Research Letter

An Energy-Work Relationship Integration Scheme for Nonconservative Hamiltonian Systems

Fu Jingli,¹ Chen Benyong,² and Xie Fengping¹

¹ Institute of Mathematical Physics, Zhejiang Sci-Tech University, Hangzhou 310018, China

² College of Mechanics and Automatization Control, Zhejiang Sci-Tech University, Hangzhou 310018, China

Correspondence should be addressed to Fu Jingli, sqfujingli@163.com

Received 13 March 2008; Accepted 1 May 2008

Recommended by Eric G. Blackman

This letter focuses on studying a new energy-work relationship numerical integration scheme of nonconservative Hamiltonian systems. The signal-stage, multistage, and parallel composition numerical integration schemes are presented for this system. The high-order energy-work relation scheme of the system is constructed by a parallel connection of n multistage scheme of order 2 which its order of accuracy is $2n$. The connection, which is discrete analog of usual case, between the change of energy and work of nonconservative force is obtained for nonconservative Hamiltonian systems. This letter also shows that the more the stages of the schemes are, the less the error rate of the scheme is for nonconservative Hamiltonian systems. Finally, an applied example is discussed to illustrate these results.

Copyright © 2008 Fu Jingli et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. INTRODUCTION

Recently, there have been a great number of studies on the so-called geometric numerical integration scheme which preserve the structure of systems [1–3]. Leimkuhler and Reich pointed out that the geometric numerical integrators are time-stepping methods, designed such that they exactly satisfy conservation laws, symmetries or symplectic properties of a system of differential equations [1]. Hairer et al. presented the symplectic integration of separable Hamiltonian ordinary and partial differential equations. In this way, the symplectic scheme is performed prior to the spatial step as opposed to the standard approach of spatially discretizing the PDE to form a system of Hamiltonian ODEs to which a symplectic integrator can be applied [2]. An energy-conserving scheme is one of such geometric numerical integration scheme [4–8]. It is very known that a high-order scheme can be constructed by connecting low-order scheme in series (hereafter we will call it series composition) [1–3]. Now, the high-order energy-conserving scheme has been constructed with method [9]. In this letter, we will present a new numerical integration scheme, which is energy-work relation integration scheme, of nonconservative Hamiltonian systems. This work also study that a high-order energy-work relation scheme, which it has a structure connecting the order 2 multistage scheme in parallel scheme

(hereafter we will call it parallel composition scheme), can be constructed by connecting low-order scheme in series.

2. NUMERICAL INTEGRATION FOR NONCONSERVATIVE HAMILTONIAN SYSTEMS

Let the configuration of a mechanical system is described by n generalized coordinates q_s ($s = 1, \dots, n$) and n generalized momentums p_s ($s = 1, \dots, n$). Suppose the system is subjected to n nonpotential generalized forces Q'_s . The generalized Hamiltonian canonical equations of system as

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} + Q'_s, \quad s = 1, 2, \dots, n, \quad (1)$$

here, the Hamiltonian

$$H = H(q_1, \dots, q_n, p_1, \dots, p_n), \quad (2)$$

which represents the total energy. The relationship between the change of energy and the power of nonconservative force is easily verified as

$$\begin{aligned} \frac{dH}{dt} &= \sum_{s=1}^n \left(\frac{\partial H}{\partial q_s} \dot{q}_s + \frac{\partial H}{\partial p_s} \dot{p}_s \right) \\ &= \sum_{s=1}^n \left(\frac{\partial H}{\partial q_s} \frac{\partial H}{\partial p_s} - \frac{\partial H}{\partial p_s} \frac{\partial H}{\partial q_s} + \frac{\partial H}{\partial p_s} Q'_s \right) = \sum_{s=1}^n \frac{\partial H}{\partial p_s} Q'_s. \end{aligned} \quad (3)$$

The numerical integration is considered as the discretization of

$$\begin{aligned} q_s(t^{k+1}) &= q_s(t^k) + \int_{t^k}^{t^{k+1}} \frac{\partial H(q_1, \dots, q_n, p_1, \dots, p_n)}{\partial p_s} dt, \\ p_s(t^{k+1}) &= p_s(t^k) - \int_{t^k}^{t^{k+1}} \frac{\partial H(q_1, \dots, q_n, p_1, \dots, p_n)}{\partial q_s} dt + Q'_s \Delta t, \\ t^k &= k \Delta t, \quad k = 0, 1, 2, \dots, \end{aligned} \quad (4)$$

which are obtained by integrating both sides of (1) on the interval $[t^k, t^{k+1}]$, where Δt is the step size.

