_{\ell m} \psi_x \psi_x \psi_x \psi_{x_m} \psi_{x_{x_m}} \phi^3 v^2
$$

$$
- (a_{ij} a_{\ell m} \psi_x \psi_x \psi_x \phi_{x_m}) v^3 v^2 - 3 \mu a_{ij} a_{\ell m} \psi_x \psi_x \psi_x \psi_{x_m} \phi^3 v^2) dx dt \geq -C \lambda^3 \mu^3 \int_\Omega \phi^3 v^2 dx dt + 2 \lambda^3 \mu^4 \Lambda^2 \int_\Omega \phi^3 |\nabla \psi|^4 v^2 dx dt.
$$

(3.9)
Therefore, for large $\lambda$, the following estimate holds:

$$
\int_Q \left\{ \lambda \alpha_{tt} - \lambda \sum_{i,j} \left[ ((a_{ij})_{x_t} \alpha_{x_t})_t - \lambda(a_{ij} \alpha_{x_t} \alpha_{x_t})_t + 2\lambda(a_{ij} \alpha_{x_t} \alpha_{x_t})_{x_t} + 4\lambda a_{ij} \alpha_{x_t} \alpha_{x_t} \alpha_{x_t} \right] 
+ 2\lambda \sum_{i,j,l,m} \left[ \lambda^2(a_{ij} a_{lm} \alpha_{x_t} \alpha_{x_l} \alpha_{x_m})_x - \lambda(a_{ij} a_{lm} \alpha_{x_t} \alpha_{x_l} \alpha_{x_m})_{x_x} - \lambda((a_{lm})_{x_m} a_{ij} \alpha_{x_t} \alpha_{x_l})_{x_t} 
+ 2\lambda a_{ij} a_{lm} \alpha_{x_t} \alpha_{x_l} \alpha_{x_m} 
- 2\lambda a_{ij} a_{lm} \alpha_{x_t} \alpha_{x_l} \alpha_{x_m} - (a_{ij} (a_{lm})_{x_m} a_{ij} \alpha_{x_t} \alpha_{x_l})_{x_m} \right] \right\} v^2 \, dx \, dt
\geq -C\lambda^3 \mu^3 \int_\Omega v^2 \, dx \, dt + 2\lambda^3 \mu^4 \Lambda^2 \int_Q \phi^3 |\nabla \psi|^4 v^2 \, dx \, dt.
$$

(3.10)

**Step 2.** From (3.4), one finds

$$
C \left( \int_Q \theta^2 (Lu)^2 \, dx \, dt - \int_Q \text{div} \, V \, dx \, dt + \lambda^3 \mu^3 \int_Q \phi^3 v^2 \, dx \, dt + \lambda \mu^2 \int_Q \phi |\nabla v|^2 \, dx \, dt \right)
\geq \lambda^3 \mu^4 \int_Q \phi^3 |\nabla \psi|^4 v^2 \, dx \, dt + \lambda \mu^2 \int_Q \phi |\nabla \psi|^2 |\nabla v|^2 \, dx \, dt.
$$

(3.11)

Set

$$
Q_{\omega_0} = \omega_0 \times (0, T).
$$

(3.12)

Noting that $|\nabla \psi(x)| > 0$ for all $x \in \Omega_0 = \overline{\Omega} \setminus \omega_0$, by (3.11), it is easy to see that

$$
C \left( \int_Q \theta^2 (Lu)^2 \, dx \, dt - \int_Q \text{div} \, V \, dx \, dt + \lambda^3 \mu^3 \int_{Q_{\omega_0}} \phi^3 v^2 \, dx \, dt + \lambda \mu^2 \int_{Q_{\omega_0}} \phi |\nabla v|^2 \, dx \, dt \right)
\geq \lambda^3 \mu^4 \int_Q \phi^3 v^2 \, dx \, dt + \lambda \mu^2 \int_Q \phi |\nabla v|^2 \, dx \, dt.
$$

(3.13)

Returning $v$ to $e^{\lambda t} u$ in (3.13), we arrive at

$$
C \left( \int_Q \theta^2 (Lu)^2 \, dx \, dt - \int_Q \text{div} \, V \, dx \, dt + \lambda^3 \int_{Q_{\omega_0}} \theta^2 \phi^3 u^2 \, dx \, dt + \lambda \int_{Q_{\omega_0}} \theta^2 \phi |\nabla u|^2 \, dx \, dt \right)
\geq \lambda^3 \int_Q \theta^2 \phi^3 u^2 \, dx \, dt + \lambda \int_Q \theta^2 \phi |\nabla u|^2 \, dx \, dt.
$$

