Hindawi Publishing Corporation
International Journal of Stochastic Analysis
Volume 2010, Article ID 730492, 24 pages
doi:10.1155/2010/730492

Research Article

Stochastic Navier-Stokes Equations with Artificial Compressibility in Random Durations

Hong Yin

Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA

Correspondence should be addressed to Hong Yin, hongyin@usc.edu

Received 1 December 2009; Accepted 11 May 2010

Academic Editor: Jiongmin M. Yong

The existence and uniqueness of adapted solutions to the backward stochastic Navier-Stokes equation with artificial compressibility in two-dimensional bounded domains are shown by Minty-Browder monotonicity argument, finite-dimensional projections, and truncations. Continuity of the solutions with respect to terminal conditions is given, and the convergence of the system to an incompressible flow is also established.

1. Introduction

The Navier-Stokes equation (NSE for short), named in honor of Navier and Stokes, who were responsible for its formulation, is an acknowledged model for equation of motion for Newtonian fluid. It is closely connected to the theory of hydrodynamic turbulence, the time dependent chaotic behavior seen in many fluid flows.

The well-posedness of the Navier-Stokes equation has been studied extensively by Ladyzhenskaya [1], Constantin and Foias [2], and Temam [3], among others. Although some ingenious approaches have been made, the problem has not been fully understood. The nonlinearity, part of the cause of turbulence, made the problem extraordinarily difficult. In hope of taking advantage of the noise, randomness has been introduced into the system and some pioneer work has been done by Flandoli and Gatarek [4], Mikulevicius and Rozovsky [5], Menaldi and Sritharan [6], and others. Although the introduction of randomness is not very successful in overcoming the difficulty, it provides a more realistic model than deterministic Navier-Stokes equations and is interesting in itself.

The vast majority of work on the Navier-Stokes equations is done for viscous incompressible Newtonian fluids. In a suitable Hilbert space and under the incompressibility assumption $\nabla \cdot \mathbf{u} = 0$, the two-dimensional stochastic Navier-Stokes equation in a bounded

domain $G \subset \mathbb{R}^2$ with no-slip condition reads

$$\partial \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}dt - \nu \Delta \mathbf{u}dt = -\nabla p dt + \mathbf{f}(t)dt + \sigma(t, \mathbf{u})d\mathbb{W}(t), \tag{1.1}$$

where ν is the constant viscosity, $\mathbf{u}$ is the velocity, p is the pressure, $\mathbf{f}$ is the external body force and $\mathbb{W}$ is the infinite-dimensional Wiener process. The assumption of incompressibility works well even for compressible fluids such as air at room temperature. But there are extreme phenomena, such as the diffusion of sound, that are closely related to fluid compressibility. Also the constraint caused by the incompressibility creates computational difficulties for numerical approximation of the Navier-Stokes equations. The method of artificial compressibility was first introduced by Temam [3] to surmount this obstacle. It also describes the slight compressibility existed in most fluids. The model has its own interest, and is given below with the parameter ε:

$$\begin{gathered} \partial_t \mathbf{u}_\varepsilon - \nu \Delta \mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon + \frac{1}{2}(\nabla \cdot \mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}, \\ \varepsilon \partial_t p_\varepsilon + \nabla \cdot \mathbf{u}_\varepsilon = 0. \end{gathered} \tag{1.2}$$

Backward stochastic Navier-Stokes equations (BSNSEs for short) arise as an inverse problem wherein the velocity profile at a time T is observed and given, and the noise coefficient has to be ascertained from the given terminal data. Such a motivation arises naturally when one understands the importance of inverse problems in partial differential equations (see Lions [7, 8]). Linear backward stochastic differential equations were introduced by Bismut in 1973 [9], and the systematic study of general backward stochastic differential equations (BSDEs for short) were put forward first by Pardoux and Peng [10], Ma, Protter, Yong, Zhou, and several other authors in a finite-dimensional setting. Ma and Yong [11] have studied linear degenerate backward stochastic differential equations motivated by stochastic control theory. Later, Hu et al. [12] considered the semilinear equations as well. Backward stochastic partial differential equations were shown to arise naturally in stochastic versions of the Black-Scholes formula by Ma and Yong [13]. A nice introduction to backward stochastic differential equations is presented in the book by Yong and Zhou [14], with various applications.

The usual method of proving existence and uniqueness of solutions by fixed point arguments does not apply to the stochastic system on hand since the drift coefficient in the backward stochastic Navier-Stokes equation is nonlinear, non-Lipschitz and unbounded. The drift coefficient is monotone on bounded $L^4(G)$ balls in V, which was first observed by Menaldi and Sritharan [6]. The method of monotonicity is used in this paper to prove the existence of solutions to BSNSEs. The proof of the uniqueness and continuity of solutions also relies on the monotonicity assumption of the coefficients. Existence and uniqueness of solutions are shown to hold under the $\mathbb{H}_0^1$ boundedness on the terminal values.

The structure of the paper is as follows. The functional setup of the paper is introduced and several frequently used inequalities are listed in Section 2. The a priori estimates for the solutions of projected BSNSEs are given under different assumptions of the terminal conditions and external body force in Section 3. The existence and uniqueness of solutions of projected BSNSEs are shown in Section 4. Also the existence of solutions of BSNSEs under suitable assumptions is shown by Minty-Browder monotonicity argument. The uniqueness of the solution under the assumption that terminal condition is uniformly bounded in H^1 sense is given in Section 5. The continuity of solutions and the convergence as ε approaches zero are also studied.

2. Preliminaries

Suppose that G is a domain bounded in $\mathbb{R}^2$ with smooth boundary conditions. Let ε be a positive parameter which vanishes to 0. The artificial state equation for a slightly compressible medium is defined as

$$\rho = \rho_0 + \varepsilon p, \tag{2.1}$$

where ρ is the density, p is the pressure, and ρ_0 is the first approximation of the density. By adjusting the equations of motion according to the state equation, we obtain the following family of perturbed systems associated with the parameter ε:

$$\begin{gathered}\partial_t \mathbf{u}_\varepsilon - \nu\Delta\mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon + \frac{1}{2}(\nabla\cdot\mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f},\\ \varepsilon\partial_t p_\varepsilon + \nabla\cdot\mathbf{u}_\varepsilon = 0,\end{gathered} \tag{2.2}$$

where $\mathbf{u}_\varepsilon \in \mathbb{L}^2 = \mathbb{L}^2(G)$ is the velocity, $p_\varepsilon \in L^2 = L^2(G)$ is the pressure, $\mathbf{f} \in \mathbb{L}^2$ is the external body force, and ν is the kinematic viscosity. Readers may refer to Temam [3] for details.

