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Research Article

Spectral Approximation of Infinite-Dimensional Black-Scholes Equations with Memory

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This paper considers the pricing of a European option using a (B, S)-market in which the stock price and the asset in the riskless bank account both have hereditary price structures described by the authors of this paper (1999). Under the smoothness assumption of the payoff function, it is shown that the infinite dimensional Black-Scholes equation possesses a unique classical solution. A spectral approximation scheme is developed using the Fourier series expansion in the space $C[-h, 0]$ for the Black-Scholes equation. It is also shown that the nth approximant resembles the classical Black-Scholes equation in finite dimensions.

1. Introduction

The pricing of contingent claims in the continuous-time financial market that consists of a bank account and a stock account has been a subject of extensive research for the last decades. In the literature (e.g., [1–5]), the equations that describe the bank account and the price of the stock are typically written, respectively, as

$$\begin{aligned} dB(t) &= rB(t)dt, \quad B(0) = x, \\ dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t), \quad S(0) = y, \end{aligned} \tag{1.1}$$

where $W = \{W(t),\ t \geq 0\}$ is a one-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, F, \mathbf{P}; \{F(t),\ t \geq 0\})$ and r, α, and σ are positive constants that represent, respectively, the interest rate of the bank account, the stock appreciation rate, and the stock volatility rate. The financial market that consists of one bank

account and one stock account will be referred to as a (B, S)-market, where B stands for the bank account and S stands for the stock.

A European option contract is a contract giving the buyer of the contract the right to buy (sell) a share of a particular stock at a predetermined price at a predetermined time in the future. The European option problem is, briefly, to determine the fee (called the *rational price*) that the writer of the contract should receive from the buyer for the rights of the contract and also to determine the trading strategy the writer should use to invest this fee in the (B, S)-market in such a way as to ensure that the writer will be able to cover the option if it is exercised. The fee should be large enough that the writer can, with riskless investing, cover the option, but be small enough that the writer does not make an unfair (i.e., riskless) profit.

In [6], we noted reasons to include hereditary price structures to a (B, S)-market model and then introduced such a model using a functional differential equation to describe the dynamics of the bank account and a stochastic functional differential equation to describe those of the stock account. The paper then obtained a solution to the option pricing problem in terms of conditional expectation with respect to a martingale measure. The importance of including hereditary price structure in the stock price dynamics was also recognized by other researchers in recent years (see, e.g., [7–14]).

In particular, [6] was one of the firsts that took into consideration hereditary structure in studying the pricing problem of European option. There the authors obtained a solution to the option pricing problem in terms of conditional expectation with respect to a martingale measure. The two papers [7, 9] developed an explicit formula for pricing European options when the underlying stock price follows a nonlinear stochastic delay equation with fixed delays (resp., variable delays) in the drift and diffusion terms. The paper [8] computed the logarithmic utility of an insider when the financial market is modelled by a stochastic delay equation. There the author showed that, although the market does not allow free lunches and is complete, the insider can draw more from his wealth than the regular trader. The paper also offered an alternative to the anticipating delayed Black-Scholes formula, by proving stability of European call option proces when the delay coefficients approach the nondelayed ones. The paper [10] derived the infinite-dimensional Black-Scholes equation for the (B, S)-market, where the bank account evolves according to a linear (deterministic) functional differential equation and the stock dynamics is described by a very general nonlinear stochastic functional differential equation. A power series solution is also developed for the equation. Following the same model studied in [10], the work in [11] shows that under very mild conditions the pricing function is the unique viscosity solution of the infinite-dimensional Black-Scholes equation. A finite difference approximation scheme for the solution of the equation is developed and convergence result is also obtained. We mention here that option pricing problems were also considered by [12–14] for a financial market that is more restricted than those of [10, 11].

This paper considers the pricing of a European option using a (B, S)-market, such as those in [6], in which the stock price and the asset in the riskless bank account both have hereditary price structures. Under the smoothness assumption of the payoff function, it is shown that the pricing function is the unique classical solution of the infinite-dimensional Black-Scholes equation. A spectral approximation scheme is developed using the Fourier series expansion in the space $C[-h, 0]$ for the Black-Scholes equation. It is also shown that the nth approximant resembles the celebrated classical Black-Scholes equation in finite dimensions (see, e.g., [4, 5]).

This paper is organized as follows. Section 2 summarizes the definitions and key results of [6] that will be used throughout this paper. The concepts of Fréchet derivative

and extended Fréchet derivative are introduced in Section 3, along with results needed to make use of these derivatives. In Section 4, the results regarding the infinite-dimensional Black-Scholes equation and its corollary are restated from [6, 10]. Section 5 details the spectral approximate solution scheme for this equation. Section 6 is the paper's conclusion, followed by an appendix with the proof of Proposition 3.2.

2. The European Option Problem with Hereditary Price Structures

To describe the financial model with hereditary price structures, we start by defining our probability space. Let $0 < h < \infty$ be a fixed constant. This constant will be the length of the time window in which the hereditary information is contained. If $a, b \in \Re$ with $a < b$, denote the space of continuous functions $\phi : [a, b] \to \Re$ by $C[a, b]$. Define

$$C_+[a,b] = \{\phi \in C[a,b] \mid \phi(\theta) \geq 0 \ \forall \theta \in [a,b]\}. \tag{2.1}$$

Note that $C[a, b]$ is a real separable Banach space equipped with the uniform topology defined by the sup-norm $\|\phi\| = \sup_{t\in[a,b]}|\phi(t)|$ and $C_+[a, b]$ is a closed subset of $C[a, b]$. Throughout the end of this paper, we let $\mathbf{C} = C[-h, 0]$ and

$$\mathbf{C}_+ = \{\phi \in \mathbf{C} \mid \phi(\theta) \geq 0 \ \forall \theta \in [-h,0]\} \tag{2.2}$$

for simplicity. If $\psi \in C[-h, \infty)$ and $t \in [0, \infty)$, let $\psi_t \in \mathbf{C}$ be defined by $\psi_t(\theta) = \psi(t + \theta)$, $\theta \in [-h, 0]$.

Let $\Omega = C[-h, \infty)$, the space of continuous functions $\omega : [-h, \infty) \to \Re$, and let $F = B(C[-h, \infty))$, the Borel σ-algebra of subsets of $C[-h, \infty)$ under the topology defined by the metric $d : \Omega \times \Omega \to \Re$, where

$$d(\omega,\omega') = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{-h\leq t\leq n}|\omega(t) - \omega'(t)|}{1 + \sup_{-h\leq t\leq n}|\omega(t) - \omega'(t)|}. \tag{2.3}$$

Let $\mathbf{P}$ be the Wiener measure defined on (Ω, F) with

$$\mathbf{P}\{\omega \in \Omega \mid \omega(\theta) = 0 \ \forall \theta \in [-h,0]\} = 1. \tag{2.4}$$

Note that the probability space $(\Omega, F, \mathbf{P})$ is the canonical Wiener space under which the coordinate maps $W = \{W(t),\ t \geq 0\}$, $W(t) : C[-h, \infty) \to \Re$, defined by $W(t)(\omega) = \omega(t)$ for all $t \geq -h$ and $\omega \in \Omega$ is a standard Brownian motion and $\mathbf{P}\{W_0 = 0\} = 1$. Let the filtration $F^W = \{F(t),\ t \geq -h\}$ be the $\mathbf{P}$-augmentation of the natural filtration of the Brownian motion W, defined by $F(t) = \{\emptyset, \Omega\}$ for all $t \in [-h, 0]$ and

$$F(t) = \sigma(W(s),\ 0 \leq s \leq t), \quad t \geq 0. \tag{2.5}$$

Equivalently, $F(t)$ is the smallest sub-σ-algebra of subsets of Ω with respect to which the mappings $W(s) : \Omega \to \Re$ are measurable for all $0 \leq s \leq t$. It is clear that the filtration F^W defined above is right continuous in the sense of [15].

Consider the $\mathbf{C}$-valued process $\{W_t,\ t \geq 0\}$, where $W_0 = 0$ and $W_t(\theta) = W(t+\theta)$, $\theta \in [-h, 0]$ for all $t \geq 0$. That is, for each $t \geq 0$, $W_t(\omega) = \omega_t$ and $W_0 = 0$. In [10], it is shown that $F_0 = F(t)$ for $t \in [-h, 0]$ and $F(t) = F_t$ for $t \geq 0$, where

$$F_t = \sigma(W_s,\ 0 \leq s \leq t), \quad t \geq 0. \tag{2.6}$$

The new model for the (B, S)-market introduced in [6] has a hereditary price structure in the sense that the rate of change of the unit price of the investor's assets in the bank account B and that of the stock account S depend not only on the current unit price but also on their historical prices. Specifically, we assume that B and S evolve according to the following two linear functional differential equations:

$$dB(t) = L(B_t)dt, \quad t \geq 0, \tag{2.7}$$

$$dS(t) = M(S_t)dt + N(S_t)dW(t), \quad t \geq 0, \tag{2.8}$$

with initial price functions $B_0 = \phi$ and $S_0 = \psi$, where ϕ and ψ are given functions in $\mathbf{C}_+$. In the model, L, M, and N are bounded linear functionals on the real Banach space $\mathbf{C}$. The bounded linear functionals $L, M, N : \mathbf{C} \to \mathfrak{R}$ can be represented as (see [6])

$$\begin{aligned} L(\phi) &= \int_{-h}^{0} \phi(\theta)d\eta(\theta), \\ M(\phi) &= \int_{-h}^{0} \phi(\theta)d\xi(\theta), \end{aligned} \tag{2.9}$$

$$N(\phi) = \int_{-h}^{0} \phi(\theta)d\zeta(\theta), \quad \phi \in \mathbf{C}, \tag{2.10}$$

where the above integrals are to be interpreted as Lebesgue-Stieltjes integrals and η, ξ, and ζ are functions that are assumed to satisfy the following conditions.

Assumption 2.1. The functions $\eta, \xi : [-h, 0] \to \mathfrak{R}$, are nondecreasing functions on $[-h, 0]$ such that $\eta(0) - \eta(-h) > 0$ and $\xi(0) - \xi(-h) > 0$, and the function $\zeta : [-h, 0] \to \mathfrak{R}$ is a function of bounded variation on $[-h, 0]$ such that $\int_{-h}^{0} \phi(\theta)d\zeta(\theta) \geq \sigma > 0$ for every $\phi \in \mathbf{C}_+$.

We will, throughout the end, extend the domain of the above three functions to R by defining $\eta(\theta) = \eta(-h)$ for $\theta \leq -h$ and $\eta(\theta) = \eta(0)$ for $\theta \geq 0$, and so forth.

Proposition 2.3 in [6] provides an existence and uniqueness result under mild conditions, so the model makes sense mathematically to use. Note that the equations described by (2.7)-(2.8) include (1.1) as a special case. Therefore, the model considered in this paper is a generalization of that considered in most of the existing literature (see, e.g., [5]).

For the purpose of analyzing the discount rate for the bank account, let us assume that the solution process $B(L;\phi) = \{B(t),\ -h \le t < \infty\}$ of (2.7) with the initial function $\phi \in \mathbf{C}_+$ takes the following form:

$$B(t) = \phi(0)e^{rt}, \quad t \ge 0, \tag{2.11}$$

and $B_0 = \phi \in \mathbf{C}_+$. Then the constant r satisfies the following equation:

$$r = \int_{-h}^{0} e^{r\theta} d\eta(\theta). \tag{2.12}$$

The existence and uniqueness of a positive number r that satisfies the above equation is shown in [6].

Throughout the end, we will fix the initial unit price functions ϕ, and $\psi \in \mathbf{C}_+$, and the functional $N : \mathbf{C} \to \Re$ for the stock price described in (2.8) and (2.10). For the purpose of making the distinction when we interchange the usage of $M : \mathbf{C} \to \Re$ and $L : \mathbf{C} \to \Re$ in (2.8), we write the stock price process $S(M,N;\psi)$ as $S(M) = \{S(t),\ t \ge -h\}$ for simplicity. And, when the functional $K : \mathbf{C} \to \Re$, $K(\phi_t) = r\phi(t)$ is used in place of $M : \mathbf{C} \to \Re$ in (2.8), its solution process will be written as $S(K) = \{S(t),\ t \ge -h\}$.

In [6], the basic theory of European option pricing using the (B,S)-market model described in (2.7)-(2.8) is developed. We summarize the key definitions and results below.

A *trading strategy* in the (B,S)-market is a progressively measurable vector process $\pi = \{(\pi_1(t), \pi_2(t)),\ 0 \le t < \infty\}$ defined on $(\Omega, F, \mathbf{P}; F^W)$ such that for each $a > 0$,

$$\int_0^a \mathbf{E}\left[\pi_i^2(t)\right] dt < \infty, \quad i = 1,2, \tag{2.13}$$

where $\pi_1(t)$ and $\pi_2(t)$ represent, respectively, the number of units of the bank account and the number of shares of the stock owned by the *writer* at time $t \ge 0$, and $\mathbf{E}$ is the expectation with respect to $\mathbf{P}$.

The *writer's* total asset is described by the wealth process $X^\pi(M) = \{X^\pi(t),\ 0 \le t < \infty\}$ defined by

$$X^\pi(t) = \pi_1(t)B(t) + \pi_2(t)S(t), \quad 0 \le t < \infty, \tag{2.14}$$

where again $B(L;\phi)$ and $S(M,N;\psi)$ are, respectively, the unit price of the bank account and the stock described in (2.7) and (2.8). This wealth process can clearly take both positive and negative values, since it is permissible that $(\pi_1(t), \pi_2(t)) \in \Re^2$.

We will make the following basic assumption throughout this paper.