(3.14)

**Step 3.** The purpose of this step is to get rid of the fourth term in the left-hand side of (3.14). To this end, we multiply $Lu$ by $\tilde{\chi} \theta^2 \phi u$ and then integrate it over $Q$, where $\tilde{\chi} \in C_0^\infty(\omega)$, $\tilde{\chi} = 1$ in $\omega_0$, and $\tilde{\chi} = 0$ in $\Omega \setminus \omega$. Then, we obtain

$$
\int_Q \frac{\partial u}{\partial t} \tilde{\chi} \theta^2 \phi u \, dx \, dt + \int_Q \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) \tilde{\chi} \theta^2 \phi u \, dx \, dt = \int_Q (Lu) \tilde{\chi} \theta^2 \phi u \, dx \, dt.
$$

(3.15)
Using integration by parts, it is easy to deduce from (3.15) that

\[ \lambda \int_{Q^{\omega}} \theta^{2} \varphi |\nabla u|^{2} dx \ dt \leq C \left( \int_{Q} \theta^{2}(Lu)^{2} dx \ dt + \lambda^{3} \int_{Q^{\omega}} \theta^{2} \varphi^{3} u^{2} dx \ dt \right). \quad (3.16) \]

Combining (3.13) and (3.16), we end up with

\[ C \left( \int_{Q} \theta^{2}(Lu)^{2} dx \ dt + \lambda^{3} \int_{Q^{\omega}} \theta^{2} \varphi^{3} u^{2} dx \ dt - \int_{Q} \text{div} V dx \ dt \right) \]

\[ \geq \lambda^{3} \int_{Q} \theta^{2} \varphi^{3} u^{2} dx \ dt + \lambda \int_{Q} \theta^{2} \varphi |\nabla u|^{2} dx \ dt . \quad (3.17) \]

**Step 4.** This step is to estimate the “divergence” term \( \text{div} V \). Denote the terms on the right-hand side of (3.5) by \( I_{i}, i = 1, 2, \ldots, 9 \). First,

\[ I_{1} + I_{2} = \int_{Q} \sum_{j} \left( 2 \sum_{i} \left( a_{ij} \nu_{x_{i}} \nu_{x_{j}} + \lambda^{2} a_{ij} \alpha_{x_{i}} \alpha_{x_{j}} \varphi^{2} \right) \right) \ dx \]

\[ = \lambda^{2} \mu \int_{0}^{T} \theta^{2} \varphi \psi^{2} (t) \ dt \int_{\Gamma_{0}} \sum_{i,j} a_{ij} \psi_{x_{i}} n_{j} ds - 2 \lambda^{2} \mu \int_{0}^{T} \theta^{2} \varphi \alpha_{i} \psi^{3} (t) \ dt \int_{\Gamma_{0}} \sum_{i,j} a_{ij} \psi_{x_{i}} n_{j} ds. \quad (3.18) \]

Next,

\[ I_{3} + I_{5} + I_{9} = \int_{Q} \sum_{j} \left( 2 \lambda \sum_{i, \ell, m} \left( a_{ij} a_{\ell m} \alpha_{x_{i}} \alpha_{x_{\ell}} \nu_{x_{m}} - 2 a_{ij} a_{\ell m} \alpha_{x_{i}} \alpha_{x_{\ell}} \psi_{x_{m}} - \lambda^{2} a_{ij} a_{\ell m} \alpha_{x_{i}} \alpha_{x_{m}} \varphi^{2} \right) \right) \ dx \ dt \]

\[ = -4 \lambda^{3} \mu^{3} \int_{0}^{T} \theta^{2} \varphi \psi^{2} (t) \ dt \int_{\Gamma_{0}} \sum_{i,j, \ell, m} a_{ij} a_{\ell m} \psi_{x_{i}} \psi_{x_{\ell}} \psi_{x_{m}} n_{j} ds \]

\[ -4 \lambda^{2} \mu^{2} \int_{0}^{T} \theta^{2} \varphi \psi^{2} (t) \ dt \int_{\Gamma_{0}} \sum_{i,j, \ell, m} a_{ij} a_{\ell m} \psi_{x_{i}} \psi_{x_{\ell}} u_{x_{m}} n_{j} ds \]

\[ -2 \int_{0}^{T} \lambda \mu \varphi \theta^{2} \ dt \int_{\Gamma_{0}} \sum_{i,j, \ell, m} a_{ij} a_{\ell m} \psi_{x_{i}} u_{x_{\ell}} u_{x_{m}} n_{j} ds. \quad (3.19) \]