Denote by $(\cdot,\cdot)$ the inner product of $\mathbb{L}^2$, $(\cdot,\cdot)_{\mathbb{H}_0^1}$ the inner product of $\mathbb{H}_0^1 = \mathbb{H}_0^1(G)$, $\mathbb{H}^{-1}$ the dual space of $\mathbb{H}_0^1$, and $\langle\cdot,\cdot\rangle$ the duality pairing between $\mathbb{H}_0^1$ and $\mathbb{H}^{-1}$. Let $|\cdot|$ be the norm of $\mathbb{L}^2$ and let $\|\cdot\|$ be the norm of $\mathbb{H}_0^1$. Without causing any confusion, we also use the same notations to denote the norms of L^2 and $H_0^1 = H_0^1(G)$. For any $\mathbf{x} \in \mathbb{L}^2$ and $\mathbf{y} \in \mathbb{H}_0^1$, there exists $\mathbf{x}' \in \mathbb{H}^{-1}$, such that $(\mathbf{x},\mathbf{y}) = \langle\mathbf{x}',\mathbf{y}\rangle$. Then the mapping $\mathbf{x} \mapsto \mathbf{x}'$ is linear, injective, compact and continuous. A similar result holds for H^{-1} and L^2.

Suppose that $(\Omega,\mathcal{F},P)$ is a complete probability space. Let $\mathbb{W}(t)$ be an $\mathbb{L}^2$-valued $\mathbb{Q}$-Wiener process, where $\mathbb{Q}$ is a trace class operator on $\mathbb{L}^2$. Let $\{\mathbf{e}_j\}_{j=1}^\infty \in \mathbb{L}^2 \cap \mathbb{H}_0^1 \cap \mathbb{L}^4$ be a complete orthonormal system in $\mathbb{L}^2$ such that there exists a nondecreasing sequence of positive numbers $\{\lambda_j\}_{j=1}^\infty$, $\lim_{j\to\infty}\lambda_j = \infty$ and $-\Delta\mathbf{e}_j = \lambda_j\mathbf{e}_j$ for all j. Let $\mathbb{Q}\mathbf{e}_k = q_k\mathbf{e}_k$ with $\sum_{k=1}^\infty q_k < \infty$, and $\{b^k(t)\}$ be a sequence of independent standard Brownian motions in $\mathbb{R}$. Then Wiener process $\mathbb{W}(t)$ is taken as $\mathbb{W}(t)=\sum_{k=1}^\infty \sqrt{q_k}b^k(t)\mathbf{e}_k$.

Let Q be a trace class operator on L^2. Similarly, we can define a complete orthonormal system $\{e_j\}_{j=1}^\infty$, a nondecreasing sequence of positive numbers $\{\kappa_j\}_{j=1}^\infty$ such that $-\Delta e_j = \kappa_j e_j$, and positive numbers q_j' such that $Qe_j = q_j' e_j$ and $\sum_{j=1}^\infty q_j' < \infty$. Let $W(t)=\sum_{j=1}^\infty \sqrt{q_j'}b^j(t)e_j$. Then $W(t)$ is an L^2-valued Q-Wiener process. From now on, let $\{\mathcal{F}_t\}$ be the natural filtration of $\{\mathbb{W}(t)\}$ and $\{W(t)\}$, augmented by all the P-null sets of $\mathcal{F}$. A complete definition of Hilbert space-valued Wiener processes can be found in [15].

With inner product

$$\langle\mathbf{F},\mathbf{G}\rangle_{L_\mathbb{Q}} = \operatorname{tr}(\mathbf{F}\mathbb{Q}\mathbf{G}^*) = \operatorname{tr}(\mathbf{G}\mathbb{Q}\mathbf{F}^*) \tag{2.3}$$

for all $\mathbf{F}$ and $\mathbf{G} \in L_\mathbb{Q}$, let $L_\mathbb{Q}$ denote the space of linear operators $\mathbf{E}$ such that $\mathbf{E}\mathbb{Q}^{1/2}$ is a Hilbert-Schmidt operator from $\mathbb{L}^2$ to $\mathbb{L}^2$. Similarly, we define L_Q for Q, the trace class operator on L^2.

To be realistic in nature, let us introduce randomness into the system to obtain

$$\begin{gathered}\frac{\partial \mathbf{u}_\varepsilon(t)}{\partial t} - \nu\Delta\mathbf{u}_\varepsilon(t) + (\mathbf{u}_\varepsilon(t)\cdot\nabla)\mathbf{u}_\varepsilon(t) + \frac{1}{2}(\nabla\cdot\mathbf{u}_\varepsilon(t))\mathbf{u}_\varepsilon(t) + \nabla p_\varepsilon(t) = \mathbf{f}(t) + \sigma(t)\frac{d\mathbb{W}(t)}{dt},\\ \varepsilon\partial p_\varepsilon(t) + \nabla\cdot\mathbf{u}_\varepsilon(t)dt = 0,\\ \mathbf{u}_\varepsilon(0) = \mathbf{u}_0, \quad p_\varepsilon(0) = p_0,\end{gathered} \tag{2.4}$$

where $\mathbf{u}_0$ and p_0 are initial conditions, and $\sigma(d\mathbb{W}/dt)$ is the noise term. Here $(1/2)(\nabla\cdot\mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon$ is called the stabilization term.

If a terminal time T is given and the terminal conditions are specified as $\mathbf{u}_\varepsilon(T) = \xi$ and $p_\varepsilon(T) = \eta$, one obtains a backward system:

$$\begin{gathered}d\mathbf{u}_\varepsilon(t) + \nu\widehat{\mathbf{A}}\mathbf{u}_\varepsilon(t)dt + \widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(t))dt + \nabla p_\varepsilon(t)dt = \mathbf{f}(t)dt + \mathbf{Z}_\varepsilon(t)d\mathbb{W}(t),\\ \varepsilon dp_\varepsilon(t) + \nabla\cdot\mathbf{u}_\varepsilon(t)dt = Z_\varepsilon(t)dW(t),\\ \mathbf{u}_\varepsilon(T) = \xi, \qquad p_\varepsilon(T) = \eta\end{gathered} \tag{2.5}$$

for $0 \le t \le T$, where $\widehat{\mathbf{A}}\mathbf{u} \triangleq -\Delta\mathbf{u}$ and $\widehat{\mathbf{B}}(\mathbf{u},\mathbf{v}) \triangleq (\mathbf{u}\cdot\nabla)\mathbf{v} + (1/2)(\nabla\cdot\mathbf{u})\mathbf{v}$, with the notation $\widehat{\mathbf{B}}(\mathbf{u}) \triangleq \widehat{\mathbf{B}}(\mathbf{u},\mathbf{u})$. The processes $\mathbf{Z}_\varepsilon$ and Z_ε are in spaces $L_{\mathbb{Q}}$ and L_Q, respectively.