Assumption 2.2 (self-financing condition). In the (B,S)-market, it is assumed that all trading strategies π satisfy the following *self-financing* condition:

$$X^\pi(t) = X^\pi(0) + \int_0^t \pi_1(s)dB(s) + \int_0^t \pi_2(s)dS(s), \quad 0 \le t < \infty, \text{ a.s.} \tag{2.15}$$

or equivalently,

$$dX^{\pi}(t) = \pi_1(t)dB(t) + \pi_2(t)dS(t), \quad 0 \le t < \infty. \tag{2.16}$$

Using the same notation as in [6] (see also [10]) the set of all *self-financing* trading strategies π will be denoted by $\mathrm{SF}(L, M, N; \phi, \psi)$ or simply SF if there is no danger of ambiguity.

For the unit price of the bank account $B(L;\phi) = \{B(t),\ t \ge 0\}$ and the stock $S(M, N; \psi) = \{S(t),\ t \ge 0\}$ described in (2.7) and (2.8), define

$$\widetilde{W}(t) = W(t) + \int_0^t \gamma(B_s, S_s)ds, \quad t \ge 0, \tag{2.17}$$

where $\gamma : \mathbf{C}_+ \times \mathbf{C}_+ \to \Re$ is defined by

$$\gamma(\phi, \psi) = \frac{\phi(0)M(\psi) - \psi(0)L(\phi)}{\phi(0)N(\psi)}. \tag{2.18}$$

Define the process $Z(L, M, N; \phi, \psi) = \{Z(t),\ t \ge 0\}$ by

$$Z(t) = \exp\left\{\int_0^t \gamma(B_s, S_s)dW(s) - \frac{1}{2}\int_0^t \left|\gamma(B_s, S_s)\right|^2 ds\right\}, \quad t \ge 0. \tag{2.19}$$

The following results are proven in [6].

Lemma 2.3. *The process $Z(L, M, N; \phi, \psi) = \{Z(t),\ t \ge 0\}$ defined by* (2.19) *is a martingale defined on* $(\Omega, F, \mathbf{P}; F^W)$.

Lemma 2.4. *There exists a unique probability measure $\widetilde{\mathbf{P}}$ defined on the canonical measurable space* (Ω, F) *such that*

$$\widetilde{\mathbf{P}}(A) = \mathbf{E}[\mathbf{1}_A Z(T)] \quad \forall A \in F_T,\ 0 < T < \infty, \tag{2.20}$$

where $\mathbf{1}_A$ is the indicator function of $A \in F_T$.

Lemma 2.5. *The process $\widetilde{W}$ defined by* (2.17) *is a standard Brownian motion defined on the filtered probability space* $(\Omega, F, \widetilde{\mathbf{P}}; F^W)$.

From the above, it has been shown (see [6, equation (14)]) that

$$dS(t) = rS(t)dt + N(S_t)d\widetilde{W}(t), \tag{2.21}$$

with $S_0 = \psi \in \mathbf{C}_+$. It is also clear that the probabilistic behavior of $S(M)$ under the probability measure $\mathbf{P}$ is the same as that of $S(K)$ under the probability measure $\widetilde{\mathbf{P}}$; that is, they have the same distribution.

Define the process $Y^{\pi}(L, M, N; \phi, \psi) = \{Y^{\pi}(t),\ t \geq 0\}$, called the discounted wealth process, by

$$Y^{\pi}(t) = \frac{X^{\pi}(t)}{B(t)}, \quad t \geq 0. \tag{2.22}$$

We say that a trading strategy π from $\mathrm{SF}(L, M, N; \phi, \psi)$ belongs to a subclass $\mathrm{SF}^{\varsigma} \subset \mathrm{SF}$ if $\widetilde{\mathbf{P}}$ a.s.

$$Y^{\pi}(t) \geq -\widetilde{\mathbf{E}}[\varsigma \mid F_t], \quad t \geq 0, \tag{2.23}$$

where $\widetilde{\mathbf{E}}$ is the expectation with respect to $\widetilde{\mathbf{P}}$, ς is a nonnegative F-measurable random variable such that $\widetilde{\mathbf{E}}[\varsigma] < \infty$. We say that π belongs to $\mathrm{SF}^{+} \subset \mathrm{SF}$ if $\varsigma \geq 0$.

In [6, 10], it is shown that Y^{π} is a local martingale; for $\pi \in \mathrm{SF}^{\varsigma}$, Y^{π} is a supermartingale, and is a nonnegative supermartingale if $\pi \in \mathrm{SF}^{+}$.

Throughout, we assume the reward function Λ is an F_T-measurable nonnegative random variable satisfying the following condition:

$$\mathbf{E}\left[\Lambda^{1+\epsilon}\right] < \infty, \tag{2.24}$$

for some $\epsilon > 0$. Here, $T > 0$ is the *expiration time*. (Note that the above condition on Λ implies that $\widetilde{\mathbf{E}}[\Lambda] < \infty$.)

Let Λ be a nonnegative F_T-measurable random variable satisfying (2.24). A trading strategy $\pi \in \mathrm{SF}$ is a $(M; \Lambda, x)$-*hedge* of European type if

$$X^{\pi}(0) = \pi_1(0)\phi(0) + \pi_2(0)\psi(0) = x \tag{2.25}$$

and $\widetilde{\mathbf{P}}$ a.s.

$$X^{\pi}(T) \geq \Lambda. \tag{2.26}$$

We say that a $(M; \Lambda, x)$-hedge trading strategy $\pi^{*} \in \mathrm{SF}(M)$ is *minimal* if

$$X^{\pi}(T) \geq X^{\pi^{*}}(T) \tag{2.27}$$

for any $(M; \Lambda, x)$-hedge strategy $\pi \in \mathrm{SF}(M)$.

Let $\Pi(M; \Lambda, x)$ be the set of $(M; \Lambda, x)$-hedge strategies from $\mathrm{SF}^{+}(M)$. Define

$$C(M; \Lambda) = \inf\{x \geq 0 : \Pi(M; \Lambda, x) \neq \emptyset\}. \tag{2.28}$$

The value $C(M; \Lambda)$ defined above is called the *rational price* of the contingent claim of European type. If the infimum in (2.28) is achieved, then $C(M; \Lambda)$ is the *minimal* possible initial capital for which there exists a trading strategy $\pi \in \mathrm{SF}^{+}(M)$ possessing the property that $\widetilde{\mathbf{P}}$ a.s. $X^{\pi}(T) \geq \Lambda$.

Let $Y(M) = \{Y(t),\ 0 \le t \le T\}$ be defined by

$$Y(t) = \widetilde{\mathbf{E}}\left[\frac{\Lambda}{B(T)} \mid \widetilde{F}_t\right], \quad 0 \le t \le T, \tag{2.29}$$

where $\widetilde{F}_t = \sigma(\widetilde{W}_s,\ 0 \le s \le t)$. In [10], it is shown that the process $Y(M)$ is a martingale defined on $(\Omega, F, \widetilde{\mathbf{P}}; F^{\widetilde{W}})$ and can be represented by

$$Y(t) = Y(0) + \int_0^t \beta(s) d\widetilde{W}(s), \tag{2.30}$$

where $\beta = \{\beta(t),\ 0 \le t \le T\}$ that is $F^{\widetilde{W}}$-adapted and $\int_0^T \beta^2(t)dt < \infty$ ($\mathbf{P}$ a.s.).

The following lemma and theorem provide the main results of [6, 10]. Let $\pi^* = \{(\pi_1^*(t), \pi_2^*(t)),\ 0 \le t \le T\}$ be a trading strategy, where

$$\begin{aligned} \pi_2^*(t) &= \frac{\beta(t)B(t)}{N(S_t)}, \\ \pi_1^*(t) &= Y(t) - \frac{S(t)}{B(t)}\pi_2^*(t), \quad t \in [0,T]. \end{aligned} \tag{2.31}$$

Lemma 2.6. *$\pi^* \in \mathrm{SF}(M)$ and for each $t \in [0,T]$, $Y(t) = Y^{\pi^*}(t)$ for each $t \in [0,T]$ where again Y^{π^*} is the process defined in* (2.22) *with the minimal strategy π^* defined in* (2.31).

Theorem 2.7. *Let Λ be an F_T-measurable random variable defined on the filtered probability space $(\Omega, F, \mathbf{P}; F^W)$ that satisfies* (2.24). *Then the rational price $C(M;\Lambda)$ defined in* (2.28) *is given by*

$$C(M;\Lambda) = \widetilde{\mathbf{E}}\left[e^{-rT}\Lambda\right], \tag{2.32}$$

where r is the positive constant that satisfies (2.12). *Furthermore, there exists a minimal hedge $\pi^* = \{(\pi_1^*(t), \pi_2^*(t)),\ 0 \le t \le T\}$, where*

$$\begin{aligned} \pi_2^*(t) &= \frac{\beta(t)B(t)}{N(S_t)}, \\ \pi_1^*(t) &= Y^{\pi^*}(t) - \pi_2^*(t)\frac{S(t)}{B(t)}, \end{aligned} \tag{2.33}$$

and the process $\beta = \{\beta(t),\ 0 \le t \le T\}$ is given by (2.30).

If in addition, the reward Λ is intrinsic, that is, $\Lambda = \Gamma(S(M))$ for some measurable function $\Gamma : \mathbf{C}_+ \to \Re$, then the rational price $C(M;\Lambda)$ does not depend on the mean growth rate M of the stock and

$$C(\Lambda) = \widetilde{\mathbf{E}}\left[e^{-rT}\Lambda\right]. \tag{2.34}$$

3. Fréchet and Extended Fréchet Derivatives

In this section, results are proven that allow the use of a Dynkins formula for stochastic functional differential equation as found in [16, 17]. We assume contingent claims of European type in which the F_T-measurable reward function Λ has the explicit expression $\Lambda = f(S_T)$, where again $S_T(\theta) = S(T+\theta)$, $\theta \in [-h, 0]$ and $S(K) = \{S(t),\ t \geq 0\}$ is the unit price of the stock described by the following equation:

$$dS(t) = rS(t)dt + N(S_t)d\widetilde{W}(t), \quad t \geq 0, \tag{3.1}$$

where $S_0 = \varphi \in \mathbf{C}_+$. Throughout this section, we assume that $S(t)$, and therefore $N(S_t)$, are uniformly bounded almost surely. This assumption is realistic for the price of a stock during time interval $[0, T]$ in a financial system with finite total wealth.

The remaining sections make extensive use of Fréchet derivatives. Let $\mathbf{C}^*$ be the space of bounded linear functionals $\Phi : \mathbf{C} \to \Re$. $\mathbf{C}^*$ is a real separable Banach space under the supremum operator norm

$$\|\Phi\| = \sup_{\phi \neq 0} \frac{|\Phi(\phi)|}{\|\phi\|}. \tag{3.2}$$

For $\Psi : [0, T] \times \mathbf{C} \to \Re$, we denote the Fréchet derivative of Ψ at $\phi \in \mathbf{C}$ by $D\Psi(t, \phi)$. The second Fréchet derivative at ϕ is denoted as $D^2\Psi(t, \phi)$.

Let Γ be the vector space of all simple functions of the form $v\mathbf{1}_{\{0\}}$, where $v \in R$ and $\mathbf{1}_{\{0\}} : [-h, 0] \to \Re$ is defined by

$$\mathbf{1}_{\{0\}}(\theta) = \begin{cases} 0, & \text{for } \theta \in [-h, 0), \\ 1, & \text{for } \theta = 0. \end{cases} \tag{3.3}$$

Form the direct sum $\mathbf{C} \oplus \Gamma$ and equip it with the complete norm

$$\left\|\phi + v\mathbf{1}_{\{0\}}\right\| = \sup_{\theta \in [-h,0]} |\phi(\theta)| + |v|, \quad \phi \in \mathbf{C},\ v \in \Re. \tag{3.4}$$

Then $D\Psi(t, \phi)$ has a unique continuous linear extension from $\mathbf{C} \oplus \Gamma$ to $\Re$ which we will denote by $\overline{D\Psi(t, \phi)}$, and similarly for $\overline{D^2\Psi(t, \phi)}$; see [16] or [17] for more details.

Finally, we define

$$\mathbf{G}(\Psi)(t, \tilde{\varphi}_t) = \lim_{u \to 0^+} \frac{1}{u}\left[\Psi(t, \tilde{\varphi}_{t+u}) - \Psi(t, \tilde{\varphi}_t)\right] \tag{3.5}$$

for all $t \in [0, \infty)$ and $\varphi \in \mathbf{C}_+$, where $\tilde{\varphi} : [-h, \infty) \to \Re$ is defined by

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \in [-h, 0) \\ \varphi(0) & \text{if } t \geq 0. \end{cases} \tag{3.6}$$

Let $f : \mathbf{C} \to \mathfrak{R}$. We say that $f \in C^1(\mathbf{C})$ if f has a continuous Fréchet derivative. Similarly, $f \in C^n(\mathbf{C})$ if f has a continuous nth Fréchet derivative. For $f : R_+ \times \mathbf{C} \to \mathfrak{R}$, we say that $f \in C^{\infty,n}([0,\infty) \times \mathbf{C})$ if f is infinitely differentiable in its first variable and has a continuous nth partial derivative in its second variable.

Proposition 3.1. *Let $\varphi \in \mathbf{C}$ and $f : \mathbf{C} \to \mathfrak{R}$ with $f \in C^2(\mathbf{C})$. Define $\Psi : [0,T] \times \mathbf{C} \to \mathfrak{R}$ by*

$$\Psi(t,\varphi) = e^{-r(T-t)}\widetilde{\mathbf{E}}\left[f(S_T) \mid S_t = \varphi\right]. \tag{3.7}$$

Then $\Psi \in C^{\infty,2}([0,\infty) \times \mathbf{C})$.

Proof. That $e^{-r(T-t)}$ is $C^\infty[0,\infty)$ is clear, so we have only to show that $\Upsilon \in C^2(\mathbf{C})$, where $\Upsilon(\varphi) = \widetilde{\mathbf{E}}[f(S_T) \mid S_t = \varphi]$ given that $f \in C^2(\mathbf{C})$.