Further,

\[ I_{4} = \int_{Q} \sum_{j} \left( 2 \lambda \sum_{i} \left( a_{ij} a_{\ell m} \alpha_{x_{i}} \nu_{x_{m}} \nu_{x_{j}} \right) \right) \ dx \ dt \]

\[ = 2 \lambda \mu^{2} \int_{0}^{T} \theta^{2} \varphi \psi^{2} (t) \ dt \int_{\Gamma_{0}} \sum_{i,j, \ell, m} a_{ij} \left( a_{\ell m} \right) \psi_{x_{i}} \psi_{x_{\ell}} n_{j} ds \]

\[ + 2 \lambda \mu \int_{0}^{T} \theta^{2} \varphi \psi^{2} (t) \ dt \int_{\Gamma_{0}} \sum_{i,j, \ell, m} a_{ij} \left( a_{\ell m} \right) \psi_{x_{i}} \psi_{x_{m}} n_{j} ds \]

\[ + 2 \lambda \mu \int_{0}^{T} \theta^{2} \varphi \psi^{2} (t) \ dt \int_{\Gamma_{0}} \sum_{i,j, \ell, m} a_{ij} a_{\ell m} \psi_{x_{i}} \psi_{x_{\ell}} \psi_{x_{m}} n_{j} ds. \]
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\[ + 2\lambda^2 \mu \int_0^T \theta^2 \phi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_m n_j} ds \]

\[ + 2\lambda^2 \mu \int_0^T \theta^2 \phi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_m n_j} ds \]

\[ + 2\lambda \mu \int_0^T \theta^2 \phi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_m n_j} ds \]

\[ \geq C\lambda^3 \mu^3 \int_0^T \theta^2 \phi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_m n_j} ds. \]  

(3.20)

Further,

\[ I_6 = -\int_Q \sum_j \left( 4\lambda \sum_{i,\ell,m} a_{ij} a_{\ell m} \alpha_{x_i} \alpha_{x_m n_j} \psi_{x_j} \right) dx dt \]

\[ = -4\lambda^2 \mu^3 \int_0^T \theta^2 \phi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_m n_j} ds \]

\[ - 4\lambda \mu^2 \int_0^T \theta^2 \phi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_m n_j} ds \]  

(3.21)

Finally,

\[ I_7 + I_8 = \int_Q \sum_j \left( 2\lambda^2 \sum_{i,\ell,m} \left( a_{ij} a_{\ell m} \alpha_{x_i} \alpha_{x_m n_j} \psi_{x_j} + a_{ij} (a_{\ell m}) \alpha_{x_i} \alpha_{x_m n_j} \psi_{x_j} \right) \right) dx dt \]

\[ = 2\lambda^2 \mu^3 \int_0^T \theta^2 \phi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_m n_j} ds \]

\[ + 2\lambda^2 \mu^2 \int_0^T \theta^2 \phi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_m n_j} ds \]

\[ + 2\lambda^2 \mu \int_0^T \theta^2 \phi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_m n_j} ds. \]  

(3.22)

Now, combining (3.18)–(3.22) and using the technical condition (2.1), we conclude that, for large \( \lambda \) and \( \mu \), it holds that

\[ \int_Q \text{div} V dx dt \geq C\lambda^3 \mu^3 \int_0^T \theta^2 \phi c^2(t) dt - 4\lambda \mu \int_\Gamma |\nabla u|^2 \sum_{i,j} a_{ij} \psi_{x_i} n_j ds. \]  

(3.23)
**Step 5.** It remains to estimate \( \int_Q (\lambda \varphi)^{-1} \theta^2 (u_t^2 + \sum_{i,j} (a_{ij} u_{x_i} x_j)^2) \, dx \, dt \). For this, we observe that

\[
\left( v_t + \sum_{i,j} (a_{ij} v_{x_i} x_j) \right)^2 \leq C \left[ \lambda^2 \alpha^2 v^2 + \sum_{i,j} \left( \lambda^2 (a_{ij})_{x_j} x_i v^2 + \lambda^2 a^2_{ij} \alpha^2 v^2 \right) 
+ \lambda^4 a^2_{ij} \alpha^2 x_j \alpha^2 x_i v^2 + \lambda^2 a^2_{ij} \alpha^2 x_i v^2 + (a_{ij})_{x_j} x_i v^2 \right].
\]  