Let τ be a $\mathcal{F}_t$-stopping time when the observations are available. Suppose that the observed velocity and pressure at τ are $\mathbf{u}_\varepsilon(\tau) = \xi \in L^2_{\mathcal{F}_\tau}(\Omega;\mathbb{L}^2)$ and $p(\tau) = \eta \in L^2_{\mathcal{F}_\tau}(\Omega;L^2)$, respectively. Then we introduce the backward stochastic Navier-Stokes equation with artificial compressibility and stabilization in random duration:

$$\begin{gathered}d\mathbf{u}_\varepsilon(t) + \nu\widehat{\mathbf{A}}\mathbf{u}_\varepsilon(t)dt + \widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(t))dt + \nabla p_\varepsilon(t)dt = \mathbf{f}(t)dt + \mathbf{Z}_\varepsilon(t)d\mathbb{W}(t),\\ \varepsilon dp_\varepsilon(t) + \nabla\cdot\mathbf{u}_\varepsilon(t)dt = Z_\varepsilon(t)dW(t),\\ \mathbf{u}_\varepsilon(\tau) = \xi, \qquad p_\varepsilon(\tau) = \eta\end{gathered} \tag{2.6}$$

for $0 \le t \le \tau$, where the $\mathcal{F}_t$-stopping time τ is assumed to be bounded by a time $T > 0$. Note that processes $\mathbf{Z}_\varepsilon$ and Z_ε measure the randomness that is inherent in the hydrodynamical system. It is this randomness that has possibly led us to the observations at time τ. For instance, in wind tunnel experiments, the form and the magnitude of the randomness has to be ascertained from the velocity observations. This backward system helps us to make an attempt at uncertainty quantification. Here $\mathbf{f}$ is taken to be deterministic and is always assumed to be in $L^2(0,T;\mathbb{H}^{-1})$.

Definition 2.1. A quaternion of $\mathcal{F}_t$-Adapted processes $(\mathbf{u}_\varepsilon, \mathbf{Z}_\varepsilon, p_\varepsilon, Z_\varepsilon)$ is called a *solution* of backward Navier-Stokes equation (2.6) if it satisfies the integral form of the system

$$\begin{gathered}\mathbf{u}_\varepsilon(t\wedge\tau) = \xi + \int_{t\wedge\tau}^{\tau}\left\{\nu\widehat{\mathbf{A}}\mathbf{u}_\varepsilon(s) + \widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(s)) + \nabla p_\varepsilon(s) - \mathbf{f}(s)\right\}ds - \int_{t\wedge\tau}^{\tau}\mathbf{Z}_\varepsilon(s)d\mathbb{W}(s),\\ \varepsilon p_\varepsilon(t\wedge\tau) = \varepsilon\eta + \int_{t\wedge\tau}^{\tau}\nabla\cdot\mathbf{u}_\varepsilon(s)ds - \int_{t\wedge\tau}^{\tau}Z_\varepsilon(s)dW(s),\end{gathered} \tag{2.7}$$

P-a.s., and the following holds:

(a) $\mathbf{u}_\varepsilon \in L^2_{\mathcal{F}}(\Omega; L^\infty(0,\tau;\mathbb{L}^2)) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,\tau;\mathbb{H}^1_0))$;

(b) $\mathbf{Z}_\varepsilon \in L^2_{\mathcal{F}}(\Omega; L^2(0,\tau;L_{\mathbb{Q}}))$;

(c) $p_\varepsilon \in L^2_{\mathcal{F}}(\Omega; L^\infty(0,\tau;L^2)) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,\tau;H^1_0))$;

(d) $Z_\varepsilon \in L^2_{\mathcal{F}}(\Omega; L^2(0,\tau;L_Q))$.

The following simple results are frequently used and given as lemmas. Readers may refer to Temam [3] for similar proofs.

Lemma 2.2. *For any* $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{H}^1_0$ *and* $p \in L^2$, *one has*

(1) $\langle \widehat{\mathbf{A}}\mathbf{u},\mathbf{w}\rangle = \sum_{i,j}\int_G \partial_i u_j \partial_i w_j dx = \langle \widehat{\mathbf{A}}\mathbf{w},\mathbf{u}\rangle = (\mathbf{u},\mathbf{w})_{\mathbb{H}^1_0}$,

(2) $\langle (\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}\rangle = \sum_{i,j}\int_G u_i(\partial_i v_j)w_j dx$,

(3) $\langle (\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}\rangle = -\langle (\nabla\cdot\mathbf{u})\mathbf{w},\mathbf{v}\rangle - \langle (\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}\rangle$,

(4) $\langle \widehat{\mathbf{B}}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle = -\langle \widehat{\mathbf{B}}(\mathbf{u},\mathbf{w}),\mathbf{v}\rangle$,

(5) $\langle -\nabla p,\mathbf{u}\rangle = -\sum_i \int_G \partial_i p\, u_i dx = \int_G p\partial_i u_i dx = \langle p,\nabla\cdot\mathbf{u}\rangle$.

Remark 2.3. Sometimes $\langle \widehat{\mathbf{B}}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle$ is denoted by $\widehat{b}(\mathbf{u},\mathbf{v},\mathbf{w})$.

Lemma 2.4. *The following results hold for any real-valued smooth functions* ϕ *and* ψ *with compact support in* $\mathbb{R}^2$:

$$\begin{gathered} |\phi\psi|^2 \le C\|\phi\partial_1\phi\|_{\mathbb{L}^1}\|\psi\partial_2\psi\|_{\mathbb{L}^1}, \\ \|\phi\|^4_{\mathbb{L}^4} \le C|\phi|^2|\nabla\phi|^2. \end{gathered} \tag{2.8}$$

Proposition 2.5. *For any* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{H}^1_0$ *and* $\mathbf{w} \in \mathbb{L}^4$, *one has*

$$\left|\widehat{b}(\mathbf{u},\mathbf{v},\mathbf{w})\right| \le \|\mathbf{u}\|_{\mathbb{L}^4}\|\mathbf{v}\|\|\mathbf{w}\|_{\mathbb{L}^4} + \frac{1}{2}\|\mathbf{u}\|\|\mathbf{v}\|_{\mathbb{L}^4}\|\mathbf{w}\|_{\mathbb{L}^4}. \tag{2.9}$$

Below is a backward version of the Gronwall inequality used frequently in this paper, and the proof is straightforward.