We have that

$$dS(t) = rS(t)dt + N(S_t)d\widetilde{W}(t), \quad t \geq 0, \tag{3.8}$$

with $S_0 = \psi \in \mathbf{C}_+$. Under Assumption 2.1 on $N : \mathbf{C} \to \mathfrak{R}$ and the properties of Υ, it can be shown that there exists $H : \mathfrak{R} \times \mathfrak{R} \times \mathbf{C} \to \mathbf{C}$ such that $S_t = H(t, \widetilde{W}(t), \psi)$. Therefore,

$$\begin{aligned}\Upsilon(\varphi) &= \widetilde{\mathbf{E}}\left[f(S_T) \mid S_t = \varphi\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\left(H(T-t,y,\varphi)\right)e^{-y^2/2}dy.\end{aligned} \tag{3.9}$$

By Theorem 3.2, Chapter 2 of [16], $H(t,y,\cdot) \in C^1(\mathbf{C})$ as a function of ψ. By a second application of the same theorem (since $f \in C^2(\mathbf{C})$), we have that $H(t,y,\cdot) \in C^2(\mathbf{C})$ as a function of ψ. Define $g : \mathfrak{R} \times \mathfrak{R} \times \mathbf{C} \to \mathfrak{R}$ by $g = f \circ H$. Since $f \in C^2(\mathbf{C})$ and $H(t,y,\cdot) \in C^2(\mathbf{C})$ in its third variable, $g(t,y,\cdot) \in C^2(\mathbf{C})$. Hence, for $\varphi, \phi \in \mathbf{C}$,

$$\begin{aligned}&\widetilde{\mathbf{E}}\left[f(S_T) \mid S_t = \varphi + \phi\right] - \widetilde{\mathbf{E}}\left[f(S_T) \mid S_t = \varphi\right] \\ &\quad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[f\left(H(T-t,y,\varphi+\phi)\right) - f\left(H(T-t,y,\varphi)\right)\right]e^{-y^2/2}dy \\ &\quad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Dg(T-t,y,\varphi)(\phi)e^{-y^2/2}dy + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} o(\phi)e^{-y^2/2}dy,\end{aligned} \tag{3.10}$$

where $o(\phi)$ is a function mapping continuous functions into the reals such that

$$\frac{o(\phi)}{\|\phi\|} \longrightarrow 0 \quad \text{as } \|\phi\| \longrightarrow 0. \tag{3.11}$$

The last integral is clearly $o(\phi)$ and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Dg(T-t,y,\varphi)(\phi)e^{-y^2/2}dy \tag{3.12}$$

is bounded and linear in ϕ, so this integral is the first Fréchet derivative with respect to φ. Since $g(t,y,\cdot) \in C^2(\mathbf{C})$, the process can be repeated, giving a second Fréchet derivative with respect to φ and so $\Upsilon \in C^2(\mathbf{C})$. □

Proposition 3.2. *Let $\varphi \in \mathbf{C}$ and $f : \mathbf{C} \to \mathfrak{R}$. Further assume $f \in C^2(\mathbf{C})$ and let $\Psi : [0,T] \times \mathbf{C} \to \mathfrak{R}$ be defined by*

$$\Psi(t,\varphi) = e^{-r(T-t)}\widetilde{E}\left[f(S_T) \mid S_t = \varphi\right]. \tag{3.13}$$

Then if Df and $D^2 f$ are globally Lipschitz, then so is $D^2\Psi$.

Recall from Proposition 3.1 that $g : R \times R \times \mathbf{C} \to R$ is $f \circ H$ where $S_t = H(t, \widetilde{W}(t), \varphi)$ with $S_0 = \varphi \in \mathbf{C}_+$.

Proposition 3.3. *Let $\varphi \in \mathbf{C}$ and $f : \mathbf{C} \to \mathfrak{R}$. Further assume $f \in C^2(\mathbf{C})$ and let $\Psi : [0,T] \times \mathbf{C} \to \mathfrak{R}$ be defined by*

$$\Psi(t,\varphi) = e^{-r(T-t)}\widetilde{E}\left[f(S_T) \mid S_t = \varphi\right]. \tag{3.14}$$

Then if f and $\mathbf{G}(g)(T-t, y, \widetilde{\varphi}_t)$ are globally bounded, then so is $\mathbf{G}(\Psi)(t, \widetilde{\varphi}_t)$.

Proof. We have that

$$\begin{aligned}
\mathbf{G}(\Psi)(t,\widetilde{\varphi}_t) &= \lim_{u\to 0^+} \frac{1}{u}\left[\Psi(t,\widetilde{\varphi}_{t+u}) - \Psi(t,\widetilde{\varphi}_t)\right] \\
&= \lim_{u\to 0^+} \frac{1}{u}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[g(T-t,y,\widetilde{\varphi}_{t+u}) - g(T-t,y,\widetilde{\varphi}_t)\right]e^{-y^2/2}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\lim_{u\to 0^+}\frac{1}{u}\left[g(T-t,y,\widetilde{\varphi}_{t+u}) - g(T-t,y,\widetilde{\varphi}_t)\right]e^{-y^2/2}dy \\
&\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Me^{-y^2/2}dy \\
&= M < \infty,
\end{aligned} \tag{3.15}$$

where we used the assumption that f and hence g are globally bounded to move the limit inside the integral and $\mathbf{G}(g)(T-t,y,\widetilde{\varphi}_t) \le M < \infty$. □

Remark 3.4. Note that since $\mathbf{G}(\Psi)(\widetilde{\varphi}_s)$ is bounded for all $s \in [0,T]$, $\int_0^t D\Psi(s,\widetilde{\varphi}_s)(d\widetilde{\varphi}_s)$ exits. Also, if $D^2 f$ is bounded, $\int_0^t D^2\Psi(s,\widetilde{\varphi}_s)(d\widetilde{\varphi}_s, d\widetilde{\varphi}_s)$ exits (see [18]).

4. The Infinite-Dimensional Black-Scholes Equation

It is known (e.g., [4, 5]) that the classical Black-Scholes equation is a deterministic parabolic partial differential equation (with a suitable auxiliary condition) the solution of which gives the value of the European option contract at a given time. Propositions 3.1 through 3.3 allow

us to use the Dynkin formula in [16]. With it, a generalized version of the classical Black-Scholes equation can be derived for when the (B,S)-market model is (2.7) and (2.8). The following theorem is a restatement of Theorem 3.1 in [10].

Theorem 4.1. *Let $\Psi(t,\varphi) = e^{-r(T-t)}\widetilde{\mathbf{E}}[f(S_T) \mid S_t = \varphi]$, where $S_0 = \varphi \in \mathbf{C}_+$ and $t \in [0,T]$. Let f be a $C^2(\mathbf{C})$ function with Df and $D^2 f$ globally Lipschitz and let $\Lambda = f(S_T)$ and $x = X^{\pi^*}(0)$. Finally, let f and $\mathbf{G}(g)(T-t,y,\widetilde{\varphi}_t)$ be globally bounded. Then if $X^{\pi^*}(t) = \Psi(t,S_t)$ is the wealth process for the minimal (Λ,x)-hedge, one has*

$$\begin{aligned} r\Psi(t,\varphi) = {} & \frac{\partial}{\partial t}\Psi(t,\varphi) + \mathbf{G}(\Psi)(t,\widetilde{\varphi}_t) + \overline{D\Psi(t,\varphi)}\left(r\varphi(0)\mathbf{1}_{\{0\}}\right) \\ & + \frac{1}{2}\overline{D^2\Psi(t,\varphi)}\left(N(\varphi)\mathbf{1}_{\{0\}}, N(\varphi)\mathbf{1}_{\{0\}}\right), \quad a.s.\ \forall(t,\varphi) \in [0,T) \times \mathbf{C}_+, \end{aligned} \tag{4.1}$$

where

$$\Psi(T,\varphi) = f(\varphi) \quad \forall \varphi \in \mathbf{C}_+, \tag{4.2}$$

and the trading strategy $(\pi_1^*(t), \pi_2^*(t))$ is defined by

$$\begin{aligned} \pi_2^*(t) &= \overline{D\Psi(t,\varphi)}\left(\mathbf{1}_{\{0\}}\right) \quad a.s., \\ \pi_1^*(t) &= \frac{1}{B(t)}\left[X^{\pi^*}(t) - \varphi(0)\pi_2^*(t)\right]. \end{aligned} \tag{4.3}$$

Furthermore, if (4.1) *and* (4.2) *hold, then $\Psi(t,S_t)$ is the wealth process for the (Λ,x)-hedge with $\pi_2^*(t) = \overline{D\Psi(t,S_t)}(\mathbf{1}_{\{0\}})$ and $\pi_1^*(t) = (1/B(t))[X^{\pi^*}(t) - S(t)\pi_2^*(t)]$.*

Proof. The theorem is a restatement of Theorem 3.1 in [10] and is therefore omitted. □

Note

Equations (4.1) and (4.2) are the generalized Black-Scholes equation for the (B,S)-market with hereditary price structure as described by (2.7) and (2.8).

5. Approximation of Solutions

In this section, we will solve the generalized Black-Scholes equation (4.1)-(4.2) by considering a sequence of approximations of its solution. By a (classical) solution to (4.1)-(4.2), we mean $\Psi : [0,T] \times \mathbf{C} \to \mathfrak{R}$ satisfying the following conditions:

(i) $\Psi \in C^{1,2}([0,T] \times \mathbf{C})$,

(ii) $\Psi(T,\varphi) = f(\varphi)$ for all $\varphi \in \mathbf{C}$,

(iii) Ψ satisfies (4.1).

The sequence of approximate solutions is constructed by looking at finite-dimensional subspaces of $\mathbf{C}$, solving (4.1)-(4.2) on these subspaces, and then showing that as the

dimension of the subspaces goes to infinity, the finite-dimensional solutions converge to a solution of the original problem. Theorem 5.2, Remark 5.3, and Corollary 5.4 show that the generalized Black-Scholes equation can be solved by solving two simpler equations. The first of these, a first-order partial differential equation, can be handled by traditional techniques once the second equation is solved. Theorem 5.5 provides a solution to the second. Proposition 5.7, which uses Lemma 5.6, gives a generalized Black-Scholes formula for the standard European call option when used in conjunction with Theorem 5.2.

We start by noting that $\mathbf{C} \subset L^2[-h,0]$ where $L^2[-h,0]$ is the space of all square-integrable functions on the interval $[-h,0]$. Furthermore, $\mathbf{C}$ is dense in $L^2[-h,0]$. It is well known (e.g., [19]) that even extensions of a function φ in $L^2[-h,0]$ may be represented by a cosine Fourier series where

$$\left\| \varphi - \sum_{i=0}^{N} a_i \cos\left(\frac{2\pi i \cdot}{h}\right) \right\|_2 \longrightarrow 0 \tag{5.1}$$

as $N \to \infty$ where

$$\begin{gathered} a_0 = \frac{1}{h}\int_{-h}^{0} \varphi(\theta)\,d\theta, \\ a_i = \frac{2}{h}\int_{-h}^{0} \varphi(\theta)\cos\left(\frac{2\pi i\theta}{h}\right)d\theta, \quad i = 1,2,3,\ldots. \end{gathered} \tag{5.2}$$

Here,

$$\|f\|_2^2 = \int_{-h}^{0} f^2(\theta)\,d\theta \tag{5.3}$$

for $f \in L^2[-h,0]$. If φ is Hölder-continuous, then the convergence is also point wise (see, e.g., [20]).

Throughout this section, we let $L_n^2[-h,0]$ be the subspace of $L^2[-h,0]$ consisting of functions that can be represented as a finite Fourier series, that is, $\varphi^{(n)} \in L_n^2[-h,0]$ if

$$\varphi^{(n)}(\theta) = \sum_{i=0}^{n} a_i \cos\left(\frac{2\pi i\theta}{h}\right) \tag{5.4}$$

for all $\theta \in [-h,0]$.

We will see that it is convenient having a spanning set $\{f_i\}_{i=0}^{\infty}$ for $L_n^2[-h,0]$ where $f_i : [-h,0] \to \Re$ for $i = 0,1,\ldots$ such that $f_i(0) = 1$ for all i and $N(f_i) = \int_{-h}^{0} f_i(\theta)\,d\zeta(\theta) = \delta$ for all i. Here, $\delta = \int_{-h}^{0} d\zeta(\theta) \in \Re$. Let $q : [-h,0] \to \Re$ be any function such that $N(q) = q(0) \neq 0$. For example, let

$$q(\theta) = 1 + \frac{1-\delta-d_2}{d_1}\theta + \theta^2, \tag{5.5}$$

where $d_1 = \int_{-h}^{0} \theta d\zeta(\theta)$ and $d_2 = \int_{-h}^{0} \theta^2 d\zeta(\theta)$. To this end, we define the following functions. Let

$$\begin{aligned} f_0(\theta) &= 1 \quad \forall \theta \in [-h, 0], \\ f_1(\theta) &= \alpha_{1,1} + \alpha_{1,2} q(\theta) \quad \forall \theta \in [-h, 0], \\ f_2(\theta) &= \alpha_{2,1} q(\theta) + \alpha_{2,2} \cos\left(\frac{2\pi\theta}{h}\right) \quad \forall \theta \in [-h, 0], \end{aligned} \tag{5.6}$$

and for $i = 3, 4, \ldots$,

$$f_i(\theta) = \alpha_{i,1} \cos\left(\frac{2\pi(i-2)\theta}{h}\right) + \alpha_{i,2} \cos\left(\frac{2\pi(i-1)\theta}{h}\right) \quad \forall \theta \in [-h, 0]. \tag{5.7}$$

Recall that $N : L^2[-h, 0] \to \mathfrak{R}$ is defined by

$$N(\varphi) = \int_{-h}^{0} \varphi(\theta) d\zeta(\theta), \tag{5.8}$$

and let

$$c_i = N\left(\cos\left(\frac{2\pi i \cdot}{h}\right)\right) = \int_{-h}^{0} \cos\left(\frac{2\pi i\theta}{h}\right) d\zeta(\theta). \tag{5.9}$$

Here again $q : [-h, 0] \to \mathfrak{R}$ is any function such that $N(q) = q(0) \neq 0$. For example, q can be chosen as in (5.5). In this case, the constant $\alpha_{1,2}$ is nonzero but otherwise arbitrary,

$$\begin{aligned} \alpha_{1,1} &= 1 - \alpha_{1,2} q(0), \\ \alpha_{2,1} &= \frac{\delta - c_1}{q(0)(1 - c_1)}, \\ \alpha_{2,2} &= 1 - \alpha_{2,1} q(0), \end{aligned} \tag{5.10}$$

and so on with

$$\begin{aligned} \alpha_{i,2} &= \frac{\delta - c_{i-2}}{c_{i-1} - c_{i-2}}, \\ \alpha_{i,1} &= 1 - \alpha_{i,2} \end{aligned} \tag{5.11}$$

for $i \geq 3$.