3. SECOND-ORDER SCHEMES OF NUMERICAL INTEGRATION FOR NONCONSERVATIVE HAMILTONIAN SYSTEMS

3.1. Single-stage scheme

Let p_s^k and q_s^k be the numerical approximations of $p_s(t^k)$ and $q_s(t^k)$, respectively. Then a 1-stage scheme is given by

$$q_s^{k+1} = q_s^k + I_{p_s}^{1,0}, \quad p_s^{k+1} = p_s^k - I_{q_s}^{1,0} + Q'_s \Delta t, \quad s = 1, 2, \dots, n, \quad (5)$$

with

$$\begin{aligned} I_{q_s}^{a,b} &= (a - b) \Delta t \delta_{q_s}^{a,b} \bar{\mu}_{q_s}^{a,b} H^k, \\ I_{p_s}^{a,b} &= (a - b) \Delta t \delta_{p_s}^{a,b} \bar{\mu}_{p_s}^{a,b} H^k, \\ H^k &= H(q_1^k, \dots, q_n^k, p_1^k, \dots, p_n^k). \end{aligned} \quad (6)$$

The notations $\delta_{p_s}^{a,b}$ and $\delta_{q_s}^{a,b}$ denote the partial difference quotient operators with respect to p_s and q_s , respectively, which are defined as

$$\begin{aligned} \delta_{q_s}^{a,b} F(q_1^{l_1}, \dots, q_n^{l_n}, p_1^{k_1}, \dots, p_n^{k_n}) &= \frac{(E_{q_s}^a - E_{q_s}^b) F}{(E_{q_s}^a - E_{q_s}^b) q_s^{l_s}}, \\ \delta_{p_s}^{a,b} F(q_1^{l_1}, \dots, q_n^{l_n}, p_1^{k_1}, \dots, p_n^{k_n}) &= \frac{(E_{p_s}^a - E_{p_s}^b) F}{(E_{p_s}^a - E_{p_s}^b) p_s^{k_s}}, \end{aligned} \quad (7)$$

where $E_{q_s}^a$ and $E_{p_s}^a$ are the shift operators defined as

$$\begin{aligned} E_{q_s}^a F(q_1^{l_1}, \dots, q_{s-1}^{l_{s-1}}, q_s^{l_s}, q_{s+1}^{l_{s+1}}, \dots, q_n^{l_n}, p_1^{k_1}, \dots, p_n^{k_n}) \\ = F(q_1^{l_1}, \dots, q_{s-1}^{l_{s-1}}, q_s^{l_s+a}, q_{s+1}^{l_{s+1}}, \dots, q_n^{l_n}, p_1^{k_1}, \dots, p_n^{k_n}), \\ E_{p_s}^a F(q_1^{l_1}, \dots, q_n^{l_n}, p_1^{k_1}, \dots, p_{s-1}^{k_{s-1}}, p_s^{k_s}, p_{s+1}^{k_{s+1}}, \dots, p_n^{k_n}) \\ = F(q_1^{l_1}, \dots, q_n^{l_n}, p_1^{k_1}, \dots, p_{s-1}^{k_{s-1}}, p_s^{k_s+a}, p_{s+1}^{k_{s+1}}, \dots, p_n^{k_n}). \end{aligned} \quad (8)$$