Lemma 2.6. *Suppose that* $g(t)$, $\alpha(t)$, $\beta(t)$, *and* $\gamma(t)$ *are integrable functions, and* $\beta(t)$, $\gamma(t)$ *are nonnegative functions. For* $0 \le t \le T$, *if*

$$g(t) \le \alpha(t) + \beta(t)\int_t^T \gamma(\rho)g(\rho)d\rho, \tag{2.10}$$

then

$$g(t) \le \alpha(t) + \beta(t)\int_t^T \alpha(\eta)\gamma(\eta)e^{\int_t^\eta \beta(\rho)\gamma(\rho)d\rho}d\eta. \tag{2.11}$$

In particular, if $\alpha(t) \equiv \alpha$, $\beta(t) \equiv \beta$ and $\gamma(t) \equiv 1$, then

$$g(t) \le \alpha e^{\beta(T-t)}. \tag{2.12}$$

3. A Priori Estimates

The purpose of the this paper is to show the existence and uniqueness of the randomly stopped backward stochastic Navier-Stokes equation (2.6). We employ Galerkin's method by defining orthogonal projections $P_N : \mathbb{L}^2 \to \mathbb{L}^2_N$, where $\mathbb{L}^2_N = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, for all $N \in \mathbb{N}$. An important result is that the Galerkin-type approximations converge weakly to the solution of the Navier-Stokes equation.

First of all, let us establish some a priori estimates. Let us define the projected operators $\widehat{\mathbf{A}}^N \triangleq P_N \widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}^N \triangleq P_N \widehat{\mathbf{B}}$. Under projection P_N, let us construct a finite dimensional system. Let

$$\begin{gathered}
\mathbb{W}^N(t) \triangleq P_N \mathbb{W}(t) = \sum_{i=1}^{N} \sqrt{q_i} b^i(t) \mathbf{e}_i, \quad W^N(t) \triangleq P_N W(t) = \sum_{i=1}^{N} \sqrt{q_i'} b^i(t) e_i, \\
\mathbf{f}^N(t) \triangleq P_N \mathbf{f}(t), \quad \xi^N \triangleq E\left(P_N \xi \mid \mathcal{F}^N_\tau\right), \quad \eta^N \triangleq E\left(P_N \eta \mid \mathcal{F}^N_\tau\right),
\end{gathered} \tag{3.1}$$

where $\{\mathcal{F}^N_t\}$ is the natural filtration of $\{\mathbb{W}^N(t)\}$ and $\{W^N(t)\}$. The projected system with solution $(\mathbf{u}^N_\varepsilon, \mathbf{Z}^N_\varepsilon, p^N_\varepsilon, Z^N_\varepsilon)$ is defined as follows:

$$\begin{gathered}
d\mathbf{u}^N_\varepsilon(t) = -\nu \widehat{\mathbf{A}}^N \mathbf{u}^N_\varepsilon(t) dt - \widehat{\mathbf{B}}^N\left(\mathbf{u}^N_\varepsilon(t)\right) dt - \nabla p^N_\varepsilon(t) dt + \mathbf{f}^N(t) dt + \mathbf{Z}^N_\varepsilon(t) d\mathbb{W}^N(t), \\
\varepsilon dp^N_\varepsilon(t) + \nabla \cdot \mathbf{u}^N_\varepsilon(t) dt = Z^N_\varepsilon(t) dW^N(t), \\
\mathbf{u}^N_\varepsilon(\tau) = \xi^N, \qquad p^N_\varepsilon(\tau) = \eta^N
\end{gathered} \tag{3.2}$$

for $0 \le t \le \tau$.

Proposition 3.1. *Let $\xi \in L^\infty_{\mathcal{F}_\tau}(\Omega; \mathbb{L}^2)$, $\eta \in L^\infty_{\mathcal{F}_\tau}(\Omega; L^2)$, and $\mathbf{f} \in L^2(0, T; \mathbb{H}^{-1})$. Then for any solution of system* (3.2)*, the following is true:*

$$\begin{gathered}
\left(\mathbf{u}^N_\varepsilon, \mathbf{Z}^N_\varepsilon\right) \in \left\{L^\infty_{\mathcal{F}}\left([0,\tau] \times \Omega; \mathbb{L}^2\right) \cap L^2_{\mathcal{F}}\left(\Omega; L^2\left(0, \tau; \mathbb{H}^1_0\right)\right)\right\} \times L^2_{\mathcal{F}}\left(\Omega; L^2(0, \tau; L_{\mathbb{Q}})\right), \\
\left(p^N_\varepsilon, Z^N_\varepsilon\right) \in \left\{L^\infty_{\mathcal{F}}\left([0,\tau] \times \Omega; L^2\right) \cap L^2_{\mathcal{F}}\left(\Omega; L^2\left(0, \tau; H^1_0\right)\right)\right\} \times L^2_{\mathcal{F}}\left(\Omega; L^2(0, \tau; L_{Q})\right).
\end{gathered} \tag{3.3}$$

Proof. Applying the Itô formula to $|p^N_\varepsilon(t)|^2$ to get

$$\begin{aligned}
d\left|p^N_\varepsilon(t)\right|^2 &= -\frac{2}{\varepsilon}\left\langle \nabla \cdot \mathbf{u}^N_\varepsilon(t), p^N_\varepsilon(t)\right\rangle dt + \frac{2}{\varepsilon}\left\langle Z^N_\varepsilon(t) dW^N(t), p^N_\varepsilon(t)\right\rangle + \frac{1}{\varepsilon^2} \operatorname{tr}\left[Z^N_\varepsilon(t) Q\left(Z^N_\varepsilon(t)\right)^*\right] dt \\
&= \frac{2}{\varepsilon}\left\langle \nabla p^N_\varepsilon(t), \mathbf{u}^N_\varepsilon(t)\right\rangle dt + \frac{2}{\varepsilon}\left\langle Z^N_\varepsilon(t) dW^N(t), p^N_\varepsilon(t)\right\rangle + \frac{1}{\varepsilon^2} \operatorname{tr}\left[Z^N_\varepsilon(t) Q\left(Z^N_\varepsilon(t)\right)^*\right] dt,
\end{aligned} \tag{3.4}$$

thus we get

$$2\left\langle \nabla p_\varepsilon^N(t), \mathbf{u}_\varepsilon^N(t)\right\rangle dt = \varepsilon d\left|p_\varepsilon^N(t)\right|^2 - 2\left\langle Z_\varepsilon^N(t)dW^N(t), p_\varepsilon^N(t)\right\rangle - \frac{1}{\varepsilon}\left\|Z_\varepsilon^N(t)\right\|_{L_Q}^2 dt. \tag{3.5}$$