Lemma 5.1. *The set* $\{f_i\}_{i=0}^{\infty}$ *defined in* (5.6) *and* (5.7) *forms a spanning set for* $L^2[-h,0]$ *in the sense that*

$$\left\|\varphi - \sum_{i=0}^{n+1} x_i f_i\right\|_2 \longrightarrow 0 \tag{5.12}$$

as $n \to \infty$, *where the* x_i *are defined by*

$$\begin{aligned} x_{n+1} &= \frac{a_n}{\alpha_{n+1,2}}, \\ x_n &= \frac{a_{n-1} - x_{n+1}\alpha_{n+1,1}}{\alpha_{n,2}}, \end{aligned} \tag{5.13}$$

and continuing using

$$x_i = \frac{a_{i-1} - x_{i+1}\alpha_{i+1,1}}{\alpha_{i,2}} \tag{5.14}$$

until

$$\begin{aligned} x_1 &= -\frac{x_2\alpha_{2,1}}{\alpha_{1,2}}, \\ x_0 &= a_0 - x_1\alpha_{1,1}. \end{aligned} \tag{5.15}$$

This set of functions has the properties that $f_i(0) = 1$ *and* $N(f_i) = \delta$ *for all* $i = 0, 1, \ldots$.

Proof. For any $\varphi \in L^2[-h,0]$, we can construct an even extension $\phi \in L^2[-h,h]$ where $\phi(\theta) = \varphi(\theta)$ for all $\theta \in [-h,0]$ and $\phi(\theta) = \varphi(-\theta)$ for all $\theta \in [0,h]$. The function ϕ may be represented by a Fourier series of cosine functions

$$\phi(\theta) \sim \sum_{i=0}^{N} a_i \cos\left(\frac{2\pi i\theta}{h}\right), \tag{5.16}$$

where the "$\sim$" is used to indicate that

$$\left\|\phi - \sum_{i=0}^{N} a_i \cos\left(\frac{2\pi i\cdot}{h}\right)\right\|_2 \longrightarrow 0 \tag{5.17}$$

as $N \to \infty$. In what mentioned before,

$$\begin{aligned} a_0 &= \frac{1}{h}\int_{-h}^{0} \phi(\theta)\,d\theta, \\ a_i &= \frac{2}{h}\int_{-h}^{0} \phi(\theta)\cos\left(\frac{2\pi i\theta}{h}\right)d\theta \end{aligned} \tag{5.18}$$

for all $i = 1, 2, \ldots$. For simplicity, we will replace the "~" with an equality sign knowing that mean-square convergence is implied.

For the Fourier series, the basis is

$$\left\{\cos\left(\frac{2\pi i\theta}{h}\right)\right\}_{i=0}^{\infty}, \tag{5.19}$$

so the first term of this basis and $\{f_i\}_{i=0}^{\infty}$ are the same, namely, the constant "1." Clearly $f_0(0) = 1$ and $N(f_0) = \delta$. The first part of this proof is to show that for all $i = 0, 1, \ldots$, $f_i(0) = 1$ and $N(f_i) = \delta$.

For f_1, we have that $f_1(0) = \alpha_{1,1} + \alpha_{1,2}q(0) = 1$ which implies that

$$\alpha_{1,1} = 1 - \alpha_{1,2}q(0). \tag{5.20}$$

Also, $N(f_1) = \alpha_{1,1}\delta + \alpha_{1,2}N(q) = \delta$. Since we do not want $\alpha_{1,2} = 0$, we require that

$$N(q) = q(0). \tag{5.21}$$

There are no restrictions on $\alpha_{1,2}$ other than $\alpha_{1,2} \neq 0$.

For f_2, $\alpha_{2,1}q(0) + \alpha_{2,2} = 1$ requires that

$$\alpha_{2,2} = 1 - \alpha_{2,1}q(0). \tag{5.22}$$

Since we want $\alpha_{2,1}N(q) + \alpha_{2,2}c_1 = \delta$, then

$$\alpha_{2,1} = \frac{\delta - c_1}{N(q) - q(0)c_1} = \frac{\delta - c_1}{q(0)(1 - c_1)}. \tag{5.23}$$

The rest of the f_i, that is, where $i \geq 3$, are handled alike. In order that $f_i(0) = 1$, we require that $\alpha_{i,1} = 1 - \alpha_{i,2}$. To ensure that $N(f_i) = \delta$,

$$\alpha_{i,2} = \frac{\delta - c_{i-2}}{c_{i-1} - c_{i-2}}. \tag{5.24}$$

We have now shown that the sequence of functions $\{f_i\}_{i=0}^{\infty}$ is such that $f_i(0) = 1$ and $N(f_i) = \delta$ for all $i = 0, 1, \ldots$. Now it must be shown that this sequence is a spanning set for $L^2[-h, 0]$. To do this, we will compare this sequence of functions with the cosine Fourier sequence of functions.

Consider $\varphi^{(n)} : [-h, 0] \to \Re$ where

$$\varphi^{(n)}(\theta) = \sum_{i=0}^{n} a_i \cos\left(\frac{2\pi i\theta}{h}\right). \tag{5.25}$$

We would like

$$\varphi^{(n)}(\theta) = \sum_{i=0}^{n+1} x_i f_i(\theta) \tag{5.26}$$

for some set $\{x_i\}_{i=0}^{n+1}$ of real numbers. By the Fourier expansion,

$$\varphi^{(n)}(\theta) = a_0 + a_1 \cos\left(\frac{2\pi\theta}{h}\right) + \cdots + a_n \cos\left(\frac{2\pi n\theta}{h}\right). \tag{5.27}$$

We want $\{x_i\}_{i=0}^{n+1}$ where

$$\begin{aligned}
\varphi^{(n)}(\theta) &= x_0 + x_1\left(\alpha_{1,1} + \alpha_{1,2}q(\theta)\right) \\
&\quad + x_2\left(\alpha_{2,1}q(\theta) + \alpha_{2,2}\cos\left(\frac{2\pi\theta}{h}\right)\right) \\
&\quad + x_3\left(\alpha_{3,1}\cos\left(\frac{2\pi\theta}{h}\right) + \alpha_{3,2}\cos\left(\frac{4\pi\theta}{h}\right)\right) \\
&\quad + \cdots + x_n\left(\alpha_{n,1}\cos\left(\frac{2\pi(n-2)\theta}{h}\right) + \alpha_{n,2}\cos\left(\frac{2\pi(n-1)\theta}{h}\right)\right) \\
&\quad + x_{n+1}\left(\alpha_{n+1,1}\cos\left(\frac{2\pi(n-1)\theta}{h}\right) + \alpha_{n+1,2}\cos\left(\frac{2\pi n\theta}{h}\right)\right) \\
&= (x_0 + x_1\alpha_{1,1}) + q(\theta)(x_1\alpha_{1,2} + x_2\alpha_{2,1}) \\
&\quad + \cos\left(\frac{2\pi\theta}{h}\right)(x_2\alpha_{2,2} + x_3\alpha_{3,1}) \\
&\quad + \cdots + \cos\left(\frac{2\pi i\theta}{h}\right)(x_{i+1}\alpha_{i+1,2} + x_{i+2}\alpha_{i+2,1}) \\
&\quad + \cdots + \cos\left(\frac{2\pi(n-1)\theta}{h}\right)(x_n\alpha_{n,2} + x_{n+1}\alpha_{n+1,1}) \\
&\quad + \cos\left(\frac{2\pi n\theta}{h}\right)(x_{n+1}\alpha_{n+1,2}).
\end{aligned} \tag{5.28}$$

Equating the last coefficients gives

$$\begin{aligned}
x_{n+1} &= \frac{a_n}{\alpha_{n+1,2}}, \\
x_n &= \frac{a_{n-1} - x_{n+1}\alpha_{n+1,1}}{\alpha_{n,2}}.
\end{aligned} \tag{5.29}$$

Continuing,

$$x_i = \frac{a_{i-1} - x_{i+1}\alpha_{i+1,1}}{\alpha_{i,2}}, \tag{5.30}$$

and finally

$$\begin{aligned} x_1 &= -\frac{x_2\alpha_{2,1}}{\alpha_{1,2}}, \\ x_0 &= a_0 - x_1\alpha_{1,1}. \end{aligned} \tag{5.31}$$

Hence, with the above choice of $\{x_i\}_{i=0}^{n+1}$,

$$\sum_{i=0}^{n} a_i \cos\left(\frac{2\pi i\theta}{h}\right) = \sum_{i=0}^{n+1} x_i f_i(\theta), \tag{5.32}$$

and so

$$\begin{aligned} \left\|\varphi - \sum_{i=0}^{n+1} x_i f_i\right\|_2 &= \left\|\varphi - \sum_{i=0}^{n+1} x_i f_i + \sum_{i=0}^{n} a_i \cos\left(\frac{2\pi i\cdot}{h}\right) - \sum_{i=0}^{n} a_i \cos\left(\frac{2\pi i\cdot}{h}\right)\right\|_2 \\ &= \left\|\varphi - \sum_{i=0}^{n} a_i \cos\left(\frac{2\pi i\cdot}{h}\right) + \sum_{i=0}^{n} a_i \cos\left(\frac{2\pi i\cdot}{h}\right) - \sum_{i=0}^{n+1} x_i f_i\right\|_2 \\ &\leq \left\|\varphi - \sum_{i=0}^{n} a_i \cos\left(\frac{2\pi i\cdot}{h}\right)\right\|_2 + \left\|\sum_{i=0}^{n} a_i \cos\left(\frac{2\pi i\cdot}{h}\right) - \sum_{i=0}^{n+1} x_i f_i\right\|_2 \\ &= \left\|\varphi - \sum_{i=0}^{n} a_i \cos\left(\frac{2\pi i\cdot}{h}\right)\right\|_2 \longrightarrow 0 \end{aligned} \tag{5.33}$$

as $n \to \infty$. □

To find an approximate solution to the generalized Black-Scholes equation we start by letting $X^{\pi^*}(t) = \widetilde{\mathbf{E}}[f(S_T) \mid S_t = \varphi]$ (from [6]) and approximating φ by

$$\varphi^{(n)} = \sum_{i=0}^{n+1} x_i f_i. \tag{5.34}$$

We define the space $\mathbf{C}_n$ as the set of all continuous functions that can be represented by this summation for some $\{x_i\}_{i=0}^{\infty}$. Note that $\mathbf{C}_n \subset L_n^2[-h, 0]$. Also define $e_n : \mathfrak{R}^{n+2} \to \mathfrak{R}$ by

$$e_n(\overrightarrow{x}) = \sum_{i=0}^{n+1} x_i f_i, \tag{5.35}$$

so that $\Psi(t,\varphi^{(n)}) = \Psi(t,e_n(\overrightarrow{x}))$. Define $\Psi_n : [0,T]\times\mathfrak{R}^{n+2} \to \mathfrak{R}$ by $\Psi_n(t,\overrightarrow{x}) = \Psi(t,\varphi^{(n)})$ provided that the $\overrightarrow{x}$ is formed by the coefficients of $\varphi^{(n)}$ in the spanning set $\{f_i\}_{i=0}^{\infty}$. In general, $\overrightarrow{x}(t)$ is formed by the coefficients of $\varphi_t^{(n)}$ in the spanning set $\{f_i\}_{i=0}^{\infty}$. Also, define $v_n : [-h,0] \to \mathfrak{R}$ by

$$v_n(\theta) = \begin{cases} 0, & \text{for } \theta \in \left[-h,-\dfrac{1}{n}\right), \\ n\theta+1, & \text{for } \theta = \left[-\dfrac{1}{n},0\right]. \end{cases} \tag{5.36}$$

Last, let $g_n : [0,T]\times\mathbf{C}_n\times\mathfrak{R}^{n+1}\times C^{1,2}([0,T]\times\mathbf{C}) \to \mathfrak{R}$ be defined by

$$\begin{aligned} g_n\left(t,\varphi^{(n)},\overrightarrow{x},\Psi\right) &= r\left(\sum_{i=0}^{n+1}x_i\right)\left[\overline{D\Psi(t,\varphi^{(n)})}(\mathbf{1}_{\{0\}}) - \sum_{i=0}^{n+1}k_i\frac{\partial}{\partial x_i}\Psi_n(t,\overrightarrow{x})\right] \\ &\quad + \frac{\delta^2}{2}\left(\sum_{i=0}^{n+1}x_i\right)^2\left[\overline{D^2\Psi(t,\varphi^{(n)})}(\mathbf{1}_{\{0\}},\mathbf{1}_{\{0\}}) - \sum_{i,j=0}^{n+1}k_ik_j\frac{\partial^2}{\partial x_i\partial x_j}\Psi_n(t,\overrightarrow{x})\right], \end{aligned} \tag{5.37}$$

where the k_i are the coefficients of v_n using the spanning set $\{f_i\}_{i=0}^{\infty}$. Finally, define the operator $(\cdot)_n : \mathbf{C} \to \mathbf{C}_n$ by

$$(\varphi)_n = \sum_{i=0}^{n+1}x_if_i, \tag{5.38}$$

where the right-hand side is the first $n+2$ terms of the $\{f_i\}$-expansion of φ.