The notations $\bar{\mu}_{q_s}^{a,b}$ and $\bar{\mu}_{p_s}^{a,b}$ denote the mean difference operators with respect to all variables except for q_s and p_s , respectively, which are defined as

$$\begin{aligned} \bar{\mu}_{q_s}^{a,b} F(q_1^{l_1}, \dots, q_n^{l_n}, p_1^{k_1}, \dots, p_n^{k_n}) \\ = M^{a,b}(E_{q_1}, \dots, E_{q_{s-1}}, E_{q_{s+1}}, \dots, E_{q_n}, E_{p_1}, \dots, E_{p_n}), \\ \bar{\mu}_{p_s}^{a,b} F(q_1^{l_1}, \dots, q_n^{l_n}, p_1^{k_1}, \dots, p_n^{k_n}) \\ = M^{a,b}(E_{q_1}, \dots, E_{q_n}, E_{p_1}, \dots, E_{p_{s-1}}, E_{p_{s+1}}, \dots, E_{p_n}), \end{aligned} \quad (9)$$

with

$$M^{a,b}(x_1, x_2, \dots, x_{r-1}) = \frac{1}{r!} \sum_{l=1}^r \text{per} \left(\begin{array}{cccc} x_1^a & x_2^a & \cdots & x_{r-1}^a \\ \vdots & \vdots & & \vdots \\ x_1^l & x_2^l & \cdots & x_{r-1}^l \\ \vdots & \vdots & & \vdots \\ x_1^b & x_2^b & \cdots & x_{r-1}^b \end{array} \right) \left. \begin{array}{l} \left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} r-1 \\ \left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} l-1, \end{array} \right\} \quad (10)$$

where $\text{per}(A)$ denotes the permanent or plus determinant of a matrix A [10]. For example, in the case $d = 1$, we have

$$\begin{aligned} \bar{\mu}_{q_1}^{a,b} &= M^{a,b}(E_{q_1}) = \frac{1}{2} (E_{q_1}^a + E_{q_1}^b), \\ \bar{\mu}_{p_1}^{a,b} &= M^{a,b}(E_{p_1}) = \frac{1}{2} (E_{p_1}^a + E_{p_1}^b). \end{aligned} \quad (11)$$

The operators $\delta_{q_s}^{a,b}$, $\delta_{p_s}^{a,b}$, $\bar{\mu}_{q_s}^{a,b}$, and $\bar{\mu}_{p_s}^{a,b}$ have symmetry expressed as

$$\begin{aligned} \delta_{q_s}^{a,b} &= \delta_{q_s}^{b,a}, & \delta_{p_s}^{a,b} &= \delta_{p_s}^{b,a}, \\ \bar{\mu}_{q_s}^{a,b} &= \bar{\mu}_{q_s}^{b,a}, & \bar{\mu}_{p_s}^{a,b} &= \bar{\mu}_{p_s}^{b,a}. \end{aligned} \quad (12)$$

3.2. Relation between the energy and work of nonconservative force for nonconservative Hamiltonian systems

Proposition 1. *The relation between the energy and work of nonconservative force for nonconservative Hamiltonian system holds:*

$$\begin{aligned} H^{k+a} - H^{k+b} \\ = \sum_{s=1}^n [(\delta_{q_s}^{a,b} \bar{\mu}_{q_s}^{a,b} H^k)(q_s^{k+a} - q_s^{k+b}) + (\delta_{p_s}^{a,b} \bar{\mu}_{p_s}^{a,b} H^k)(p_s^{k+a} - p_s^{k+b})]. \end{aligned} \quad (13)$$

Proof. For simplicity, we set

$$\bar{E}^a = (E_{q_1}^a, \dots, E_{q_n}^a, E_{p_1}^a, \dots, E_{p_n}^a)^T. \quad (14)$$

We first note the identity

$$\prod_s^n E_{q_s}^a E_{p_s}^a = \frac{1}{(2n)!} \text{per} \left(\begin{array}{c} \bar{E}^a, \dots, \bar{E}^a \\ 2n \end{array} \right). \quad (15)$$