By means of the Itô formula, one has

$$\begin{aligned}\left|\mathbf{u}_\varepsilon^N(t\wedge\tau)\right|^2 &= \left|\xi^N\right|^2 + 2\int_{t\wedge\tau}^{\tau}\left\langle \nu\widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(s) + \widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^N(s)\right) + \nabla p_\varepsilon^N(s) - \mathbf{f}^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle ds \\ &\quad - 2\int_{t\wedge\tau}^{\tau}\langle \mathbf{Z}_\varepsilon^N(s)d\mathbb{W}^N(s), \mathbf{u}_\varepsilon^N(s)\rangle - \int_{t\wedge\tau}^{\tau}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_{\mathbb{Q}}}^2 ds.\end{aligned} \tag{3.6}$$

Clearly,

$$\left\langle \widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^N(s)\right), \mathbf{u}_\varepsilon^N(s)\right\rangle = 0, \tag{3.7}$$

and Lemma 2.2 yields

$$\left|2\left\langle \mathbf{f}^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle\right| \le \left\|\mathbf{f}^N(s)\right\|_{\mathbb{H}^{-1}}^2 + \left\|\mathbf{u}_\varepsilon^N(s)\right\|^2 = \left|\mathbf{f}^N(s)\right|_{\mathbb{H}^{-1}}^2 + \langle \widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(s), \mathbf{u}_\varepsilon^N(s)\rangle. \tag{3.8}$$

For $0 < r \le t$, taking the conditional expectation with respect to $\mathcal{F}_{r\wedge\tau}$, and by (3.5), the above two equation and along with the fact that $\|\mathbf{u}_\varepsilon(s)\|^2 = \langle \widehat{\mathbf{A}}\mathbf{u}_\varepsilon(s), \mathbf{u}_\varepsilon(s)\rangle$, one gets

$$\begin{aligned}&E^{\mathcal{F}_{r\wedge\tau}}\left|\mathbf{u}_\varepsilon^N(t\wedge\tau)\right|^2 + E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_{\mathbb{Q}}}^2 ds + E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|\mathbf{u}_\varepsilon^N(s)\right\|^2 ds \\ &\quad \le E^{\mathcal{F}_{r\wedge\tau}}\left|\xi^N\right|^2 + 2(\nu+1)E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\langle \widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle ds + E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|\mathbf{f}^N(s)\right\|_{\mathbb{H}^{-1}}^2 ds \\ &\quad + \varepsilon E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau} d\left|p_\varepsilon^N(s)\right|^2 - \frac{1}{\varepsilon}E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|Z_\varepsilon^N(s)\right\|_{L_Q}^2 ds,\end{aligned} \tag{3.9}$$

P-a.s. Since $\widehat{\mathbf{A}}\mathbf{e}_j = \lambda_j\mathbf{e}_j$ and $\lambda_i \le \lambda_j$ for $i < j$, one gets

$$\left\langle \widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle \le \lambda_N\left|\mathbf{u}_\varepsilon^N(s)\right|^2. \tag{3.10}$$

Thus

$$\begin{aligned}
&E^{\mathcal{F}_{r\wedge\tau}}\left|\mathbf{u}_\varepsilon^N(t\wedge\tau)\right|^2+\varepsilon E^{\mathcal{F}_{r\wedge\tau}}\left|p_\varepsilon^N(t\wedge\tau)\right|^2+E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|\mathbf{u}_\varepsilon^N(s)\right\|^2ds\\
&\quad+E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_{\mathbb{Q}}}^2ds+\frac{1}{\varepsilon}E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|Z_\varepsilon^N(s)\right\|_{L_Q}^2ds\\
&\le E^{\mathcal{F}_{r\wedge\tau}}\left|\xi^N\right|^2+\varepsilon E^{\mathcal{F}_{r\wedge\tau}}\left|\eta^N\right|^2+2(\nu+1)\lambda_N\int_t^T E^{\mathcal{F}_{r\wedge\tau}}\left|\mathbf{u}_\varepsilon^N(s\wedge\tau)\right|^2ds\\
&\quad+E^{\mathcal{F}_{r\wedge\tau}}\int_0^{\tau}\left\|\mathbf{f}^N(s)\right\|_{\mathbb{H}^{-1}}^2ds,
\end{aligned}\tag{3.11}$$

P-a.s., and by Lemma 2.6, the backward Gronwall inequality, and letting $r=t$, we get

$$\begin{aligned}
&\left|\mathbf{u}_\varepsilon^N(t\wedge\tau)\right|^2+\varepsilon\left|p_\varepsilon^N(t\wedge\tau)\right|^2+E^{\mathcal{F}_{t\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|\mathbf{u}_\varepsilon^N(s)\right\|^2ds\\
&\quad+E^{\mathcal{F}_{t\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_{\mathbb{Q}}}^2ds+\frac{1}{\varepsilon}E^{\mathcal{F}_{t\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|Z_\varepsilon^N(s)\right\|_{L_Q}^2ds\\
&\le\left(E^{\mathcal{F}_{t\wedge\tau}}\left|\xi^N\right|^2+\varepsilon E^{\mathcal{F}_{t\wedge\tau}}\left|\eta^N\right|^2+E^{\mathcal{F}_{t\wedge\tau}}\int_0^{\tau}\left\|\mathbf{f}^N(s)\right\|_{\mathbb{H}^{-1}}^2ds\right)e^{2(\nu+1)\lambda_N(T-t)},
\end{aligned}\tag{3.12}$$

P-a.s. Because of the integrability of ξ, η, and $\mathbf{f}$, there exists a constant K_N, depending on N only, s.t.

$$\left|\mathbf{u}_\varepsilon^N(t)\right|^2+\varepsilon\left|p_\varepsilon^N(t)\right|^2+E\int_0^{\tau}\left\|\mathbf{u}_\varepsilon^N(s)\right\|^2ds+E\int_0^{\tau}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_{\mathbb{Q}}}^2ds+E\int_0^{\tau}\left\|Z_\varepsilon^N(s)\right\|_{L_Q}^2ds\le K_N,\tag{3.13}$$

for all $t\in[0,\tau]$, P-a.s.

