Research Article

Parametric Set-Valued Vector Quasi-Equilibrium Problems

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Two kinds of parametric set-valued vector quasi-equilibrium problems are introduced. The existence of solutions to these problems is studied. The upper and lower semicontinuities of their solution maps with respect to the parameters are investigated.

1. Introduction and Preliminaries

Equilibrium problems are a class of general problems that contains many other problems, such as optimization problems, variational inequality problems, saddle point problems, and complementarity problems, as special cases. Up to now, the main efforts for equilibrium problems have been made for the solution existence; see for example [1–6] and the references therein. A few results have been obtained for properties of solution sets, see [7–12].

Motivated and inspired by works in [1, 5, 8–12], in this paper, we will introduce two kinds of parametric set-valued vector quasi-equilibrium problems and study the solution existence of these problems. In addition, we will investigate the upper and lower semicontinuities of their solution maps with respect to the parameters.

Throughout this paper, let $X, Y$ be real Hausdorff topological vector spaces, $\Lambda, M$ real topological vector spaces, and $A$ a nonempty compact convex subset of $X$. We denote by $\text{co} A, \text{int} A, \partial A$, and $\text{cl} A$ the convex hull, interior, boundary, and closed hull of $A$, respectively. Let $K : A \times M \to 2^X$, $T : A \times \Lambda \to 2^Y$, $F : A \times X \times Y \to 2^Y$, and $C : A \to 2^Y$ be set-valued mappings such that $A \cap K(x, \mu) \neq \emptyset$ for all $x \in A$ and $\mu \in M$ and $C(x)$ be a closed convex pointed cone of $Y$ with $\text{int} C(x) \neq \emptyset$ for each $x \in A$.

The mapping $F$ is said to be $Y \setminus \text{int} C$ quasiconvex of type 2 with respect to $T$ (see [1]) if for any nonempty finite subset $\{y_1, \ldots, y_n\} \subseteq A$ and any $x \in \text{co} \{y_1, \ldots, y_n\}$, there exist
\[ i \in \{1, \ldots, n\} \text{ and } z \in T(x) \text{ such that } F(x, y, z) \subseteq Y \setminus \text{int } C(x). \text{ F is said to be } Y \setminus \text{int } C\text{ quasi
}

\text{convex-like of type 2 with respect to } T \text{ (see [1]) if for any nonempty finite subset } \{y_1, \ldots, y_n\} \subseteq A \text{ and any } x \in \text{co}\{y_1, \ldots, y_n\}, \text{ there exist } i \in \{1, \ldots, n\} \text{ and } z \in T(x) \text{ such that}

\[ F(x, y, z) \cap (Y \setminus \text{int } C(x)) \neq \emptyset. \] (1.1)

Let \( B \) be a nonempty subset of \( X \). A set-valued mapping \( G : B \rightarrow 2^Y \) is said to be

upper semicontinuous (shortly, u.s.c) at \( x_0 \in B \) if for any open set \( V \supseteq G(x_0) \), there exists

an open neighborhood \( U \) of \( x_0 \) such that \( G(x) \subseteq V \) for each \( x \in U \cap B \). \( G \) is said to be u.s.c on \( B \)

if it is u.s.c at each point in \( B \).

The mapping \( G : B \rightarrow 2^Y \) is said to be lower semicontinuous (shortly, l.s.c) at \( x_0 \in B \) if

for each \( y \in G(x_0) \) and any open neighborhood \( V \) of \( y \) there exists an open neighborhood

\( U \) of \( x_0 \) such that \( G(z) \cap V \neq \emptyset \) for each \( z \in U \cap B \), or, equivalently, if for any net \( \{x_{\alpha}\} \) with

\( x_{\alpha} \rightarrow x_0 \) and any \( y \in G(x_0) \), there exists a net \( \{y_{\alpha}\} \) with \( y_{\alpha} \in G(x_{\alpha}) \) for each \( \alpha \) such that

\( y_{\alpha} \rightarrow y \). \( G \) is said to be l.s.c at each point in \( B \).

The mapping \( G : B \rightarrow 2^Y \) is said to be closed at \( x_0 \in B \) if for any net \( \{(x_{\alpha}, y_{\alpha})\} : (x_{\alpha}, y_{\alpha}) \rightarrow (x_0, y_0) \) and \( y_{\alpha} \in G(x_{\alpha}) \) for each \( \alpha \), one has \( y_0 \in G(x_0) \). \( G \) is said to be a closed

set-valued mapping if its graph, denoted by graph\( G \), is a closed set in \( X \times Y \), where \( \text{graph} G = \{(x, y) : x \in B, y \in G(x)\} \). \( G \) is said to have closed values if \( G(x) \) is a closed set for each \( x \in B \).