We are now ready for a theorem which enables us to approximate the solution of the infinite-dimensional Black-Scholes equation by solving a first-order real-valued partial differential equation and an equation similar to the generalized Black-Scholes equation but without the $\mathbf{G}(\Psi)(t,\widetilde{\varphi}_t)$ term. The lack of this term allows approximate solutions to be found using traditional techniques.

Theorem 5.2. *Let $S_0 = \psi \in \mathbf{C}_+$ and $t \in [0,T]$. Let f be a $C^2(\mathbf{C})$ function satisfying the conditions of Theorem 4.1 and let $\Lambda = f(S_T)$. Then*

$$\begin{aligned} r\Psi\left(t,\varphi^{(n)}\right) &= \frac{\partial}{\partial t}\Psi\left(t,\varphi^{(n)}\right) + \mathbf{G}(\Psi)\left(t,(\widetilde{\varphi}_t)_n\right) + \overline{D\Psi(t,\varphi^{(n)})}\left(r\varphi^{(n)}(0)\mathbf{1}_{\{0\}}\right) \\ &\quad + \frac{1}{2}\overline{D^2\Psi(t,\varphi^{(n)})}\left(N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}},N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}\right), \quad \forall\left(t,\varphi^{(n)}\right) \in [0,T)\times\mathbf{C}_n, \end{aligned} \tag{5.39}$$

where

$$\Psi\left(T,\varphi^{(n)}\right) = f\left(\varphi^{(n)}\right) \quad \forall\varphi^{(n)} \in \mathbf{C}_n \tag{5.40}$$

has a solution of the form

$$V(t,(\tilde{\varphi}_t)_n)F\left(t,\varphi^{(n)}\right). \tag{5.41}$$

Here, $V(t,(\tilde{\varphi}_t)_n) = w_n(t,0)$ *is a solution to*

$$F\left(t,\varphi^{(n)}\right)\frac{\partial w_n}{\partial t}(t,u) + F(t,(\tilde{\varphi}_t)_n)\frac{\partial w_n}{\partial u}(t,u) \tag{5.42}$$

$$+\mathbf{G}(F)(t,(\tilde{\varphi}_t)_n)w_n(t,u) = 0 \tag{5.43}$$

for $t \in [0,T]$ *and* $u \in [0,\epsilon)$ *for some* $\epsilon > 0$ *and* $w_n(T,0) = 1$, *and* $F : \Re_+ \times \mathbf{C}_n \to \Re$ *is a solution of*

$$\begin{aligned} rF\left(t,\varphi^{(n)}\right) &= \frac{\partial}{\partial t}F\left(t,\varphi^{(n)}\right) + \overline{DF(t,\varphi^{(n)})}\left(r\varphi^{(n)}(0)\mathbf{1}_{\{0\}}\right) \\ &\quad + \frac{1}{2}\overline{D^2F(t,\varphi^{(n)})}\left(N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}, N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}\right) \quad \forall\left(t,\varphi^{(n)}\right) \in [0,T)\times\mathbf{C}_n, \end{aligned} \tag{5.44}$$

where

$$F\left(T,\varphi^{(n)}\right) = f\left(\varphi^{(n)}\right) \quad \forall\varphi^{(n)} \in \mathbf{C}_n, \tag{5.45}$$

and f *is a uniformly bounded* $C^2(\mathbf{C})$ *function satisfying the conditions of Theorem 4.1.*

Proof. We assume a solution of the form $\Psi(t,\varphi^{(n)}) = V(t,(\tilde{\varphi}_t)_n)F(t,\varphi^{(n)})$, then

$$\begin{aligned} &rV(t,(\tilde{\varphi}_t)_n)F\left(t,\varphi^{(n)}\right) \\ &\quad = \frac{\partial}{\partial t}\left(V(t,(\tilde{\varphi}_t)_n)F\left(t,\varphi^{(n)}\right)\right) + \mathbf{G}(VF)(t,(\tilde{\varphi}_t)_n) + \overline{D(VF)(t,\varphi^{(n)})}\left(r\varphi^{(n)}(0)\mathbf{1}_{\{0\}}\right) \\ &\qquad + \frac{1}{2}\overline{D^2(VF)(t,\varphi^{(n)})}\left(N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}, N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}\right) \\ &\quad = F\left(t,\varphi^{(n)}\right)\frac{\partial V}{\partial t}(t,(\tilde{\varphi}_t)_n) + V(t,(\tilde{\varphi}_t)_n)\frac{\partial}{\partial t}F\left(t,\varphi^{(n)}\right) \\ &\qquad + F\left(t,\varphi^{(n)}\right)\mathbf{G}(V)(t,(\tilde{\varphi}_t)_n) + V(t,(\tilde{\varphi}_t)_n)\mathbf{G}(F)\left(t,\varphi^{(n)}\right) + V(t,(\tilde{\varphi}_t)_n) \\ &\qquad \times\left\{\overline{D(F)(t,\varphi^{(n)})}\left(r\varphi^{(n)}(0)\mathbf{1}_{\{0\}}\right)\right. \\ &\qquad\quad \left. + \frac{1}{2}\overline{D^2(F)(t,\varphi^{(n)})}\left(N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}, N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}\right)\right\}, \quad \forall\left(t,\varphi^{(n)}\right) \in [0,T)\times\mathbf{C}_n. \end{aligned} \tag{5.46}$$

If $F(t,\varphi^{(n)})$ is the solution to (5.44), then

$$F\left(t,\varphi^{(n)}\right)\frac{\partial V}{\partial t}\left(t,\left(\widetilde{\varphi}_t\right)_n\right)+F\left(t,\left(\widetilde{\varphi}_t\right)_n\right)\mathbf{G}(V)\left(t,\left(\widetilde{\varphi}_t\right)_n\right)+\mathbf{G}(F)\left(t,\left(\widetilde{\varphi}_t\right)_n\right)V\left(t,\left(\widetilde{\varphi}_t\right)_n\right)=0. \quad (5.47)$$

Define $T_u : \mathbf{C} \to \mathbf{C}$ by $T_u(\varphi) = \varphi_u$, that is, T_u is a shift operator. Now let $V(t,(\widetilde{\varphi}_{t+u})_n) = V(t,(T_u(\widetilde{\varphi}_t))_n) = w_n(t,u)$ for a fixed $\varphi \in \mathbf{C}$. Then

$$\mathbf{G}(V)\left(t,\left(\widetilde{\varphi}_t\right)_n\right)=\frac{\partial w_n}{\partial u}(t,0^+), \quad (5.48)$$

where the superscript + denotes a right-hand derivative with respect to u. Then

$$F\left(t,\varphi^{(n)}\right)\frac{\partial w_n}{\partial t}(t,0)+F\left(t,\left(\widetilde{\varphi}_t\right)_n\right)\frac{\partial w_n}{\partial u}(t,0^+)+\mathbf{G}(F)\left(t,\left(\widetilde{\varphi}_t\right)_n\right)w_n(t,0)=0. \quad (5.49)$$

A slightly more restrictive, but more familiar form is

$$F\left(t,\varphi^{(n)}\right)\frac{\partial w_n}{\partial t}(t,u)+F\left(t,\left(\widetilde{\varphi}_t\right)_n\right)\frac{\partial w_n}{\partial u}(t,u)+\mathbf{G}(F)\left(t,\left(\widetilde{\varphi}_t\right)_n\right)w_n(t,u)=0, \quad (5.50)$$

where $t \in [0,h]$ and $u \in [0,\epsilon)$ for some $\epsilon > 0$. There is the additional requirement that $w_n(T,0) = 1$ so that (5.44) holds. □

Remark 5.3. It can be easily shown that S_t is α-Hölder continuous a.s. for $0 < \alpha < 1/2$ provided that S_0 is α-Hölder continuous for the same α. Therefore,

$$\left|F_n(t,\overrightarrow{x})-F(t,\varphi)\right| \longrightarrow 0 \quad (5.51)$$

for each t as $n \to \infty$ where $F(t,\varphi)$ is a solution to (5.44) and $F_n(t,\overrightarrow{x}) = F(t,\varphi^{(n)})$ is an approximate solution, since F is $C^2(\mathbf{C})$ in its second variable and

$$F_n(t,\overrightarrow{x}) = F(t,e_n(\overrightarrow{x})) = F\left(t,\varphi^{(n)}\right). \quad (5.52)$$

The proof of the following corollary is identical to that of Theorem 5.2, with the use of Remark 5.3 to obtain $\Psi(t,\varphi)$.

Corollary 5.4. *If S_0 is Hölder continuous, then*

$$\begin{aligned} r\Psi(t,\varphi) &= \frac{\partial}{\partial t}\Psi(t,\varphi)+\mathbf{G}(\Psi)(t,\widetilde{\varphi}_t)+\overline{D\Psi(t,\varphi)}\left(r\varphi(0)\mathbf{1}_{\{0\}}\right) \\ &\quad +\frac{1}{2}\overline{D^2\Psi(t,\varphi)}\left(N(\varphi)\mathbf{1}_{\{0\}},N(\varphi)\mathbf{1}_{\{0\}}\right), \quad \forall(t,\varphi)\in[0,T)\times\mathbf{C}_+, \end{aligned} \quad (5.53)$$

where

$$\Psi(T,\varphi) = f(\varphi) \quad \forall \varphi \in \mathbf{C}_+ \tag{5.54}$$

has a solution of the form $V(t,\tilde{\varphi}_t)F(t,\varphi)$. *Here,* $V(t,\tilde{\varphi}_t) = w(t,0)$ *is a solution to*

$$F(t,\varphi)\frac{\partial w}{\partial t}(t,u) + F(t,\tilde{\varphi}_t)\frac{\partial w}{\partial u}(t,u) + \mathbf{G}(F)(t,\tilde{\varphi}_t)w(t,u) = 0 \tag{5.55}$$

for $t \in [0,T]$ *and* $u \in [0,\epsilon)$ *for some* $\epsilon > 0$. $F(t,\varphi)$ *is the solution to* (5.44) *where one lets* $n \to \infty$. *In addition,* $w(T,0) = 1$.

Now we must solve (5.44), which is done in the following theorem. With this solution, the first-order partial differential equation can be solved by traditional means.

Theorem 5.5. *Let*

$$\begin{aligned} r\Psi\left(t,\varphi^{(n)}\right) &= \frac{\partial}{\partial t}\Psi\left(t,\varphi^{(n)}\right) + \overline{D\Psi\left(t,\varphi^{(n)}\right)}\left(r\varphi^{(n)}(0)\mathbf{1}_{\{0\}}\right) \\ &\quad + \frac{1}{2}\overline{D^2\Psi\left(t,\varphi^{(n)}\right)}\left(N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}, N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}\right) \quad \forall\left(t,\varphi^{(n)}\right) \in [0,T)\times\mathbf{C}_n, \end{aligned} \tag{5.56}$$

where

$$\Psi\left(T,\varphi^{(n)}\right) = f\left(\varphi^{(n)}\right) \quad \forall \varphi^{(n)} \in \mathbf{C}_n, \tag{5.57}$$

and f *is a uniformly bounded* $C^2(\mathbf{C})$ *function satisfying the conditions of Theorem 4.1. Let* $f_n : \mathfrak{R}^{n+2} \to \mathfrak{R}$ *be defined by* $f_n = f \circ e_n$, *then*

$$\begin{aligned} \Psi_n(t,\overrightarrow{x}) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_n\left(\exp\left[\left(rB - \frac{\delta^2}{2}B^2\right)(T-t) + \delta By\sqrt{T-t}\right]\overrightarrow{x}\right)e^{-y^2/2}dy \\ &\quad + \int_t^T g_n\left(s,\varphi_s^{(n)},\overrightarrow{x}(s),\Psi\right)e^{-r(s-t)}ds. \end{aligned} \tag{5.58}$$

Here,

$$B = \begin{bmatrix} k_0 & k_0 & \cdots & k_0 \\ k_1 & k_1 & \cdots & k_1 \\ \vdots & \vdots & \vdots & \vdots \\ k_{n+1} & k_{n+1} & \cdots & k_{n+1} \end{bmatrix}, \qquad v_n = \sum_{i=0}^{n+1} k_i f_i \tag{5.59}$$

from (5.36).