It follows that

$$\begin{aligned} H^{k+a} - H^{k+b} \\ = \left(\prod_{s=1}^n E_{q_s}^a E_{p_s}^a - \prod_{s=1}^n E_{q_s}^b E_{p_s}^b \right) H^k \\ = \sum_{s=1}^n [(E_{q_s}^a - E_{q_s}^b) M^{a,b}(E_{q_1}, \dots, E_{q_n}, E_{p_1}, \dots, E_{p_{s-1}}, E_{p_{s+1}}, \dots, E_{p_n}) \\ + (E_{p_s}^a - E_{p_s}^b) M^{a,b}(E_{q_1}, \dots, E_{q_{s-1}}, E_{q_{s+1}}, \dots, E_{q_n}, E_{p_1}, \dots, E_{p_n})] H^k \\ = \sum_{s=1}^n [(\delta_{q_s}^{a,b} \bar{\mu}_{q_s}^{a,b} H^k)(q_s^{k+a} - q_s^{k+b}) + (\delta_{p_s}^{a,b} \bar{\mu}_{p_s}^{a,b} H^k)(p_s^{k+a} - p_s^{k+b})], \end{aligned} \quad (16)$$

where we have used the properties of the permanent and the definitions of operators (7)–(10) [10]. $\square$

Proposition 2. *The scheme $\{(1), \dots, (6)\}$ satisfies the relation between the change of energy and the work of nonconservative force for the system.*

Proof. We see from the chain rule (13) that the change of energy H^k is equivalent to the work of nonconservative force:

$$\begin{aligned}
 H^{k+1} - H^k &= \sum_{s=1}^n [(\delta_{q_s}^{1,0} \bar{\mu}_{q_s}^{1,0} H^k)(q_s^{k+1} - q_s^k) + (\delta_{p_s}^{1,0} \bar{\mu}_{p_s}^{1,0} H^k)(p_s^{k+1} - p_s^k)] \\
 &= \Delta t \sum_{s=1}^n [(\delta_{q_s}^{1,0} \bar{\mu}_{q_s}^{1,0} H^k)(\delta_{p_s}^{1,0} \bar{\mu}_{p_s}^{1,0} H^k) \\
 &\quad + (\delta_{p_s}^{1,0} \bar{\mu}_{p_s}^{1,0} H^k)(-\delta_{q_s}^{1,0} \bar{\mu}_{q_s}^{1,0} H^k + Q'_s)] \\
 &= \sum_{s=1}^n \Delta t (\delta_{p_s}^{1,0} \bar{\mu}_{p_s}^{1,0} H^k) Q'_s,
 \end{aligned} \tag{17}$$

which is a discrete analog of (3). $\square$

3.3. Order of accuracy

The local errors involved in the determination of $\{p_s^{k+1}, q_s^{k+1}\}_{s=1}^n$ from $\{p_s^k, q_s^k\}_{s=1}^n$ are $O(\Delta t^3)$, that is, $I_{q_s}^{1,0}$ and $I_{p_s}^{1,0}$ in the scheme (5) are the second-order approximations of the integrals in (4), respectively. Although this can be proved by the Taylor expansions, it is obvious because the scheme is symmetric (see Section 4.3).

3.4. Multistage scheme

An c -stage scheme is constructed by connecting the second-order scheme with small integration interval of length $\Delta t/c$ in series:

$$\begin{aligned}
 p_s^{k+m/c} &= p_s^{k+(m-1)/c} - I_{Q_s}^{m/c, (m-1)/c} + Q'_s(Q_1^k, \dots, Q_n^k, p_1^k, \dots, p_n^k) \frac{\Delta t}{c}, \\
 Q_s^{k+m/c} &= Q_s^{k+(m-1)/c} + I_{P_s}^{m/c, (m-1)/c} \\
 p_s^{k+1} p_s^{k+1}, \quad p_s^k &= p_s^k, \quad Q_s^{k+1} = q_s^{k+1}, \quad Q_s^k = q_s^k, \\
 s &= 1, 2, \dots, n, \quad m = 1, 2, \dots, c,
 \end{aligned} \tag{18}$$