A set-valued mapping \( G : B \rightarrow 2^B \) is said to be a KKM mapping if for each nonempty

finite subset \( \{x_1, \ldots, x_n\} \) of \( B \), one has \( \text{co}\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i) \).

**Lemma 1.1** (Fan-KKM Theorem). Let \( B \) a nonempty convex subset of \( X \) and \( G : B \rightarrow 2^B \) be a

KKM mapping. If \( G(x) \) is a compact set for every \( x \in B \) and there exists \( x_0 \in B \) such that \( G(x_0) \) is a

compact set, then \( \bigcap_{x \in B} G(x) \neq \emptyset \).

**Lemma 1.2** (see [13]). If a set-valued mapping \( G : X \rightarrow 2^Y \) is u.s.c and has closed values, then it is a

closed set-valued mapping.

**Lemma 1.3** (see [14]). Let the set-valued mapping \( G : X \rightarrow 2^Y \) have a compact value at \( x \). Then \( G \)

is u.s.c at \( x \in X \) if and only if for any nets \( \{x_{\alpha}\} \subseteq X : x_{\alpha} \rightarrow x \) and \( \{y_{\alpha}\} : y_{\alpha} \in G(x_{\alpha}) \) for each \( \alpha \)

there exist \( y \in G(x) \) and a subnet \( \{y_{\beta}\} \) of \( \{y_{\alpha}\} \) such that \( y_{\beta} \rightarrow y \).

For any given parameters \( \lambda \in \Lambda \) and \( \mu \in M \), in this paper, we consider the following

two parametric set-valued vector quasi-equilibrium problems.

**PSVQEP 1.** Find \( x \in A \cap \text{cl } K(x, \mu) \) such that for each \( y \in K(x, \mu) \) there exists \( z \in T(x, \lambda) \)

satisfying

\[ F(x, y, z) \subseteq Y \setminus \text{int } C(x). \] (1.2)

**PSVQEP 2.** Find \( x \in A \cap \text{cl } K(x, \mu) \) such that for each \( y \in K(x, \mu) \) there exists \( z \in T(x, \lambda) \)

satisfying

\[ F(x, y, z) \cap (Y \setminus \text{int } C(x)) = \emptyset. \] (1.3)

We denote their solution sets by \( S_1(\lambda, \mu) \) and \( S_2(\lambda, \mu) \), respectively. Obviously, \( S_1(\lambda, \mu) \subseteq S_2(\lambda, \mu) \).
2. Solution Existence

In this section, we will study the existence of solutions for PSVQEP 1 and PSVQEP 2 without any monotonicity. Since parameters play no role in considering solution existence, for the sake of convenience, we state and prove existence results without parameters. We denote the above problems without parameters by SVQEP1 and SVQEP2, and their solution sets by $S_1$ and $S_2$, respectively.

**Theorem 2.1.** Let

(i) $\text{co } K(x) \subseteq \text{cl } K(x)$ for all $x \in A$,  
(ii) $\{x \in A : y \in K(x)\}$ be an open set,  
(iii) $F$ be $Y \setminus \text{int } C$ quasi convex of type 2 with respect to $T$,  
(iv) $\{y \in A : \exists z \in T(x) \text{ s.t. } F(x,y,z) \subseteq Y \setminus \text{int } C(x)\}$ be a closed set for each $x \in A$. Then (SVQEP1) has at least a solution.

**Proof.** Put $E := \{x \in A : x \in \text{cl } K(x)\}$ and define three set-valued mappings $P : A \to 2^A, H : A \to 2^A$, and $Q : A \to 2^A$ by

$$P(x) = \{y \in A : F(x,y,z) \cap \text{int } C(x) \neq \emptyset, \forall z \in T(x)\}, \quad \forall x \in A,$$

$$H(x) = \begin{cases} 
K(x) \cap P(x), & x \in E, \\
A \cap K(x), & x \in A \setminus E, 
\end{cases} \quad (2.1)$$

$$Q(y) = A \setminus \{x \in A : y \in H(x)\}, \quad \forall y \in A.$$  

Firstly, we show that $Q$ is a KKM mapping. Suppose to the contrary that $Q$ is not a KKM mapping. Then there exist a nonempty finite subset $\{y_1, \ldots, y_n\} \subseteq A$ and a point $\bar{x} = \sum_{j=1}^{n} a_j y_j \in \text{co} \{y_1, \ldots, y_n\}$, where $a_j \geq 0, j = 1, \ldots, n$ and $\sum_{j=1}^{n} a_j = 1$, such that $\bar{x} \notin \bigcup_{j=1}^{n} Q(y_j)$, which implies that $y_j \notin H(\bar{x}), j = 1, \ldots, n$.