Proof. Since $\Psi_n : [0,T]\times\mathfrak{R}^{n+2} \to \mathfrak{R}$, the definition of the Fréchet derivatives and the properties of the set $\{f_i\}_{i=0}^{\infty}$ give

$$\begin{aligned} r\Psi_n(t,\overrightarrow{x}) &= \frac{\partial}{\partial t}\Psi_n(t,\overrightarrow{x}) + r\left(\sum_{i=0}^{n+1}x_i\right)\sum_{i=0}^{n+1}k_i\frac{\partial}{\partial x_i}\Psi_n(t,\overrightarrow{x}) \\ &\quad + \frac{\delta^2}{2}\left(\sum_{i=0}^{n+1}x_i\right)^2\sum_{i,j=0}^{n+1}k_ik_j\frac{\partial^2}{\partial x_i\partial x_j}\Psi_n(t,\overrightarrow{x}) \\ &\quad + g_n\left(t,\varphi_s^{(n)},\overrightarrow{x},\Psi\right), \quad \forall(t,\overrightarrow{x})\in[0,T)\times\mathfrak{R}^{n+2}, \\ &\quad \Psi(T,\overrightarrow{x}(T)) = f_n(\overrightarrow{x}(T)). \end{aligned} \tag{5.60}$$

The $\overrightarrow{x}(T)$ consists of the first $n+2$ coefficients of S_T in the set of functions $\{f_i\}$. By the Feynman-Kac theorem (see [15, Theorem 5.7.6]),

$$\Psi_n(t,\overrightarrow{x}) = e^{-r(T-t)}\widetilde{\mathbf{E}}\left[f(\overrightarrow{x}(T)) \mid \overrightarrow{x}(t)=\overrightarrow{x}\right] + \int_t^T g_n\left(s,\varphi_s^{(n)},\overrightarrow{x}(s),\Psi\right)e^{-r(s-t)}ds, \tag{5.61}$$

where $\overrightarrow{x}(t)$ is the solution to

$$dx_i(t) = b_i(t,\overrightarrow{x}(t))dt + \sum_{j=0}^{n+1}\sigma_{ij}(t,\overrightarrow{x}(t))d\widetilde{W}^{(j)}(t) \tag{5.62}$$

for $i = 0,1,\ldots,n+1$. Noting that $\overrightarrow{x}(t) = \overrightarrow{x}$,

$$\begin{aligned} b_i(t,\overrightarrow{x}) &= rk_i\left(\sum_{i=0}^{n+1}x_i\right), \\ \delta^2k_ik_j\left(\sum_{i=0}^{n+1}x_i\right)^2 &= \sum_{k=0}^{n+1}\sigma_{ik}(t,\overrightarrow{x})\sigma_{jk}(t,\overrightarrow{x}) \end{aligned} \tag{5.63}$$

with $0\le i,j\le n+1$. Hence,

$$\overrightarrow{b}(t,\overrightarrow{x}) = r\begin{bmatrix} k_0\left(\sum_{i=0}^{n+1}x_i\right) \\ k_1\left(\sum_{i=0}^{n+1}x_i\right) \\ \vdots \\ k_{n+1}\left(\sum_{i=0}^{n+1}x_i\right) \end{bmatrix}, \tag{5.64}$$

and so

$$\overrightarrow{b}(t,\overrightarrow{x}) = rB\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n+1} \end{bmatrix}. \tag{5.65}$$

Also,

$$\sigma(t,\overrightarrow{x})\sigma^{\top}(t,\overrightarrow{x}) = \delta^2\left(\sum_{i=0}^{n+1} x_i\right)^2 \begin{bmatrix} k_0^2 & k_0k_1 & \cdots & k_0k_{n+1} \\ k_0k_1 & k_1^2 & \cdots & k_1k_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ k_0k_{n+1} & k_1k_{n+1} & \cdots & k_{n+1}^2 \end{bmatrix}. \tag{5.66}$$

Therefore,

$$\sigma(t,\overrightarrow{x}) = \delta B\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} \tag{5.67}$$

as well. Thus,

$$d\overrightarrow{x}(t) = rB\overrightarrow{x}(t)dt + \delta B\overrightarrow{x}(t)d\widetilde{W}(t) \tag{5.68}$$

which has the solution

$$\overrightarrow{x}(t) = \exp\left[\left(rB - \frac{\delta^2}{2}B^2\right)t + \delta B\widetilde{W}(t)\right]\overrightarrow{x}(0). \tag{5.69}$$

Notice that $\overrightarrow{b}(t,\overrightarrow{x})$ and $\sigma(t,\overrightarrow{x})$ satisfy the conditions necessary for applying the Feynman-Kac formula. Also note that (5.58) satisfies the polynomial growth condition

$$\max_{0\le t\le T}\left|\Psi_n(t,\overrightarrow{x})\right| \le M\left(1 + \left\|\overrightarrow{x}\right\|^{2\mu}\right) \tag{5.70}$$

for $M > 0$ and $\mu \ge 1$ due to the boundedness of f_n and g_n for n sufficiently large. Therefore, by (5.61), equation (5.58) follows. □

Lemma 5.6. *Let* $A : [0,T] \times \mathfrak{R} \to \mathfrak{R}^{(n+2)\times(n+2)}$ *be defined by*

$$A(t,y) = \left(rB - \frac{\delta^2}{2}B^2\right)(T-t) + \delta By\sqrt{T-t}, \tag{5.71}$$

where

$$B = \begin{bmatrix} k_0 & k_0 & \cdots & k_0 \\ k_1 & k_1 & \cdots & k_1 \\ \vdots & \vdots & \vdots & \vdots \\ k_{n+1} & k_{n+1} & \cdots & k_{n+1} \end{bmatrix}. \tag{5.72}$$

Then $A(t,y) = QD(t,y)Q^{-1}$ *where* Q *is the* $(n+2)\times(n+2)$ *matrix defined by*

$$Q = \begin{bmatrix} \frac{k_0}{\alpha} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \frac{k_1}{\alpha} & 0 & 0 & \cdots & -\frac{1}{\sqrt{2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{k_{n+1}}{\alpha} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \end{bmatrix},$$

$$Q^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{\sqrt{2}k_{n+1}}{\alpha} & \frac{\sqrt{2}k_{n+1}}{\alpha} & \frac{\sqrt{2}k_{n+1}}{\alpha} & \cdots & -\sqrt{2}\frac{\alpha - k_{n+1}}{\alpha} \\ \frac{\sqrt{2}k_n}{\alpha} & \frac{\sqrt{2}k_n}{\alpha} & \frac{\sqrt{2}k_n}{\alpha} & \cdots & \frac{\sqrt{2}k_n}{\alpha} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\sqrt{2}k_1}{\alpha} & -\sqrt{2}\frac{\alpha - k_1}{\alpha} & \frac{\sqrt{2}k_1}{\alpha} & \cdots & \frac{\sqrt{2}k_1}{\alpha} \end{bmatrix}, \tag{5.73}$$

$$D(t,y) = \begin{bmatrix} \left(r - \frac{\delta^2}{2}\alpha\right)\alpha(T-t) + \delta\alpha y\sqrt{T-t} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Here, $\alpha = \sum_{i=0}^{n+1} k_i$.

Proof. It is clear that $B^2 = \alpha B$, so

$$A(t,y) = \left[\left(r - \frac{\delta^2}{2}\alpha\right)(T-t) + \delta y\sqrt{T-t}\right]B. \tag{5.74}$$

The matrix $A(t,y)$ is an $(n+2)\times(n+2)$ matrix with $n+2$ eigenvalues. The $n+1$ eigenvectors associated with the eigenvalue 0 are

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{5.75}$$

and so forth. The remaining eigenvalue is

$$\left[\left(r - \frac{\delta^2}{2}\alpha\right)(T-t) + \delta y\sqrt{T-t}\right]\alpha \tag{5.76}$$

with the associated eigenvector of

$$\frac{1}{\alpha}\begin{bmatrix} k_0 \\ k_1 \\ k_2 \\ \vdots \\ k_{n+1} \end{bmatrix}. \tag{5.77}$$

The matrix Q is simply a matrix with these eigenvectors for columns, the expression for Q^{-1} can easily be verified, and $D(t,y) = Q^{-1}A(t,y)Q$. □

Next, we will derive an approximate solution to (5.44) with the auxiliary condition being that for the standard European call option. By using Theorem 5.2, a significant piece of the approximate solution is then known.

Proposition 5.7. *Let*

$$r\Psi(t,\varphi) = \frac{\partial}{\partial t}\Psi(t,\varphi) + \overline{D\Psi(t,\varphi)}\left(r\varphi(0)1_{\{0\}}\right) + \frac{1}{2}\overline{D^2\Psi(t,\varphi)}\left(N(\varphi)1_{\{0\}}, N(\varphi)1_{\{0\}}\right), \quad \forall(t,\varphi) \in [0,T)\times \mathbf{C}_+, \tag{5.78}$$

where

$$\Psi(T,\varphi) = (\varphi(0)-K)^+ \quad \forall\varphi \in \mathbf{C}_+, \tag{5.79}$$

and K is the strike price of the option contract. Then

$$\Psi_n(t,\overrightarrow{x}) = e^{-r(T-t)}\left\{\frac{k_{n+1}}{\alpha}\exp[r\alpha(T-t)]\sum_{i=0}^{n+1}x_i\left(\Phi\left(\delta\alpha\sqrt{T-t}-\Upsilon_0\right)\right) - K\Phi(-\Upsilon_0)\right\} + \int_t^T g_n\left(s,\varphi_s^{(n)},\overrightarrow{x}(s),\Psi\right)e^{-r(s-t)}ds, \tag{5.80}$$

where $\Psi_n(t,\overrightarrow{x}) \to \Psi(t,\varphi)$ pointwise as $n \to \infty$. Here,

$$\Phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-y^2/2}dy, \qquad \Upsilon_0 = \frac{\ln\left[\alpha K/k_{n+1}\sum_{i=0}^{n+1}x_i\right]}{\delta\alpha\sqrt{T-t}} - \left(\frac{r}{\delta}-\frac{\delta\alpha}{2}\right)\sqrt{T-t}. \tag{5.81}$$

Proof. By Theorem 5.5, the equation

$$r\Psi\left(t,\varphi^{(n)}\right) = \frac{\partial}{\partial t}\Psi\left(t,\varphi^{(n)}\right) + \overline{D\Psi\left(t,\varphi^{(n)}\right)}\left(r\varphi^{(n)}(0)\mathbf{1}_{\{0\}}\right) + \frac{1}{2}\overline{D^2\Psi\left(t,\varphi^{(n)}\right)}\left(N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}, N\left(\varphi^{(n)}\right)\mathbf{1}_{\{0\}}\right), \quad \forall\left(t,\varphi^{(n)}\right) \in [0,T)\times \mathbf{C}_n, \tag{5.82}$$

with

$$\Psi\left(T,\varphi^{(n)}\right) = f\left(\varphi^{(n)}\right) \quad \forall\varphi^{(n)} \in \mathbf{C}_n, \tag{5.83}$$

and f being a $C^2(\mathbf{C})$ function has the solution

$$\Psi_n(t,\overrightarrow{x}) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_n\left(\exp\left[\left(rB-\frac{\delta^2}{2}B^2\right)(T-t)+\delta By\sqrt{T-t}\right]\overrightarrow{x}\right)e^{-y^2/2}dy + \int_t^T g_n\left(s,\varphi_s^{(n)},\overrightarrow{x}(s),\Psi\right)e^{-r(s-t)}ds. \tag{5.84}$$

By Lemma 5.6,

$$A(t,y) = \left(rB - \frac{\delta^2}{2}B^2\right)(T-t) + \delta By\sqrt{T-t} \tag{5.85}$$

may be expressed as $A(t,y) = QD(t,y)Q^{-1}$ where Q, Q^{-1}, and $D(t,y)$ are defined in the lemma. Since $D(t,y)$ is diagonal, $e^{D(t,y)}$ is straightforward to find and $e^{A(t,y)} = Qe^{D(t,y)}Q^{-1}$. Let $\overrightarrow{v}(t,y)$ be the $(n+2)$th row of $e^{A(t,y)}$, expressed as a column vector. Then

$$\overrightarrow{v}(t,y) = \frac{k_{n+1}}{\alpha}\exp\left[\left(r - \frac{\delta^2}{2}\alpha\right)\alpha(T-t) + \delta\alpha y\sqrt{T-t}\right]\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}. \tag{5.86}$$

Let $u_{n+1}(t,y,\overrightarrow{x})$ be the $(n+2)$th component of $e^{A(t,y)}\overrightarrow{x}$, we have that

$$\begin{aligned} u_{n+1}(t,y,\overrightarrow{x}) &= \overrightarrow{v}(t,y)\cdot\overrightarrow{x} \\ &= \frac{k_{n+1}}{\alpha}\left(\exp\left[\left(r - \frac{\delta^2}{2}\alpha\right)\alpha(T-t) + \delta\alpha y\sqrt{T-t}\right]\right)\sum_{i=0}^{n+1}x_i. \end{aligned} \tag{5.87}$$

Therefore,

$$\begin{aligned} \Psi_n(t,\overrightarrow{x}) = {} & \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{Y_0}^{\infty}\left[u_{n+1}(t,y,\overrightarrow{x}) - K\right]e^{-y^2/2}dy \\ & + \int_t^T g_n\left(s,\varphi_s^{(n)},\overrightarrow{x}(s),\Psi\right)e^{-r(s-t)}ds \end{aligned} \tag{5.88}$$

with

$$\frac{k_{n+1}}{\alpha}\left(\exp\left[\left(r - \frac{\delta^2}{2}\alpha\right)\alpha(T-t) + \delta\alpha Y_0\sqrt{T-t}\right]\right)\sum_{i=0}^{n+1}x_i = K \tag{5.89}$$

since $[u_{n+1}(t,y,\overrightarrow{x}) - K]^+$ is C^2 in $\overrightarrow{x}$ on the interval (Y_0,∞), with the necessary Lipschitz properties. Solving for Y_0 we have

$$Y_0 = \frac{\ln\left[\alpha K/k_{n+1}\sum_{i=0}^{n+1}x_i\right]}{\delta\alpha\sqrt{T-t}} - \left(\frac{r}{\delta} - \frac{\delta\alpha}{2}\right)\sqrt{T-t}. \tag{5.90}$$