(3.24)

This implies

\[
(\lambda \varphi)^{-1} \left( v_t + \sum_{i,j} (a_{ij} v_{x_i} x_j) \right)^2 \leq C(\lambda \varphi)^{-1} \left[ \lambda^2 \alpha^2 v^2 + \sum_{i,j} \left( \lambda^2 (a_{ij})_{x_j} x_i v^2 + \lambda^2 a^2_{ij} \alpha^2 v^2 \right) 
+ \lambda^4 a^2_{ij} \alpha^2 x_j \alpha^2 x_i v^2 + \lambda^2 a^2_{ij} \alpha^2 x_i v^2 + (a_{ij})_{x_j} x_i v^2 \right].
\]  

(3.25)

Noting that

\[
2 \int_Q \sum_{i,j} (\lambda \varphi)^{-1} v_{ij} (a_{ij} v_{x_i} x_j) \, dx \, dt
= 2 \sum_{i,j} \left( \int_Q (a_{ij} (\lambda \varphi)^{-1} v_{x_i} x_j) \, dx \, dt - 2 \int_Q a_{ij} (\lambda \varphi)^{-1} v_{x_i} x_j v_t \, dx \, dt + \int_Q a_{ij} (\lambda \varphi)^{-1} v_{x_i} x_j v_t \, dx \, dt \right)
+ \int_Q (a_{ij})_{i} (\lambda \varphi)^{-1} v_{x_i} x_j \, dx \, dt + \int_Q a_{ij} ((\lambda \varphi)^{-1} v_{i} )_{x_j} \, dx \, dt, \]

(3.26)

we get

\[
2 \int_Q (\lambda \varphi)^{-1} \sum_{i,j} (a_{ij} v_{x_i} x_j) v_t \, dx \, dt \leq \frac{1}{2} \int_Q (\lambda \varphi)^{-1} v_t^2 \, dx \, dt + C\lambda \int_Q |\nabla v|^2 \, dx \, dt
+ \lambda \mu \int_0^T \theta \alpha c^2(t) \, dt \int_{\Gamma_0} \sum_{i,j} a_{ij} \psi_{x_i} n_j ds.
\]  

(3.27)

By (3.24)–(3.27), we obtain that

\[
\int_Q (\lambda \varphi)^{-1} \left[ v_t^2 + (v_{x_i} x_j)^2 \right] \, dx \, dt
\leq C \left( \int_Q (Lu)^2 \, dx \, dt + \lambda^3 \int_{Q_{T_0}} \mu^3 \varphi^3 v^2 \, dx \, dt + \lambda \int_{Q_{T_0}} \mu \varphi |\nabla v|^2 \, dx \, dt \right)
+ \lambda \mu \int_0^T \theta \alpha c^2(t) \, dt \int_{\Gamma_0} \sum_{i,j} a_{ij} \psi_{x_i} n_j ds.
\]  

(3.28)
This gives
\begin{align*}
\int_Q (\lambda \varphi)^{-1} [v_t^2 + (\Delta v)^2] \, dx \, dt \\
\leq C \left( \int_Q (Lu)^2 \theta^2 \, dx \, dt + \lambda^3 \int_{Q_T} \mu^3 \varphi^3 v^2 \, dx \, dt + \lambda \int_{Q_T^0} \mu \varphi | \nabla v|^2 \, dx \, dt \right) \\
+ \lambda \mu \int_0^T \theta a_{ij} c^2(t) \, dt \int_{t_0} a_{ij} \psi x_i n_j \, ds.
\end{align*}

(3.29)

It is easy to see that the last term of the above inequality can be absorbed by (3.23) for large $\lambda$ and $\mu$. Finally, replacing $v$ by $e^{\lambda u}$ and combining (2.2), (3.16), (3.17), (3.23), and (3.29), we get the desired estimate (3.2). This completes the proof of Theorem 3.1. □

Acknowledgments

This work is partially supported by FANEDD of China (Project No 200119), NCET of China under Grant NCET-04-0882, and the NSF of China under Grants 10371084 and 10525105. The author gratefully acknowledges Professor Xu Zhang for his guidance, encouragement, and suggestions.

References


Zhongqi Yin: Department of Engineering and Technology of Caotang School, Sichuan Normal University, Chengdu 610072, China
E-mail address: zhongqiyin@sohu.com
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