If $\bar{x} \in E$, then $F(\bar{x}, y_j, z) \cap \text{int } C(\bar{x}) \neq \emptyset$ for all $z \in T(\bar{x})$ and $j = 1, \ldots, n$, which contradicts (iii).

If $\bar{x} \notin E$, then $y_j \notin K(\bar{x}), j = 1, \ldots, n$, which indicates that $\bar{x} = \sum_{j=1}^{n} a_j y_j \in \text{co } K(\bar{x}) \subseteq \text{cl } K(\bar{x})$ and then $\bar{x} \in E$. This is a contradiction.

Thus, $Q$ is a KKM mapping.

Secondly, we show that $\bigcap_{y \in A} Q(y) \neq \emptyset$.

For any given $y \in A$, we can deduce that

$$Q(y) = A \setminus \{x \in A : y \in H(x)\}$$

$$= A \setminus (\{x \in E : y \in K(x) \cap P(x)\} \cup \{x \in A \setminus E : y \in A \cap K(x)\})$$

$$= A \setminus (\{x \in A : y \in K(x)\} \cap (A \setminus E) \cup \{x \in A : y \in P(x)\})$$

$$= (A \setminus (\{x \in A : y \in K(x)\})) \cup \{x \in A : \exists z \in T(x), F(x,y,z) \subseteq Y \setminus \text{int } C(x)\}).$$  

(2.2)
By (ii) and (iv), we can conclude that \( Q(y) \) is a closed set. Since \( X \) is a Hausdorff topological vector space and \( A \) is a compact set, we have that \( Q(y) \) is compact for each \( y \in A \). By Lemma 1.1, we get \( \bigcap_{y \in A} Q(y) \neq \emptyset \).

Finally, we prove that the assertion of the theorem holds.

Taking arbitrarily \( \bar{x} \in \bigcap_{y \in A} Q(y) \), we have \( \bar{x} \notin \{ x \in A : y \in H(x) \} \) for all \( y \in A \), which indicates that \( H(\bar{x}) = \emptyset \). As \( A \cap K(x) \neq \emptyset \) for all \( x \), we know that \( \bar{x} \in E \) and then \( K(\bar{x}) \cap P(\bar{x}) = \emptyset \). Consequently, for each \( \bar{y} \in K(\bar{x}) \), there exists \( \bar{z} \in T(\bar{x}) \) such that \( F(\bar{x}, \bar{y}, \bar{z}) \subseteq Y \setminus \text{int} C(\bar{x}) \), which shows that \( \bar{x} \in S_1 \).

By a similar proof as for Theorem 2.1, we obtain the following result.

**Theorem 2.2.** Let hypotheses (i) and (ii) in Theorem 2.1 hold and let

(iii) \( F \) be \( Y \setminus \text{int} C \) quasi convex-like of type 2 with respect to \( T \),

(iv) \( \{ y \in A : \exists z \in T(x) \text{ s.t. } F(x, y, z) \cap (Y \setminus \text{int} C(x)) \neq \emptyset \} \) be a closed set for each \( x \in A \).

Then (SVQEP2) has at least a solution.

### 3. Upper Semicontinuity of Solution Sets

In this section, we will study the upper semicontinuity of the solution sets \( S_1(\lambda, \mu) \) and \( S_2(\lambda, \mu) \) with respect to parameters \((\lambda, \mu)\). For this end, we assume that \( S_1(\lambda, \mu) \) and \( S_2(\lambda, \mu) \) are nonempty for any \((\lambda, \mu) \in \Lambda \times M\). Let \( x_0 \in A, \lambda_0, \lambda \in \Lambda \) and \( \mu_0, \mu \in M \).