with

$$\begin{aligned}
 I_{Q_s}^{a,b} &= (a-b) \Delta t \delta_{Q_s}^{a,b} \bar{\mu}_{Q_s}^{a,b} H(Q_1^k, \dots, Q_n^k, p_1^k, \dots, p_n^k), \\
 I_{P_s}^{a,b} &= (a-b) \Delta t \delta_{P_s}^{a,b} \bar{\mu}_{P_s}^{a,b} H(Q_1^k, \dots, Q_n^k, p_1^k, \dots, p_n^k),
 \end{aligned} \tag{19}$$

where $p_s^{k+m/c}$ and $Q_s^{k+m/c}$ are the internal stagevariables. It

should be noted that the above scheme is equivalent to the scheme:

$$\begin{aligned}
 p_s^{k+1} &= p_s^k - \sum_{l=1}^c I_{Q_s}^{l/c, (l-1)/c} + Q'_s(q_1^k, \dots, q_n^k, p_1^k, \dots, p_n^k) \frac{\Delta t}{c}, \\
 q_s^{k+1} &= q_s^k - \sum_{l=1}^c I_{P_s}^{l/c, (l-1)/c}, \\
 p_s^{k+m/c} &= \frac{c-m}{c} \left(p_s^k - \sum_{l=1}^m I_{Q_s}^{l/c, (l-1)/c} + Q'_s \frac{\Delta t}{c} \right) \\
 &\quad + \frac{m}{c} \left(p_s^{k+1} + \sum_{l=m+1}^c I_{Q_s}^{l/c, (l-1)/c} + Q'_s \frac{\Delta t}{c} \right), \\
 Q_s^{k+m/c} &= \frac{c-m}{c} \left(q_s^k + \sum_{l=1}^m I_{P_s}^{l/c, (l-1)/c} \right) + \frac{m}{c} \left(q_s^{k+1} - \sum_{l=m+1}^c I_{P_s}^{l/c, (l-1)/c} \right), \\
 s &= 1, 2, \dots, n, \quad m = 1, 2, \dots, c-1.
 \end{aligned} \tag{20}$$

The latter scheme (20) will be used in the next section to construct a higher-order scheme.

It is obvious for the c -stage that the relationship between the change of energy and the work of nonconservative force is exactly equivalent and that the order of accuracy is 2. We point out here that the local error is expressed as $c \times O[(\Delta t/c)^3] = c^{-2} O(\Delta t)^3$.

4. HIGHER-ORDER SCHEMES OF NUMERICAL INTEGRATION FOR NONCONSERVATIVE HAMILTONIAN SYSTEMS

4.1. Parallel composition scheme

Let $c_1, c_2, \dots, c_n$ be arbitrary positive integers satisfying

$$c_1 < c_2 < \dots < c_n, \tag{21}$$

then a new scheme is constructed by connection c_1 -stage, c_2 -stage, $\dots$, c_n -stage schemes of order 2 in parallel:

$$\begin{aligned}
 p_s^{k+1} &= p_s^k - \sum_{j=1}^u d_j \sum_{l=1}^{c_j} I_{Q_{sj}}^{l/c_j, (l-1)/c_j} + Q'_s(q_1^k, \dots, q_n^k, p_1^k, \dots, p_n^k) \Delta t, \\
 q_s^{k+1} &= q_s^k + \sum_{j=1}^u d_j \sum_{l=1}^{c_j} I_{P_{sj}}^{l/c_j, (l-1)/c_j}, \\
 P_{sj}^{k+m/c_j} &= \frac{c_j-m}{c_j} \left(p_s^k - \sum_{l=1}^m I_{Q_{sj}}^{l/c_j, (l-1)/c_j} + Q'_{sj} \Delta t \right) \\
 &\quad + \frac{m}{c_j} \left(p_s^{k+1} - \sum_{l=m+1}^m I_{Q_{sj}}^{l/c_j, (l-1)/c_j} + Q'_{sj} \Delta t \right), \\
 Q_{sj}^{k+m/c_j} &= \frac{c_j-m}{c_j} \left(q_s^k + \sum_{l=1}^m I_{P_{sj}}^{l/c_j, (l-1)/c_j} \right) \\
 &\quad + \frac{m}{c_j} \left(q_s^{k+1} + \sum_{l=m+1}^m I_{P_{sj}}^{l/c_j, (l-1)/c_j} \right), \\
 p_{sj}^{k+1} &= p_s^{k+1}, \quad p_{sj}^k = p_s^k, \quad Q_{sj}^{k+1} = q_s^{k+1}, \quad Q_{sj}^k = q_s^k, \\
 s &= 1, 2, \dots, n, \quad j = 1, 2, \dots, u, \quad m = 1, 2, \dots, c_j - 1,
 \end{aligned} \tag{22}$$