Similarly, making use of (3.4), it follows that $p_\varepsilon^N\in L_{\mathcal{F}}^2(\Omega;L^2(0,\tau;H_0^1))$. □

Proposition 3.2. *Let $\xi\in L_{\mathcal{F}_\tau}^n(\Omega;\mathbb{L}^2)$, $\eta\in L_{\mathcal{F}_\tau}^n(\Omega;L^2)$, and $\mathbf{f}\in L^2(0,T;\mathbb{H}^{-1})$, for all $n\in\mathbb{N}$ and $n\ge 2$. The following is true for any solution of system* (3.2)*:*

$$\begin{aligned}
&\mathbf{u}_\varepsilon^N\in L^\infty\left(0,\tau;L_{\mathcal{F}}^n\left(\Omega;\mathbb{L}^2\right)\right)\cap L_{\mathcal{F}}^n\left(\Omega;L^n\left(0,\tau;\mathbb{H}_0^1\right)\right),\\
&p_\varepsilon^N\in L^\infty\left(0,\tau;L_{\mathcal{F}}^n\left(\Omega;L^2\right)\right)\cap L_{\mathcal{F}}^n\left(\Omega;L^n\left(0,\tau;H_0^1\right)\right).
\end{aligned}\tag{3.14}$$

Proof. Let us prove it by the method of mathematical induction. Similar to Proposition 3.1, it is easy to obtain the result for $n=2$. Suppose that it is true for all $m\le n-1$. Let us show that the proposition holds for $m=n$.

An application of the Itô formula to $|\mathbf{u}_\varepsilon^N(t)|^n$ yields

$$\begin{aligned}
&\left|\mathbf{u}_\varepsilon^N(t\wedge\tau)\right|^n \\
&\quad= \left|\xi^N\right|^n + n\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^{n-2}\left\langle \nu\widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(s) + \widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^N(s)\right) + \nabla p_\varepsilon^N(s) - \mathbf{f}^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle ds \\
&\qquad - n\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^{n-2}\left\langle \mathbf{Z}_\varepsilon^N(s)d\mathbb{W}^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle - \frac{n^2-n}{2}\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^{n-2}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_{\mathbb{Q}}}^2 ds.
\end{aligned} \tag{3.15}$$

Clearly $|\nabla p_\varepsilon^N(s)| \le C\|p_\varepsilon^N(s)\| \le C\sqrt{\kappa_N}|p_\varepsilon^N(s)|$, where κ_N, as stated in Section 2, is the eigenvalue of $-\Delta$ for e_N. Taking the expectation, one obtains

$$\begin{aligned}
&E\left|\mathbf{u}_\varepsilon^N(t\wedge\tau)\right|^n + E\int_{t\wedge\tau}^{\tau}\left\|\mathbf{u}_\varepsilon^N(s)\right\|^n ds \le E\left|\xi^N\right|^n + \lambda_N^{n/2}E\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^n ds \\
&\quad + nE\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^{n-2}\left\langle \nu\widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(s) + \nabla p_\varepsilon^N(s) - \mathbf{f}^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle ds \\
&\le E\left|\xi^N\right|^n + \left(\nu\lambda_N n + \lambda_N^{n/2}\right)E\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^n ds + nE\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^{n-1}\left|\nabla p_\varepsilon^N(s)\right| ds \\
&\quad + \frac{n}{2}E\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^{n-2}\left(\left\|\mathbf{f}^N(s)\right\|_{\mathbb{H}^{-1}}^2 + \lambda_N\left|\mathbf{u}_\varepsilon^N(s)\right|^2\right) ds \\
&\le E\left|\xi^N\right|^n + \left(\nu\lambda_N n + \lambda_N^{n/2} + \frac{n}{2}\lambda_N\right)E\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^n ds \\
&\quad + nC\sqrt{\kappa_N}\left\{E\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^n ds\right\}^{(n-1)/n}\left\{E\int_{t\wedge\tau}^{\tau}\left|p_\varepsilon^N(s)\right|^n ds\right\}^{1/n} \\
&\quad + \frac{n}{2}\int_t^T\left\|\mathbf{f}^N(s)\right\|_{\mathbb{H}^{-1}}^2 E\left\{1_{[t\wedge\tau,\tau]}\left|\mathbf{u}_\varepsilon^N(s)\right|^{n-2}\right\} ds \\
&\le E\left|\xi^N\right|^n + \frac{n}{2}\left\{\sup_{0\le t\le\tau}E\left|\mathbf{u}_\varepsilon^N(t)\right|^{n-2}\right\}\int_t^T\left\|\mathbf{f}^N(s)\right\|_{\mathbb{H}^{-1}}^2 ds \\
&\quad + \left(\nu\lambda_N n + \lambda_N^{n/2} + \frac{n}{2}\lambda_N + (n-1)C\sqrt{\kappa_N}\right)\int_t^T E\left|\mathbf{u}_\varepsilon^N(s\wedge\tau)\right|^n ds \\
&\quad + C\sqrt{\kappa_N}\int_t^T E\left|p_\varepsilon^N(s\wedge\tau)\right|^n ds \\
&\le K + K(n,N)\int_t^T E\left|\mathbf{u}_\varepsilon^N(s\wedge\tau)\right|^n ds + K(n,N)\int_t^T E\left|p_\varepsilon^N(s\wedge\tau)\right|^n ds,
\end{aligned} \tag{3.16}$$

where K is a constant, and $K(n,N)$ is a constant depending on n and N. Both constants may vary throughout the proof. But we keep the same notations for simplicity. Applying the Itô formula to $|p_\varepsilon^N(t)|^n$, one obtains