So,

$$\begin{aligned}
\Psi_n(t,\vec{x}) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\Upsilon_0}^{\infty}\left[\frac{k_{n+1}}{\alpha}\exp\left[\left(r-\frac{\delta^2}{2}\alpha\right)\alpha(T-t)+\delta\alpha y\sqrt{T-t}\right]\sum_{i=0}^{n+1}x_i-K\right]e^{-y^2/2}dy \\
&\quad+\int_t^T g_n\left(s,\varphi_s^{(n)},\vec{x}(s),\Psi\right)e^{-r(s-t)}ds \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\Upsilon_0}^{\infty}\left[\frac{k_{n+1}}{\alpha}\exp\left[\left(r-\frac{\delta^2}{2}\alpha\right)\alpha(T-t)+\delta\alpha y\sqrt{T-t}\right]\sum_{i=0}^{n+1}x_i\right]e^{-y^2/2}dy \\
&\quad-Ke^{-r(T-t)}(1-\Phi(\Upsilon_0))+\int_t^T g_n\left(s,\varphi_s^{(n)},\vec{x}(s),\Psi\right)e^{-r(s-t)}ds \\
&= e^{-r(T-t)}\left\{\frac{k_{n+1}}{\alpha}\exp\left[\left(r-\frac{\delta^2}{2}\alpha\right)\alpha(T-t)\right]\right. \\
&\qquad\left.\times\sum_{i=0}^{n+1}x_i\frac{1}{\sqrt{2\pi}}\int_{\Upsilon_0}^{\infty}e^{-y^2/2+\delta\alpha y\sqrt{T-t}}dy-K(1-\Phi(\Upsilon_0))\right\} \\
&\quad+\int_t^T g_n\left(s,\varphi_s^{(n)},\vec{x}(s),\Psi\right)e^{-r(s-t)}ds \\
&= e^{-r(T-t)}\left\{(1-\Phi(\Upsilon_0))(-K)+\frac{k_{n+1}}{\alpha}\exp\left[\left(r-\frac{\delta^2}{2}\alpha\right)\alpha(T-t)\right]\right. \\
&\qquad\left.\times\sum_{i=0}^{n+1}x_ie^{\delta^2\alpha^2(T-t)/2}\left(1-\Phi\left(\Upsilon_0-\delta\alpha\sqrt{T-t}\right)\right)\right\} \\
&\quad+\int_t^T g_n\left(s,\varphi_s^{(n)},\vec{x}(s),\Psi\right)e^{-r(s-t)}ds \\
&= e^{-r(T-t)}\left\{\frac{k_{n+1}}{\alpha}\exp[r\alpha(T-t)]\sum_{i=0}^{n+1}x_i\left(\Phi\left(\delta\alpha\sqrt{T-t}-\Upsilon_0\right)\right)-K\Phi(-\Upsilon_0)\right\} \\
&\quad+\int_t^T g_n\left(s,\varphi_s^{(n)},\vec{x}(s),\Psi\right)e^{-r(s-t)}ds.
\end{aligned} \tag{5.91}$$

This result is the solution given $\varphi^{(n)}\in\mathbf{C}_n$. Notice that

$$g_n\left(s,\varphi_s^{(n)},\vec{x}(s),\Psi\right)\longrightarrow 0 \tag{5.92}$$

pointwise as $n\to\infty$ since

$$\lim_{n\to\infty}\overline{D\Psi\left(t,\varphi^{(n)}\right)}\left(\mathbf{1}_{\{0\}}\right)=\lim_{n\to\infty}D\Psi_n(t,\vec{x})(v_n)=\lim_{n\to\infty}\sum_{i=0}^{n+1}k_i\frac{\partial}{\partial x_i}\Psi_n(t,\vec{x}), \tag{5.93}$$

and similarly for the second derivatives. By dominated convergence, we have that

$$\int_t^T g_n\left(s, \varphi_s^{(n)}, \overrightarrow{x}(s), \Psi\right) e^{-r(s-t)} ds \longrightarrow 0 \tag{5.94}$$

pointwise as $n \to \infty$. By Remark 5.3, we have that $\Psi_n(t, \overrightarrow{x}) \to \Psi(t, \varphi)$ pointwise as $n \to \infty$. □

6. Summary and Conclusions

In this paper, we have continued [6] by deriving an infinite-dimensional Black-Scholes equation for the European option problem, where the (B, S)-market model is given by (2.7) and (2.8). The resulting deterministic partial differential equation is a new type of equation, one where the partial differentiation contains extended Fréchet derivatives. Given the (B, S)-market model equations, a spanning set for the space of square-integrable function is developed which simplifies finding an approximate solution to this equation. The solution method detail in this paper consists of the following steps.

Step 1. Given r and $N(\cdot)$, use (5.6)-(5.7) with (5.5) and (5.10)-(5.11) to find the spanning set $\{f_i\}_{i=0}^{n+1}$ for n sufficiently large. Coefficients in this spanning set of functions are found using (5.13)–(5.15).

Step 2. Use Theorem 5.5 or Proposition 5.7, depending on the reward function, to find $\Psi_n(t, \overrightarrow{x})$. The term

$$\int_t^T g_n\left(s, \varphi_s^{(n)}, \overrightarrow{x}(s), \Psi\right) e^{-r(s-t)} ds \tag{6.1}$$

approaches zero as n approaches infinity, so this term may be assumed small for sufficiently large n. The vector $\overrightarrow{x}$ is found from (5.69).

Step 3. Having found $\Psi_n(t, \overrightarrow{x})$, solve (5.43) for $w_n(t, 0)$, then $w_n(t, 0)\Psi_n(t, \overrightarrow{x})$ is an approximate solution to the generalized Black-Scholes equation. By Corollary 5.4, $w_n(t, 0)\Psi_n(t, \overrightarrow{x}) \to \Psi(t, \varphi)$ pointwise as $n \to \infty$.

Appendix

Proof of Proposition 3.2

In this appendix, we prove Proposition 3.2, which is stated again as follows.

Proposition 3.2. *Let $\varphi \in \mathbf{C}$ and $f : \mathbf{C} \to \mathfrak{R}$. Further assume $f \in C^2(\mathbf{C})$ and let $\Psi : [0, T] \times \mathbf{C} \to \mathfrak{R}$ be defined by*

$$\Psi(t, \varphi) = e^{-r(T-t)} \widetilde{E}\left[f(S_T) \mid S_t = \varphi\right]. \tag{A.1}$$

Then if Df and $D^2 f$ are globally Lipschitz, then so is $D^2\Psi$.

Before proving the proposition, an additional result for Fréchet derivatives is needed.

Lemma A.1. *Let $H : [0,T] \times \mathfrak{R} \times \mathbf{C} \to \mathbf{C}$ and $f : \mathbf{C} \to \mathfrak{R}$. Define $g : [0,T] \times R \times \mathbf{C} \to \mathfrak{R}$ by $g(t,y,\varphi) = (f \circ H)(t,y,\varphi)$ for any $t \in [0,T]$, $y \in \mathfrak{R}$, and $\varphi \in \mathbf{C}$. Assume that f has a second Fréchet derivative and likewise for H (with respect to the third variable.) Then*

$$\begin{aligned} D^2 g(t,y,\varphi)(\phi,\phi) &= D^2 f(H(t,y,\varphi))(DH(t,y,\varphi)(\phi), DH(t,y,\varphi+\phi)(\phi)) \\ &\quad + Df(H(t,y,\varphi))\Big(D^2 H(t,y,\phi)(\phi,\phi)\Big). \end{aligned} \tag{A.2}$$

Proof. We start by considering

$$\begin{aligned} &Dg(t,y,\varphi+\phi)(\phi) - Dg(t,y,\varphi)(\phi) \\ &\quad = Df(H(t,y,\varphi+\phi))(DH(t,y,\varphi+\phi)(\phi)) - Df(H(t,y,\varphi))(DH(t,y,\varphi)(\phi)) \\ &\quad = Df(H(t,y,\varphi+\phi))(DH(t,y,\varphi+\phi)(\phi)) - Df(H(t,y,\varphi))(DH(t,y,\varphi+\phi)(\phi)) \\ &\qquad + Df(H(t,y,\varphi))(DH(t,y,\varphi+\phi)(\phi)) - Df(H(t,y,\varphi))(DH(t,y,\varphi)(\phi)). \end{aligned} \tag{A.3}$$

From here we have that

$$\begin{aligned} &Df(H(t,y,\varphi+\phi))(DH(t,y,\varphi+\phi)(\phi)) \\ &\qquad - Df(H(t,y,\varphi)(DH(t,y,\varphi+\phi)(\phi))) \\ &\quad = D^2 f(H(t,y,\varphi))(H(t,y,\varphi+\phi) - H(t,y,\varphi), DH(t,y,\varphi+\phi)(\phi)) \\ &\qquad + o_1(H(t,y,\varphi+\phi) - H(t,y,\varphi), DH(t,y,\varphi+\phi)(\phi)) \\ &\quad = D^2 f(H(t,y,\varphi))(DH(t,y,\varphi)(\phi), DH(t,y,\varphi+\phi)(\phi)) \\ &\qquad + D^2 f(H(t,y,\varphi))(o_2(\phi), DH(t,y,\varphi+\phi)(\phi)) \\ &\qquad + o_1(H(t,y,\varphi+\phi) - H(t,y,\varphi), DH(t,y,\varphi+\phi)(\phi)) \\ &\quad = D^2 f(H(t,y,\varphi))(DH(t,y,\varphi)(\phi), DH(t,y,\varphi+\phi)(\phi)) \\ &\qquad + D^2 f(H(t,y,\varphi))(o_2(\phi), DH(t,y,\varphi+\phi)(\phi)) \\ &\qquad + o_1(DH(t,y,\varphi)(\phi) + o_3(\phi), DH(t,y,\varphi+\phi)(\phi)). \end{aligned} \tag{A.4}$$

Also,

$$\begin{aligned} &Df(H(t,y,\varphi))(DH(t,y,\varphi+\phi)(\phi) - DH(t,y,\varphi)(\phi)) \\ &\quad = Df(H(t,y,\varphi))\Big(D^2(H(t,y,\varphi)(\phi,\phi)) + o_4(\phi,\phi)\Big). \end{aligned} \tag{A.5}$$

Clearly,

$$D^2 f(H(t,y,\varphi))(o_2(\phi), DH(t,y,\varphi+\phi)(\phi)) = o_5(\phi,\phi), \tag{A.6}$$

since the second derivative is linear. Also

$$\frac{o_1(DH(t,y,\varphi)(\phi) + o_3(\phi), DH(t,y,\varphi+\phi)(\phi))}{\|DH(t,y,\varphi+\phi)(\phi)\|} \longrightarrow 0 \tag{A.7}$$

as $\|DH(t,y,\varphi+\phi)(\phi)\| \to 0$ and

$$\begin{aligned} \frac{\|DH(t,y,\varphi+\phi)(\phi)\|}{\|\phi\|} &\leq \frac{\|DH(t,y,\varphi+\phi)\| \cdot \|\phi\|}{\|\phi\|} \\ &= \|DH(t,y,\varphi+\phi)(\phi)\| \\ &\leq K_1 \end{aligned} \tag{A.8}$$

for some $K_1 < \infty$. Furthermore,

$$\frac{o_1(DH(t,y,\varphi)(\phi) + o_3(\phi), DH(t,y,\varphi+\phi)(\phi))}{\|DH(t,y,\varphi+\phi)(\phi) + o_3(\phi)\|} \longrightarrow 0 \tag{A.9}$$

as $\|DH(t,y,\varphi+\phi)(\phi) + o_3(\phi)\| \to 0$, and

$$\begin{aligned} &\frac{\|DH(t,y,\varphi+\phi)(\phi) + o_3(\phi)\|}{\|\phi\|} \\ &\quad \leq \frac{\|DH(t,y,\varphi+\phi)\| \cdot \|\phi\|}{\|\phi\|} + \frac{\|o_3(\phi)\|}{\|\phi\|} \\ &\quad = \|DH(t,y,\varphi+\phi)\| + \frac{\|o_3(\phi)\|}{\|\phi\|} \\ &\quad \leq K_2 + \frac{\|o_3(\phi)\|}{\|\phi\|} \end{aligned} \tag{A.10}$$

for some $K_2 < \infty$. Therefore

$$o_1(DH(t,y,\varphi)(\phi) + o_3(\phi), DH(t,y,\varphi+\phi)(\phi)) = o_6(\phi,\phi). \tag{A.11}$$

Finally,

$$\begin{aligned} Df(H(t,y,\varphi))(o_4(\phi,\phi)) &\leq \|Df(H(t,y,\varphi))\| \cdot \|o_4(\phi,\phi)\| \\ &\leq K_3\|o_4(\phi,\phi)\| \end{aligned} \tag{A.12}$$

for some $K_3 < \infty$, so

$$Df(H(t,y,\varphi))(o_4(\phi,\phi)) = o_7(\phi,\phi). \tag{A.13}$$

Since

$$\begin{aligned} &D^2 f(H(t,y,\varphi))(DH(t,y,\varphi)(\phi), DH(t,y,\varphi+\phi)(\phi)) \\ &\quad + Df(H(t,y,\varphi))\left(D^2 H(t,y,\phi)(\phi,\phi)\right) \end{aligned} \tag{A.14}$$

is bounded and linear, we are done. □

Proof of Proposition 3.2. From Proposition 3.1, we have that

$$\begin{aligned} \Psi(t,\varphi) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(H(T-t,y,\varphi)) e^{-y^2/2} dy \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(T-t,y,\varphi) e^{-y^2/2} dy, \end{aligned} \tag{A.15}$$

where $H : [0,T] \times \mathfrak{R} \times \mathbf{C} \to \mathbf{C}$ is defined by $S_t = H(t, \widetilde{W}(t), S_0)$, and $g : [0,T] \times \mathfrak{R} \times \mathbf{C} \to \mathfrak{R}$ is defined by $g(t, \widetilde{W}(t), S_0) = (f \circ H)(t, \widetilde{W}(t), S_0)$. In the same proposition, we see that

$$\begin{aligned} D\Psi(t,\varphi)(\phi) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Dg(T-t,y,\varphi)(\phi) e^{-y^2/2} dy, \\ D^2\Psi(t,\varphi)(\phi,\phi) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} D^2 g(T-t,y,\varphi)(\phi,\phi) e^{-y^2/2} dy. \end{aligned} \tag{A.16}$$

If $D^2 g(T-t,y,\varphi)$ is globally Lipschitz, then it is clear that $D^2\Psi(t,\varphi)$ is also globally Lipschitz. We will now look at $D^2 g(T-t,y,\varphi)(\phi,\phi)$.