**Theorem 3.1.** Let

(i) \( E(\cdot) = \{ x \in A : x \in \text{cl} K(x, \cdot) \} \) and \( W(\cdot) \) be closed set-valued mappings, where \( W(x) := Y \setminus \text{int} C(x) \) for each \( x \in A \);

(ii) for any nets \( \{ \lambda_{\alpha} \} : \lambda_{\alpha} \to \lambda_0, \{ \mu_{\alpha} \} : \mu_{\alpha} \to \mu_0, \{ x_{\alpha} \} : x_{\alpha} \to x_0, \{ z_{\alpha} \} : z_{\alpha} \in T(x_{\alpha}, \lambda_{\alpha}) \) for each \( \alpha \) and any \( y_0 \in K(x_0, \mu_0) \), there exist nets \( \{ y_{\alpha} \} : y_{\alpha} \in K(x_{\alpha}, \mu_{\alpha}) \) for each \( \alpha \), \( \{ z_{\beta} \} \subseteq \{ z_{\alpha} \} \) and \( z_0 \in T(x_0, \lambda_0) \) such that \( y_{\alpha} \to y_0 \) and \( z_{\beta} \to z_0 \);

(iii) \( F \) be l.s.c on \( A \times X \times Y \). Then \( S_1(\cdot, \cdot) \) is both closed and u.s.c at \((\lambda_0, \mu_0)\).

**Proof.** We first show that \( S_1(\cdot, \cdot) \) is closed at \((\lambda_0, \mu_0)\).

Suppose to the contrary that \( S_1(\cdot, \cdot) \) is not closed at \((\lambda_0, \mu_0)\). Then there exist nets \( \{ (\lambda_{\alpha}, \mu_{\alpha}) \} : (\lambda_{\alpha}, \mu_{\alpha}) \to (\lambda_0, \mu_0) \) and \( \{ x_{\alpha} \} : x_{\alpha} \to x_0 \) and \( x_{\alpha} \in S_1(\lambda_{\alpha}, \mu_{\alpha}) \) for each \( \alpha \) such that \( x_0 \notin S_1(\lambda_0, \mu_0) \).

\( x_{\alpha} \in S_1(\lambda_{\alpha}, \mu_{\alpha}) \) implies that \( (\mu_{\alpha}, x_{\alpha}) \in \text{graph} E \) for each \( \alpha \). By the closedness of \( A \cap \text{cl} K(\cdot, \cdot) \), we get \( x_0 \in A \cap \text{cl} K(x_0, \mu_0) \), which together with \( x_0 \notin S_1(\lambda_0, \mu_0) \) indicates that there exists \( y_0 \in K(x_0, \mu_0) \) such that

\[
F(x_0, y_0, z) \cap -\text{int} C(x_0) \neq \emptyset, \quad \forall z \in T(x_0, \lambda_0). \tag{3.1}
\]

For \( y_0 \in K(x_0, \mu_0) \), by (ii), there exists \( y_{\alpha} \in K(x_{\alpha}, \mu_{\alpha}) \) for each \( \alpha \) such that \( y_{\alpha} \to y_0 \). Due to \( x_{\alpha} \in S_1(\lambda_{\alpha}, \mu_{\alpha}) \), for each \( y_{\alpha} \in K(x_{\alpha}, \mu_{\alpha}) \), there exists \( z_{\alpha} \in T(x_{\alpha}, \lambda_{\alpha}) \) such that \( F(x_{\alpha}, y_{\alpha}, z_{\alpha}) \subseteq Y \setminus \text{int} C(x_{\alpha}) \). Again by (iii), there exist a subnet \( \{ z_{\beta} \} \subseteq \{ z_{\alpha} \} \) and a point \( z_0 \in T(x_0, \lambda_0) \) such that \( z_{\beta} \to z_0 \) and

\[
F(x_{\alpha}, y_{\alpha}, z_{\beta}) \subseteq W(x_{\alpha}). \tag{3.2}
\]
For $z_0 \in T(x_0, \lambda_0)$, by (3.1), there exists $f_0 \in F(x_0, y_0, z_0)$ such that

$$f_0 \in -\operatorname{int} C(x_0). \quad (3.3)$$

By the lower semicontinuity of $F$, there exists $f_\beta \in F(x_\beta, y_\beta, z_\beta)$ for each $\beta$ such that $f_\beta \rightharpoondown f_0$, which together with the closedness of $W(\cdot)$ and (3.2) implies that $f_0 \in Y \setminus -\operatorname{int} C(x_0)$. This contradicts (3.3). Hence, $S_1(\cdot, \cdot)$ is closed at $(\lambda_0, \mu_0)$.