with the weights

$$d_j = \begin{cases} 1 & \text{for } u = 1, \\ \frac{c_j^{2u-2}}{\prod_{l=1, l \neq j}^u (c_j^2 - c_l^2)} & \text{for } u \geq 2, \quad j = 1, 2, \dots, u, \end{cases} \quad (23)$$

where

$$\begin{aligned} I_{p_{sj}}^{a,b} &= (a-b)\Delta t \delta_{p_{sj}}^{a,b} \bar{\mu}_{p_{sj}}^{a,b} H_j^k, \\ I_{Q_{sj}}^{a,b} &= (a-b)\Delta t \delta_{Q_{sj}}^{a,b} \bar{\mu}_{Q_{sj}}^{a,b} H_j^k, \\ H_j^k &= H(Q_{1j}^k, \dots, Q_{nj}^k, P_{1j}^k, \dots, P_{nj}^k), \\ \Lambda_{sj}^k &= \Lambda_s(Q_{1j}^k, \dots, Q_{nj}^k, P_{1j}^k, \dots, P_{nj}^k). \end{aligned} \quad (24)$$

4.2. Relation between the change of energy and work of nonconservative force

Proposition 3. The scheme (23) with the condition

$$\sum_{j=1}^u d_j = 1 \quad (25)$$

satisfies relationship between the change of energy and work of nonconservative force for nonconservative Hamiltonian systems

Proof. We first note

$$H^{k+1} = H_j^{k+1}, \quad H^k = H_j^k, \quad j = 1, 2, \dots, u. \quad (26)$$

We see from Proposition 1 that

$$\begin{aligned} H_j^{k+a} - H_j^{k+b} &= \sum_{s=1}^n (\delta_{p_{sj}}^{a,b} \bar{\mu}_{p_{sj}}^{a,b} H_j^k) (P_{sj}^{k+a} - P_{sj}^{k+b}) + (\delta_{Q_{sj}}^{a,b} \bar{\mu}_{Q_{sj}}^{a,b} H_j^k) (Q_{sj}^{k+a} - Q_{sj}^{k+b}). \end{aligned} \quad (27)$$

It follows from (25)–(27) that

$$\begin{aligned} H^{k+1} - H^k &= \sum_{j=1}^u d_j (H_j^{k+1} - H_j^k) \\ &= \sum_{j=1}^u d_j \sum_{m=1}^{c_j} (H_j^{k+m/c_j} - H_j^{k+(m-1)/c_j}) \\ &= \sum_{j=1}^u \frac{d_j c_j}{\Delta t} \sum_{m=1}^{c_j} \sum_{s=1}^n [I_{p_{sj}}^{k+m/c_j, k+(m-1)/c_j} \times (P_{sj}^{k+m/c_j} - P_{sj}^{k+(m-1)/c_j}) \\ &\quad + I_{Q_{sj}}^{k+m/c_j, k+(m-1)/c_j} \times (Q_{sj}^{k+m/c_j} - Q_{sj}^{k+(m-1)/c_j})]. \end{aligned} \quad (28)$$

We obtain from (22)

$$\begin{aligned} P_{sj}^{k+m/c_j} - P_{sj}^{k+(m-1)/c_j} &= \frac{1}{c_j} \left(\sum_{r=1}^u c_r \sum_{l=1}^{c_r} I_{p_{sr}}^{l/c_r, (l-1)/c_r} - \sum_{l=1}^{c_j} I_{p_{sj}}^{l/c_j, (l-1)/c_j} \right) + I_{p_{sj}}^{m/c_j, (m-1)/c_j} \end{aligned} \quad (29)$$