By Lemma A.1, we have

$$
\begin{aligned}
&D^2 g(T-t,y,\varphi)(\phi,\phi)\\
&\quad = D^2 f(H(T-t,y,\varphi))(DH(T-t,y,\varphi)(\phi), DH(T-t,y,\varphi+\phi)(\phi))\\
&\qquad + Df(H(T-t,y,\varphi))\left(D^2 H(T-t,y,\phi)(\phi,\phi)\right),\\
&\left\|D^2 g(T-t,y,\varphi+\psi) - D^2 g(T-t,y,\varphi)\right\|\\
&\quad \le \left\|D^2 f(H(T-t,y,\varphi+\psi))(DH(T-t,y,\varphi+\psi), DH(T-t,y,\varphi+\phi+\psi))\right.\\
&\qquad \left. -D^2 f(H(T-t,y,\varphi))(DH(T-t,y,\varphi), DH(T-t,y,\varphi+\phi))\right\|\\
&\qquad + \left\|Df(H(T-t,y,\varphi+\psi))\left(D^2 H(T-t,y,\varphi+\psi)\right)\right.\\
&\qquad\quad \left. -Df(H(T-t,y,\varphi))\left(D^2 H(T-t,y,\varphi)\right)\right\|\\
&\quad = \left\|D^2 f(H(T-t,y,\varphi+\psi))(DH(T-t,y,\varphi+\psi), DH(T-t,y,\varphi+\phi+\psi))\right.\\
&\qquad - D^2 f(H(T-t,y,\varphi))(DH(T-t,y,\varphi+\psi), DH(T-t,y,\varphi+\phi+\psi))\\
&\qquad + D^2 f(H(T-t,y,\varphi))(DH(T-t,y,\varphi+\psi), DH(T-t,y,\varphi+\phi+\psi))\\
&\qquad \left. -D^2 f(H(T-t,y,\varphi))(DH(T-t,y,\varphi), DH(T-t,y,\varphi+\phi))\right\|\\
&\qquad + \left\|Df(H(T-t,y,\varphi+\psi))\left(D^2 H(T-t,y,\varphi+\psi)\right)\right.\\
&\qquad\quad - Df(H(T-t,y,\varphi))\left(D^2 H(T-t,y,\varphi+\psi)\right)\\
&\qquad\quad + Df(H(T-t,y,\varphi))\left(D^2 H(T-t,y,\varphi+\psi)\right)\\
&\qquad\quad \left. -Df(H(T-t,y,\varphi))\left(D^2 H(T-t,y,\varphi)\right)\right\|\\
&\quad \le \left\|D^2 f(H(T-t,y,\varphi+\psi))(DH(T-t,y,\varphi+\psi), DH(T-t,y,\varphi+\phi+\psi))\right.\\
&\qquad \left. -D^2 f(H(T-t,y,\varphi))(DH(T-t,y,\varphi+\psi), DH(T-t,y,\varphi+\phi+\psi))\right\|\\
&\qquad + \left\|D^2 f(H(T-t,y,\varphi))(DH(T-t,y,\varphi+\psi), DH(T-t,y,\varphi+\phi+\psi))\right.\\
&\qquad\quad \left. -D^2 f(H(T-t,y,\varphi))(DH(T-t,y,\varphi), DH(T-t,y,\varphi+\phi))\right\|\\
&\qquad + \left\|Df(H(T-t,y,\varphi+\psi))\left(D^2 H(T-t,y,\varphi+\psi)\right)\right.\\
&\qquad\quad \left. -Df(H(T-t,y,\varphi))\left(D^2 H(T-t,y,\varphi+\psi)\right)\right\|\\
&\qquad + \left\|Df(H(T-t,y,\varphi))\left(D^2 H(T-t,y,\varphi+\psi)\right) - Df(H(T-t,y,\varphi))\left(D^2 H(T-t,y,\varphi)\right)\right\|\\
&\quad \le \left\|D^2 f(H(T-t,y,\varphi+\psi)) - D^2 f(H(T-t,y,\varphi))\right\|
\end{aligned}
$$

$$\begin{aligned}
&\times \left\|DH(T-t,y,\varphi+\phi+\psi)\right\| \cdot \left\|DH(T-t,y,\varphi+\psi)\right\| + \left\|D^2 f(H(T-t,y,\varphi))\right\| \\
&\times \left\|DH(T-t,y,\varphi+\phi+\psi) - DH(T-t,y,\varphi+\phi)\right\| \\
&\times \left\|DH(T-t,y,\varphi+\psi) - DH(T-t,y,\varphi)\right\| \\
&+ \left\|Df(H(T-t,y,\varphi+\psi)) - Df(H(T-t,y,\varphi))\right\| \left\|D^2 H(T-t,y,\varphi+\psi)\right\| \\
&+ \left\|Df(H(T-t,y,\varphi))\right\| \cdot \left\|D^2 H(T-t,y,\varphi+\psi) - D^2 H(T-t,y,\varphi)\right\|.
\end{aligned} \tag{A.17}$$

Taking the terms one at a time,

$$\begin{aligned}
&\left\|D^2 f(H(T-t,y,\varphi+\psi)) - D^2 f(H(T-t,y,\varphi))\right\| \\
&\qquad \times \left\|DH(T-t,y,\varphi+\phi+\psi)\right\| \cdot \left\|DH(T-t,y,\varphi+\psi)\right\| \\
&\qquad \le \tilde{C}_1 \|\psi\|,
\end{aligned} \tag{A.18}$$

since $D^2 f$ is globally Lipschitz and there is some $C_1 < \infty$ such that $\|DH(T-t,y,\varphi+\phi+\psi)\| \le C_1$ and likewise for $\|DH(T-t,y,\varphi+\phi+\psi)\|$.

For the next term,

$$\begin{aligned}
&\left\|D^2 f(H(T-t,y,\varphi))\right\| \\
&\qquad \times \left\|DH(T-t,y,\varphi+\phi+\psi) - DH(T-t,y,\varphi+\phi)\right\| \\
&\qquad \times \left\|DH(T-t,y,\varphi+\psi) - DH(T-t,y,\varphi)\right\| \\
&\qquad \le C_2 \left\|DH(T-t,y,\varphi+\phi+\psi) - DH(T-t,y,\varphi+\phi)\right\| \\
&\qquad \times \left\|DH(T-t,y,\varphi+\psi) - DH(T-t,y,\varphi)\right\|.
\end{aligned} \tag{A.19}$$

But $H(T-t,y,\varphi)$ is linear in φ as we will now show.

Again using the operator define by $(\cdot)_n : \mathbf{C} \to \mathbf{C}_n$ with

$$(\varphi)_n = \sum_{i=0}^{n+1} x_i f_i, \tag{A.20}$$

where the right-hand side is the first $n+2$ terms of the $\{f_i\}$-expansion of φ, we have that

$$(S_t)_n = e_n^{-1}(\overrightarrow{x}(t)), \tag{A.21}$$

where e_n^{-1} is linear. Recall that

$$\begin{aligned}\overrightarrow{x}(t) &= \exp\left[\left(rB - \frac{\delta^2}{2}B^2\right)t + \delta B\widetilde{W}(t)\right]\overrightarrow{x}(0)\\ &= A\left(t, \widetilde{W}(t)\right)\overrightarrow{x}(0),\end{aligned} \tag{A.22}$$

where $\overrightarrow{x}(0) = (S_0)_n$. Therefore,

$$\overrightarrow{x}(t) = \begin{bmatrix} \sum_{i=0}^{n+1} A_{0,i}\left(t, \widetilde{W}(t)\right)x_i \\ \sum_{i=0}^{n+1} A_{1,i}\left(t, \widetilde{W}(t)\right)x_i \\ \sum_{i=0}^{n+1} A_{2,i}\left(t, \widetilde{W}(t)\right)x_i \\ \vdots \\ \sum_{i=0}^{n+1} A_{n+2,i}\left(t, \widetilde{W}(t)\right)x_i \end{bmatrix}. \tag{A.23}$$

So,

$$S_t = \lim_{n\to\infty}\sum_{i=0}^{n}\left(\lim_{k\to\infty}\sum_{j=0}^{k} A_{i,j}\left(t, \widetilde{W}(t)\right)x_j\right)f_i \tag{A.24}$$

is linear in $S_0 = \lim_{n\to\infty}\sum_{i=0}^{n} x_i f_i$.

Since $H(T-t, y, \varphi)$ is linear in φ,

$$\begin{aligned}\left\|DH(T-t, y, \varphi+\phi+\psi) - DH(T-t, y, \varphi+\phi)\right\| &\le C_3\|\psi\|,\\ \left\|DH(T-t, y, \varphi+\psi) - DH(T-t, y, \varphi)\right\| &\le C_4\|\psi\|.\end{aligned} \tag{A.25}$$

Since $\|\psi\| \le M < \infty$, we have that

$$\begin{aligned}&\left\|D^2 f(H(T-t, y, \varphi))\right\| \cdot \left\|DH(T-t, y, \varphi+\phi+\psi) - DH(T-t, y, \varphi+\phi)\right\|\\ &\quad\times \left\|DH(T-t, y, \varphi+\psi) - DH(T-t, y, \varphi)\right\| \le C_5\|\psi\|.\end{aligned} \tag{A.26}$$

We also have that

$$\left\|Df(H(T-t, y, \varphi+\psi)) - Df(H(T-t, y, \varphi))\right\| \cdot \left\|D^2 H(T-t, y, \varphi+\psi)\right\| \le C_6\|\psi\| \tag{A.27}$$

for some $C_6 < \infty$ since Df is globally Lipschitz. Finally,

$$\|Df(H(T-t,y,\varphi))\| \cdot \left\|D^2H(T-t,y,\varphi+\psi) - D^2H(T-t,y,\varphi)\right\| \leq C_7\|\psi\| \tag{A.28}$$

for some $C_7 < \infty$ since $H(T-t,y,\varphi)$ is linear in φ. Combining these, we have that $D^2g(T-t,y,\varphi)$ is globally Lipschitz and therefore so is $D^2\Psi(t,\varphi)$. □

References

[1] J. M. Harrison and D. M. Kreps, "Martingales and arbitrage in multiperiod securities markets," *Journal of Economic Theory*, vol. 20, no. 3, pp. 381–408, 1979.

[2] J. M. Harrison and S. R. Pliska, "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and Their Applications*, vol. 11, no. 3, pp. 215–260, 1981.

[3] R. C. Merton, "Theory of rational option pricing," *Bell Journal of Economics and Management Science*, vol. 4, no. 1, pp. 141–183, 1973.

[4] R. C. Merton, *Continuous-Time Finance*, Basil Blackwell, Oxford, UK, 1990.

[5] A. N. Shiryaev, Yu. M. Kabanov, D. O. Kramkov, and A. V. Melnikov, "Toward a theory of pricing options of European and American types—II: continuous time," *Theory of Probability and Applications*, vol. 39, no. 1, pp. 61–102, 1994.

[6] M.-H. Chang and R. K. Youree, "The European option with hereditary price structures: basic theory," *Applied Mathematics and Computation*, vol. 102, no. 2-3, pp. 279–296, 1999.

[7] M. Arriojas, Y. Hu, S.-E. A. Mohammed, and G. Pap, "A delayed Black and Scholes formula," *Journal of Stochastic Analysis and Applications*, vol. 25, no. 2, pp. 471–492, 2007.

[8] G. Stoica, "A stochastic delay financial model," *Proceedings of the American Mathematical Society*, vol. 133, no. 6, pp. 1837–1841, 2005.

[9] M. Arriojas, Y. Hu, S.-E. A. Mohammed, and G. Pap, "A delayed Black and Scholes formula II," preprint, 2003.

[10] M.-H. Chang and R. K. Youree, "Infinite-dimensional Black-Scholes equation with hereditary structure," *Applied Mathematics and Optimization*, vol. 56, no. 3, pp. 395–424, 2007.

[11] M. H. Chang, T. Pang, and M. Pemy, "An approximation scheme for Black-Scholes with delays," preprint, 2009.

[12] Y. I. Kazmerchuk, A. V. Swishchuk, and J. H. Wu, "The pricing of options for securities markets with delayed response," *Mathematics and Computers in Simulation*, vol. 75, no. 3-4, pp. 69–79, 2007.

[13] Y. I. Kazmerchuk, A. V. Swishchuk, and J. H. Wu, "Black-Scholes formula revisited: securities markets with delayed response," preprint, 2004.

[14] Y. I. Kazmerchuk, A. V. Swishchuk, and J. H. Wu, "A continuous-time GARCH model for stochastic volatility with delay," *Canadian Applied Mathematics Quarterly*, vol. 13, no. 2, pp. 123–149, 2005.

[15] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, vol. 113 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1991.

[16] S.-E. A. Mohammed, *Stochastic Functional Differential Equations*, vol. 99 of *Research Notes in Mathematics*, Pitman, Boston, Mass, USA, 1984.

[17] S.-E. A. Mohammed, "Stochastic differential systems with memory: theory, examples and applications," in *Stochastic Analysis and Related Topics, 6 (Geilo, 1996)*, vol. 42 of *Progress in Probability*, pp. 1–77, Birkhäuser, Boston, Mass, USA, 1998.

[18] K. D. Elworthy, *Stochastic Differential Equations on Manifolds*, vol. 70 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, UK, 1982.

[19] D. L. Powers, *Boundary Value Problems*, Academic Press, New York, NY, USA, 2nd edition, 1979.

[20] H. F. Weinberger, *A First Course in Partial Differential Equations*, Xerox College Publishing, Lexington, Mass, USA, 1965.