Next, we show that $S_1(\cdot, \cdot)$ is u.s.c at $(\lambda_0, \mu_0)$.

Suppose to the contrary that $S_1(\cdot, \cdot)$ is not u.s.c at $(\lambda_0, \mu_0)$. By Lemma 1.3, there exist nets $\{\lambda_\alpha, \mu_\alpha\} : \lambda_\alpha, \mu_\alpha \rightarrow (\lambda_0, \mu_0)$ and $\{x_\alpha\} : x_\alpha \rightarrow x_0$ such that for any $x_0 \in S_1(\lambda_0, \mu_0)$ and any subnet $\{x_\beta\} \subseteq \{x_\alpha\}$ one has

$$x_\beta \rightharpoondown x_0. \quad (3.4)$$

$x_\alpha \in S_1(\lambda_\alpha, \mu_\alpha)$ implies that $x_\alpha \in E(\mu_\alpha)$ for each $\alpha$ and $\{x_\alpha\} \subseteq A$. By the compactness of $A$, there exists a convergent subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $x_\beta \rightarrow \bar{x} \in A$. By the closedness of $E(\cdot)$, we have $\bar{x} \in E(\mu_0)$. By (3.4), we get $\bar{x} \notin S_1(\lambda_0, \mu_0)$, that is,

$$F(\bar{x}, \bar{y}, z) \cap -\operatorname{int} C(\bar{x}) \neq \emptyset, \quad \forall z \in T(\bar{x}, \lambda_0). \quad (3.5)$$

By using a similar argument as in part one, we can complete the proof.

**Theorem 3.2.** Let hypotheses (i) and (ii) in Theorem 3.1 hold and let

(iii) $F$ be u.s.c on $A \times X \times Y$.

Then $S_2(\cdot, \cdot)$ is both u.s.c and closed at $(\lambda_0, \mu_0)$.

**Proof.** We first prove that $S_2(\cdot, \cdot)$ is closed at $(\lambda_0, \mu_0)$.

Suppose to the contrary that $S_2(\cdot, \cdot)$ is not closed at $(\lambda_0, \mu_0)$. Then there exist nets $\{\lambda_\alpha, \mu_\alpha\} : \lambda_\alpha, \mu_\alpha \rightarrow (\lambda_0, \mu_0)$ and $\{x_\alpha\} : x_\alpha \rightarrow x_0$ and $x_\alpha \in S_2(\lambda_\alpha, \mu_\alpha)$ for each $\alpha$ such that $x_0 \notin S_2(\lambda_0, \mu_0)$. By using a similar reasoning as in part one of the proof of Theorem 3.1, we can conclude that there exists a net $\{(x_\beta, y_\beta, z_\beta)\}$ such that $(x_\beta, y_\beta, z_\beta) \rightarrow (x_0, y_0, z_0)$ and

$$F(x_0, y_0, z_0) \subseteq -\operatorname{int} C(x_0), \quad (3.6)$$

$$F(x_\beta, y_\beta, z_\beta) \cap (Y \setminus -\operatorname{int} C(x_\beta)) \neq \emptyset, \quad \forall \beta, \quad (3.7)$$

where $y_\beta \in K(x_\beta, \mu_\beta)$ with $y_\beta \rightarrow y_0 \in K(x_0, \mu_0)$ and $z_\beta \in T(x_\beta, \lambda_\beta)$ with $z_\beta \rightarrow z_0 \in T(x_0, \lambda_0)$. By the upper semicontinuity of $F$ and (3.6), we know that there exists $\beta_0$ such that

$$F(x_\beta, y_\beta, z_\beta) \subseteq -\operatorname{int} C(x_0), \quad \forall \beta \geq \beta_0, \quad (3.8)$$

which contradicts (3.7). Hence, $S_2(\cdot, \cdot)$ is closed at $(\lambda_0, \mu_0)$.

Next, we prove that $S_2(\cdot, \cdot)$ is u.s.c at $(\lambda_0, \mu_0)$.