$$\begin{aligned}
&\varepsilon E\left|p_\varepsilon^N(t\wedge\tau)\right|^n + E\int_{t\wedge\tau}^{\tau}\left\|p_\varepsilon^N(s)\right\|^n ds \\
&\quad\le \varepsilon E\left|\eta^N\right|^n + \kappa_N^{n/2}E\int_{t\wedge\tau}^{\tau}\left|p_\varepsilon^N(s)\right|^n ds - nE\int_{t\wedge\tau}^{\tau}\left|p_\varepsilon^N(s)\right|^{n-2}\left\langle\nabla p_\varepsilon^N(s),\mathbf{u}_\varepsilon^N(s)\right\rangle ds \\
&\quad\le \varepsilon E\left|\eta^N\right|^n + \kappa_N^{n/2}E\int_{t\wedge\tau}^{\tau}\left|p_\varepsilon^N(s)\right|^n ds + nC\sqrt{\kappa_N}E\int_{t\wedge\tau}^{\tau}\left|p_\varepsilon^N(s)\right|^{n-1}\left|\mathbf{u}_\varepsilon^N(s)\right| ds \\
&\quad\le \varepsilon E\left|\eta^N\right|^n + \kappa_N^{n/2}E\int_{t\wedge\tau}^{\tau}\left|p_\varepsilon^N(s)\right|^n ds + nC\sqrt{\kappa_N}\left\{E\int_{t\wedge\tau}^{\tau}\left|p_\varepsilon^N(s)\right|^n ds\right\}^{(n-1)/n} \\
&\qquad\times\left\{E\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s)\right|^n ds\right\}^{1/n} \\
&\quad\le K + K(n,N)\int_t^T E\left|p_\varepsilon^N(s\wedge\tau)\right|^n ds + K(n,N)\int_t^T E\left|\mathbf{u}_\varepsilon^N(s\wedge\tau)\right|^n ds.
\end{aligned} \tag{3.17}$$

Adding up (3.16) and (3.17), one gets

$$\begin{aligned}
&E\left\{\left|\mathbf{u}_\varepsilon^N(t\wedge\tau)\right|^n + \varepsilon\left|p_\varepsilon^N(t\wedge\tau)\right|^n\right\} + E\int_{t\wedge\tau}^{\tau}\left\{\left\|\mathbf{u}_\varepsilon^N(s)\right\|^n + \left\|p_\varepsilon^N(s)\right\|^n\right\}ds \\
&\qquad\le K + K(n,N)\int_t^T E\left\{\left|\mathbf{u}_\varepsilon^N(s\wedge\tau)\right|^n + \left|p_\varepsilon^N(s\wedge\tau)\right|^n\right\}ds.
\end{aligned} \tag{3.18}$$

An application of the Gronwall inequality (2.11) yields the result. □

4. Existence of Solutions

The following lemma states the monotonicity of drift coefficients. The proof involves Proposition 2.5 and is straightforward.

Lemma 4.1. *Assume* $\mathbf{u},\mathbf{v}\in\mathbb{H}_0^1$ *and* $\mathbf{w}\in\mathbb{L}^4$. *The following inequalities are true:*

(a) $|\langle\widehat{\mathbf{B}}(\mathbf{u}),\mathbf{w}\rangle| \le 2\|\mathbf{u}\|^{3/2}|\mathbf{u}|^{1/2}\|\mathbf{w}\|_{\mathbb{L}^4}$,

(b) $|\langle\widehat{\mathbf{B}}(\mathbf{u})-\widehat{\mathbf{B}}(\mathbf{v}),\mathbf{u}-\mathbf{v}\rangle| \le (\nu/2)\|\mathbf{u}-\mathbf{v}\|^2 + (27/2\nu^3)|\mathbf{u}-\mathbf{v}|^2\|\mathbf{v}\|_{\mathbb{L}^4}^4$,

(c) $\langle\nu\widehat{\mathbf{A}}(\mathbf{u}-\mathbf{v})+\widehat{\mathbf{B}}(\mathbf{u})-\widehat{\mathbf{B}}(\mathbf{v}),\mathbf{u}-\mathbf{v}\rangle + (27/2\nu^3)\|\mathbf{v}\|_{\mathbb{L}^4}^4|\mathbf{u}-\mathbf{v}|^2 \ge (\nu/2)\|\mathbf{u}-\mathbf{v}\|^2$.

Furthermore, if $\mathbf{w}\in\mathbb{H}_0^1$, *then there exists a constant* C *depending on* ν, *such that*

(d) $\langle\nu\widehat{\mathbf{A}}(\mathbf{u}-\mathbf{v})+\widehat{\mathbf{B}}(\mathbf{u})-\widehat{\mathbf{B}}(\mathbf{v}),\mathbf{u}-\mathbf{v}\rangle + C\|\mathbf{v}\|^2|\mathbf{u}-\mathbf{v}|^2 \ge (\nu/2)\|\mathbf{u}-\mathbf{v}\|^2$.

Corollary 4.2. *For any* $\mathbf{u}$ *and* $\mathbf{v} \in \mathbb{L}^4$, *let*

$$h_1(t) = \frac{27}{\nu^3}\int_0^t \|\mathbf{u}(s)\|_{\mathbb{L}^4}^4 ds,$$
$$h_2(t) = \frac{27}{\nu^3}\int_0^t \|\mathbf{v}(s)\|_{\mathbb{L}^4}^4 ds. \tag{4.1}$$

Then

$$\left\langle \nu\widehat{\mathbf{A}}(\mathbf{u}-\mathbf{v}) + \widehat{\mathbf{B}}(\mathbf{u}) - \widehat{\mathbf{B}}(\mathbf{v}) + \frac{1}{2}\dot{h}_i(t)(\mathbf{u}-\mathbf{v}), \mathbf{u}-\mathbf{v}\right\rangle \geq 0, \quad i = 1,2. \tag{4.2}$$

The proposition below is used in the proof of the existence, and we provide a brief proof. Readers may refer to [14, 16] for a similar and detailed proof.