Substituting (29) into (28) yields

$$H^{k+1} - H^k = \sum_{j=1}^u d_j \sum_{m=1}^{c_j} \sum_{s=1}^n I_{p_{sj}}^{m/c_j, (m-1)/c_j} Q_{sj}^{'k}, \quad (30)$$

which is a discrete analog of that relation between the change of energy and work of nonconservative force for the systems (17). $\square$

4.3. Order of a symmetric scheme

Proposition 4. Consider the scheme (22) as mapping

$$\phi_{\Delta t} : (q_1^k, \dots, q_n^k, p_1^k, \dots, p_n^k) = (q_1^{k+1}, \dots, q_n^{k+1}, p_1^{k+1}, \dots, p_n^{k+1}), \quad (31)$$

and let $\phi_{\Delta t}^{-1}$ be the inverse mapping of $\phi_{\Delta t}$. Then, one has

$$\phi_{-\Delta t}^{-1} = \phi_{\Delta t}. \quad (32)$$

That is, the scheme is symmetric

Proof. The inverse $\phi_{\Delta t}^{-1}$ is obtained by exchanging (p_s^k, q_s^k) and (p_s^{k+1}, q_s^{k+1}) . Replacing Δt by $-\Delta t$ and rearranging terms in $\phi_{\Delta t}^{-1}$ leads to the mapping $\phi_{-\Delta t}^{-1}$. For this $\phi_{-\Delta t}^{-1}$, setting

$$\bar{P}_{sj}^{k+m/c_j} = P_{sj}^{k+1-m/c_j}, \quad (33)$$

$$s = 1, 2, \dots, n, \quad j = 1, 2, \dots, u, \quad m = 0, 1, \dots, c_j,$$

and omitting the tilde, we can obtain $\phi_{\Delta t}$. Therefore, form (31) holds. $\square$

Proposition 5. If one chooses the weights $d_1, d_2, \dots, d_n$ as (23), the accuracy of the scheme (22) is at least of order $2n$.

Proof. It is known that if a one-step scheme is symmetric, its order of accuracy is even [1, 2]. Therefore, the local error of the scheme $\phi_{\Delta t}$ is $O(\Delta t^{2r+1})$ with a positive integer r . We first choose $\{d_j\}_{j=1}^u$ such that

$$\sum_{j=1}^u d_j = 1. \quad (34)$$

Since the error of $I_{p_{sj}}^{m/c_j, (m-1)/c_j}$, $I_{Q_{sj}}^{m/c_j, (m-1)/c_j}$, and $Q_{sj}^{'k}$ are $O[(\Delta t/c_j)^3]$, the error of $\phi_{\Delta t}$ is expressed as

$$\sum_{j=1}^u d_j \times c_j \times O\left[\left(\frac{\Delta t}{c_j}\right)^3\right] = \sum_{j=1}^u c_j^{-2} d_j O(\Delta t^3). \quad (35)$$

If one chooses $\{d_j\}_{j=1}^u$ such that

$$\sum_{j=1}^n c_j^{-2} d_j = 0, \quad (36)$$

then the $O(\Delta t^3)$ -term in the error of $\phi_{\Delta t}$ vanishes. Since the error of $\phi_{\Delta t}$ is of odd order, it becomes $O(\Delta t^5)$. The $O(\Delta t^5)$ -term in the error $\{d_j\}_{j=1}^u$ -term in the error of $\phi_{\Delta t}$ is expressed as

$$\sum_{j=1}^u d_j \times c_j \times O\left[\left(\frac{\Delta t}{c_j}\right)^5\right] = \sum_{j=1}^u c_j^{-4} d_j O(\Delta t^5). \quad (37)$$

These procedures can be repeated. The final condition for $\{d_j\}_{j=1}^u$ is

$$\sum_{j=1}^u c_j^{-2(u-1)} d_j = 0. \quad (38)$$

Therefore, if one chooses $\{d_j\}_{j=1}^u$ such that they satisfy the n simultaneous linear equations:

$$\sum_{j=1}^u c_j^{-2l} d_j = \begin{cases} 1 & \text{for } l = 0, \\ 0 & \text{for } l = 1, 2, \dots, u-1, \end{cases} \quad (39)$$

then the error of $\phi_{\Delta t}$ is $O(\Delta t^{2u+1})$. Since the solution of (39) is given by (23), the order of accuracy is $2u$. $\square$

5. A NUMERICAL EXAMPLE

Consider the motion of a particle with unit mass whose Hamiltonian is

$$H(q, p) = \frac{1}{2} p^2, \quad (40)$$

and the motion of the system is subjected to nonpotential force

$$Q'_s = \cos bt, \quad (41)$$

where b is a constant.