By the closedness of $S_2(\cdot, \cdot)$ at $(\lambda_0, \mu_0)$, $S_2(\lambda_0, \mu_0)$ is closed and hence compact as is $A$. 
Suppose to the contrary that $S_2(\cdot,\cdot)$ is not u.s.c at $(\lambda_0,\mu_0)$. By Lemma 1.3, there exist nets \{$(\lambda_{\alpha},\mu_{\alpha})$\} : $(\lambda_{\alpha},\mu_{\alpha}) \rightarrow (\lambda_0,\mu_0)$ and \{x\} : x_{\alpha} \in S_2(\lambda_{\alpha},\mu_{\alpha}) for each $\alpha$ such that (3.4) holds for any x_{0} \in S_2(\lambda_0,\mu_0) and any subnet \{x_{\beta}\} \subseteq \{x_{\alpha}\}.

$x_{\alpha} \in S_2(\lambda_{\alpha},\mu_{\alpha})$ implies that $x_{\alpha} \in E(\mu_{\alpha})$ for each $\alpha$ and \{x_{\alpha}\} \subseteq A$. By the compactness of $A$ and the closedness of $E(\cdot)$, it follows that there exists a convergent subnet \{x_{\beta}\} of \{x_{\alpha}\} such that $x_{\beta} \rightarrow \overline{x} \in E(\mu_0)$. By (3.4), we get $\overline{x} \notin S_2(\lambda_0,\mu_0)$, that is,

$$F(\overline{x},\overline{y},z) \subseteq \text{Int}(C(\overline{x})), \quad \forall z \in T(\overline{x},\lambda_0). \quad (3.9)$$

By using a similar argument as in part one, we can complete the proof. \qed

4. Lower Semicontinuity of Solution Sets

In this section, we will consider the lower semicontinuity of the solution sets $S_1(\cdot,\cdot)$ and $S_2(\cdot,\cdot)$ with respect to parameters $(\lambda,\mu)$.

Theorem 4.1. Let

(i) $E(\cdot) := \{x \in A|x \in \text{Cl}K(x,\cdot)\}$ be l.s.c on $M$ and $C(\cdot)$u.s.c at $x_0$;

(ii) for any nets \{x_{\alpha}\} : $\lambda_{\alpha} \rightarrow \lambda_0,\{x_{\alpha}\} : x_{\alpha} \rightarrow x_0,\{y_{\alpha}\} \subseteq K(x_0,\mu_0)$ for each $\alpha$, \{y\} \subseteq \{y_{\alpha}\} and a point $y_0 \in K(x_0,\mu_0)$ such that $z_{\alpha} \rightarrow z_0$ and $y_{\beta} \rightarrow y_0$;

(iii) $F$ be u.s.c and have compact values on $A \times X \times Y$;

(iv) $F(x_0,y_0,z_0) \cap -\partial C(x_0) = \emptyset$ for all $x_0 \in S_1(\lambda_0,\mu_0), y_0 \in K(x_0,\mu_0)$, and $z_0 \in T(x_0,\lambda_0)$.

Then $S_1(\cdot,\cdot)$ is l.s.c at $(\lambda_0,\mu_0)$.

Proof. Suppose to the contrary that $S_1(\cdot,\cdot)$ is not l.s.c at $(\lambda_0,\mu_0)$. Then there exist a net \{$(\lambda_{\alpha},\mu_{\alpha})$\} : $(\lambda_{\alpha},\mu_{\alpha}) \rightarrow (\lambda_0,\mu_0)$ and a point $x_0 \in S_1(\lambda_0,\mu_0)$ such that for any net \{x_{\alpha}\} : $x_{\alpha} \in S_1(\lambda_{\alpha},\mu_{\alpha})$ for each $\alpha$ one has

$$\tilde{x}_{\alpha} \rightarrow x_0. \quad (4.1)$$

$x_0 \in S_1(\lambda_0,\mu_0)$ implies that $x_0 \in E(\mu_0)$. By the lower semicontinuity of $E$, there exists a net \{x_{\alpha}\} : $x_{\alpha} \in E(\mu_{\alpha})$ for each $\alpha$ such that $x_{\alpha} \rightarrow x_0$, which combining with (4.1) shows that there exists a subnet \{x_{\beta}\} of \{x_{\alpha}\} such that $x_{\beta} \notin S_1(\lambda_{\beta},\mu_{\beta})$ for all $\beta$. Consequently, for each $\beta$, there exists $y_{\beta} \in K(x_{\beta},\mu_{\beta})$ satisfying

$$F(x_{\beta},y_{\beta},z_{\beta}) \cap -\text{Int}C(x_{\beta}) \neq \emptyset, \quad \forall z_{\beta} \in T(x_{\beta},\lambda_{\beta}). \quad (4.2)$$

By (ii), there exist a subnet \{y_{\beta}\} \subseteq \{y_{\alpha}\} and a point $y_0 \in K(x_0,\mu_0)$ such that $y_{\beta} \rightarrow y_0$, which together with $x_0 \in S_1(\lambda_0,\mu_0)$ and (ii) indicates that there exist $z_0 \in T(x_0,\lambda_0)$ and $z_{\beta} \in T(\tilde{x}_{\beta},\tilde{y}_{\beta})$ such that $z_{\beta} \rightarrow z_0, F(\tilde{x}_{\beta},\tilde{y}_{\beta},z_{\beta}) \cap -\text{Int}C(\tilde{x}_{\beta}) \neq \emptyset$ for all $\beta$ and

$$F(x_0,y_0,z_0) \subseteq Y \setminus -\text{Int}C(x_0). \quad (4.3)$$
Taking arbitrarily $f_{\beta} \in F(\bar{x}_{\beta}, \bar{y}_{\beta}, \bar{z}_{\beta}) \cap -\text{int} C(\bar{x}_{\beta})$ for each $\beta$. By Lemma 1.3, there exist $f_0 \in F(x_0, y_0, z_0)$ and a subset $\{f_{\beta}\}$ of $\{f_{\beta}\}$ such that $f_{\beta} \to f_0$.

Since $f_{\beta} \in -C(\bar{x}_{\beta})$ for each $\beta$, by the upper semicontinuity of $C(\cdot)$ and Lemma 1.2, we know that $f_0 \in -C(x_0)$, which together with (iv) shows that $f_0 \in -\text{int} C(x_0)$. This contradicts (4.3). Hence, $S_1(\cdot, \cdot)$ is l.s.c at $(\lambda_0, \mu_0)$.

**Theorem 4.2.** Let hypotheses (i) and (ii) in Theorem 4.1 hold and let

(iii) $F(\cdot, \cdot, \cdot)$ be l.s.c on $A \times X \times Y$;
(iv) $F(x_0, y_0, z_0) \cap -\partial C(x_0) = \emptyset$ for all $x_0 \in S_2(\lambda_0, \mu_0), y_0 \in K(x_0, \mu_0)$, and $z_0 \in T(x_0, \lambda_0)$.

Then $S_2(\cdot, \cdot)$ is l.s.c at $(\lambda_0, \mu_0)$.

**Proof.** By arguments similar to those for Theorem 4.1, we can conclude that there exists a net $\{(\bar{x}_{\beta}, \bar{y}_{\beta}, \bar{z}_{\beta})\}$ such that $(\bar{x}_{\beta}, \bar{y}_{\beta}, \bar{z}_{\beta}) \to (x_0, y_0, z_0)$, $F(x_0, y_0, z_0) \cap (Y \setminus \text{int} C(x_0)) \neq \emptyset$, and

$$F(\bar{x}_{\beta}, \bar{y}_{\beta}, \bar{z}_{\beta}) \subseteq -\text{int} C(\bar{x}_{\beta}), \quad \forall \beta,$$

where $\bar{x}_{\beta} \in E(\bar{y}_{\beta}), \bar{y}_{\beta} \in K(\bar{x}_{\beta}, \bar{y}_{\beta}), \bar{z}_{\beta} \in T(\bar{x}_{\beta}, \bar{y}_{\beta})$ for all $\beta$ and $x_0 \in S_2(\lambda_0, \mu_0), y_0 \in K(x_0, \mu_0)$ and $z_0 \in T(x_0, \lambda_0)$.

For any given $f_0 \in F(x_0, y_0, z_0) \cap (Y \setminus \text{int} C(x_0))$, by the lower semicontinuity of $F$, there exists $\bar{f}_{\beta} \in F(\bar{x}_{\beta}, \bar{y}_{\beta}, \bar{z}_{\beta})$ for each $\beta$ such that $\bar{f}_{\beta} \to f_0$. By (4.4), we have $\bar{f}_{\beta} \in -C(\bar{x}_{\beta})$ for each $\beta$. By the upper semicontinuity of $C(\cdot)$ and Lemma 1.2, it follows that $\bar{f}_{0} \in -C(\bar{x}_{0})$, which together with (iv) implies that $f_0 \in -\text{int} C(x_0)$. This is a contradiction. Hence, $S_2(\cdot, \cdot)$ is l.s.c at $(\lambda_0, \mu_0)$.

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**References**