The equation of motion of nonconservative particle is

$$\frac{dp}{dt} = \cos bt, \quad \frac{dq}{dt} = p. \quad (42)$$

The analytic solution of (42) is given by

$$p = \frac{A}{b} \sin bt, \quad q = -\frac{A}{b^2} \cos(bt + \varphi), \quad (43)$$

which have the period $T = 2\pi/b$. We take the initial conditions:

$$p\left(\frac{\pi}{2b}\right) = 1, \quad q\left(\frac{\pi}{2}\right) = 1, \quad (44)$$

and the calculation time $t = T$. The parallel composition scheme with

$$c_j = j, \quad j = 1, 2, \dots, u \quad (45)$$

was used. We calculated the global error given by

$$e(t) = \sqrt{(p^K - p(T))^2 + (q^K - q(T))^2}, \quad (46)$$

where $K = T/\Delta t$. Since the global error $e(t)$ is about $T/\Delta t$ times the local error, $e(t)$ is expressed as $e(T) = O(\Delta t^{2n})$.

We should point out that the local error of the parallel composition is expressed as

$$\sum_{j=1}^u c_j^{-2u} d_j O(\Delta t^{2n+1}) = \frac{1}{c_1^2 c_2^2 \cdots c_n^2} O(\Delta t^{2n+1}), \quad (47)$$

the more the stages of the schemes are, the smaller the error of the scheme for nonconservative Hamiltonian systems.

6. CONCLUSION

In this paper, the new numerical integration schemes of nonconservative Hamilton systems are established. This study has given that the numerical connection between energy of system and work of nonconservative force is an analog of usual energy-work connection, and the numerical connection between the high-order energy-work is also contented. Numerical results showed that the more the stages of the schemes are, the smaller the error of the scheme for nonconservative Hamiltonian systems.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China(10672143; 60575055) and the Natural Science Foundation of Henan Province, China (Grant no. 0511022200)

REFERENCES

- [1] B. Leimkuhler and S. Reich, *Simulating Hamiltonian Dynamics*, Cambridge University Press, Cambridge, UK, 2004.
- [2] E. Hairer, C. Lubich, and G. Wanner, *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*, Springer, Berlin, Germany, 2006.
- [3] J. M. Sanz-Serna and M. P. Calvo, *Numerical Hamiltonian Problems*, Chapman & Hall, London, UK, 1994.
- [4] D. Greenspan, "Conservative numerical methods for $\ddot{x} = f(x)$," *Journal of Computational Physics*, vol. 56, no. 1, pp. 28–41, 1984.
- [5] D. Greenspan, *N-Body Problems and Models*, World Scientific, Singapore, 2004.
- [6] T. Itoh and K. Abe, "Hamiltonian-conserving discrete canonical equations based on variational difference quotients," *Journal of Computational Physics*, vol. 76, no. 1, pp. 85–102, 1988.
- [7] C. W. Gear, "Invariants and numerical methods for ODEs," *Physica D*, vol. 60, no. 1–4, pp. 303–310, 1992.
- [8] T. Matsuo, "High-order schemes for conservative or dissipative systems," *Journal of Computational and Applied Mathematics*, vol. 152, no. 1–2, pp. 305–317, 2003.
- [9] I. Yuji, "A high-order energy-conserving integration scheme for Hamiltonian systems," *Physics Letters A*, vol. 372, no. 10, pp. 1562–1573, 2008.
- [10] H. Minc, *Permanents*, Addison-Wesley, Reading, Mass, USA, 1978.