Proposition 4.3. *Let* $\xi \in L^\infty_{\mathcal{F}_\tau}(\Omega;\mathbb{L}^2)$, $\eta \in L^\infty_{\mathcal{F}_\tau}(\Omega;L^2)$, *and* $\mathbf{f} \in L^2(0,T;\mathbb{H}^{-1})$. *Then the projected system* (3.2) *admits a unique adapted solution* $(\mathbf{u}_\varepsilon^N, \mathbf{Z}_\varepsilon^N, p_\varepsilon^N, Z_\varepsilon^N)$ *in*

$$\begin{aligned}&\left\{L^\infty_{\mathcal{F}}\left([0,\tau]\times\Omega;\mathbb{L}^2\right)\cap L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,\tau;\mathbb{H}^1_0\right)\right)\right\}\times L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_{\mathbb{Q}})\right)\\ &\quad\times\left\{L^\infty_{\mathcal{F}}\left([0,\tau]\times\Omega;L^2\right)\cap L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,\tau;H^1_0\right)\right)\right\}\times L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_Q)\right).\end{aligned} \tag{4.3}$$

Proof. For every $M \in \mathbb{N}$, let L_M be a Lipschitz C^∞ function which has the following property:

$$L_M(\|\mathbf{u}\|) = \begin{cases} 1 & \text{if } \|\mathbf{u}\| < M, \\ 0 & \text{if } \|\mathbf{u}\| > M+1, \\ 0 \leq L_M(\|\mathbf{u}\|) \leq 1 & \text{otherwise.}\end{cases} \tag{4.4}$$

Applying the truncation L_M to $\widehat{\mathbf{B}}$, it is easy to show that $L_M\widehat{\mathbf{B}}$ is Lipschitz and

$$\left|L_M(\|\mathbf{x}\|)\widehat{\mathbf{B}}^N(\mathbf{x}) - L_M(\|\mathbf{y}\|)\widehat{\mathbf{B}}^N(\mathbf{y})\right| \leq C_{N,M}\|\mathbf{x}-\mathbf{y}\| \tag{4.5}$$

for any $\mathbf{x},\mathbf{y} \in \mathbb{L}^2_N$ and $M \in \mathbb{N}$. Let us define a truncated projected system:

$$\begin{aligned} d\mathbf{u}_\varepsilon^{N,M}(t) &= -\nu\widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^{N,M}(t)dt - L_M\left(\left\|\mathbf{u}_\varepsilon^{N,M}(t)\right\|\right)\widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^{N,M}(t)\right)dt - \nabla p_\varepsilon^{N,M}(t)dt \\ &\quad + \mathbf{f}^N(t)dt + \mathbf{Z}_\varepsilon^{N,M}(t)d\mathbb{W}^N(t), \\ \varepsilon dp_\varepsilon^{N,M}(t) &+ \nabla\cdot\mathbf{u}_\varepsilon^{N,M}(t)dt = Z_\varepsilon^{N,M}(t)dW^N(t), \\ \mathbf{u}_\varepsilon^{N,M}(\tau) &= \xi^N, \qquad p_\varepsilon^{N,M}(\tau) = \eta^N. \end{aligned} \tag{4.6}$$

For fixed $p \in L^{\infty}_{\mathcal{F}}([0,\tau]\times\Omega; L^2) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,\tau; H^1_0))$, let us map

$$\begin{aligned} d\mathbf{u}_\varepsilon^{N,M}(t) &= -\nu\widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^{N,M}(t)dt - L_M\left(\left\|\mathbf{u}_\varepsilon^{N,M}(t)\right\|\right)\widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^{N,M}(t)\right)dt - \nabla p^{N,M}(t)dt \\ &\quad + \mathbf{f}^N(t)dt + \mathbf{Z}_\varepsilon^{N,M}(t)d\mathbb{W}^N(t), \\ \mathbf{u}_\varepsilon^{N,M}(\tau) &= \xi^N \end{aligned} \tag{4.7}$$

to $\mathbb{R}^N$, and the image of the system is equivalent to the system. Since the coefficients in the image system are Lipschitz, a well-known result in $\mathbb{R}^N$ (see [14, page 355]) guarantees the existence of a unique adapted solution. Let the solution be $(\mathbf{u}_\varepsilon^{N,M}, \mathbf{Z}_\varepsilon^{N,M})$. Then for

$$\begin{aligned} \varepsilon dp_\varepsilon^{N,M}(t) + \nabla\cdot\mathbf{u}_\varepsilon^{N,M}(t)dt &= Z_\varepsilon^{N,M}(t)dW^N(t) \\ p_\varepsilon^{N,M}(\tau) &= \eta^N, \end{aligned} \tag{4.8}$$

there is a unique adapted solution $(p_\varepsilon^{N,M}, Z_\varepsilon^{N,M})$. Thus we can define an operator Ψ, such that $\Psi(p) = p_\varepsilon^{N,M}$. It can be shown that Ψ is a contraction mapping. Thus the unique adapted solution of (4.6) can be obtained. Let us take the limit of the solution as M approaches infinity. It can be shown that the limit is the unique solution of the projected system (3.2). □

From now on, let us assume the external body force to be an operator and denote it by $\mathbf{F}$. We also assume the following coercivity and monotonicity hypotheses in this paper. Such an approach is commonly used in studying the stochastic Euler equations so that a dissipative effect arises. Also they are standard hypotheses in the theory of stochastic PDEs in infinite dimensional spaces (see Chow [15], Kallianpur and Xiong [17], Prévôt and Röckner [18]).

Assumption A. (A.1) $\mathbf{F}$: $\mathbb{H}^1_0 \to \mathbb{H}^{-1}$ is a continuous operator.

(A.2) There exist positive constants α and β, such that

$$\begin{aligned} \left\langle \nu\widehat{\mathbf{A}}\mathbf{u} - \mathbf{F}(\mathbf{u}), \mathbf{u}\right\rangle &\le \alpha|\mathbf{u}|^2 - \beta\|\mathbf{u}\|^2; \\ \left\langle \nu\widehat{\mathbf{A}}\mathbf{u} - \mathbf{F}(\mathbf{u}), \widehat{\mathbf{A}}\mathbf{u}\right\rangle &\le \alpha\|\mathbf{u}\|^2 - \beta\left\|\widehat{\mathbf{A}}\mathbf{u}\right\|^2. \end{aligned} \tag{4.9}$$

(A.3) For any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{H}^1_0$, a constant $\kappa > \nu$, and a positive constant α,

$$\left\langle \kappa\widehat{\mathbf{A}}(\mathbf{u}-\mathbf{v}) - (\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{v})), \mathbf{u}-\mathbf{v}\right\rangle \le \alpha|\mathbf{u}-\mathbf{v}|^2. \tag{4.10}$$

(A.4) For any $\mathbf{u} \in \mathbb{H}^1_0$ and some positive constant α,

$$|\langle \mathbf{F}(\mathbf{u}), \mathbf{u}\rangle| \le \alpha\|\mathbf{u}\|^2. \tag{4.11}$$

Remark 4.4. Assumption (A.2) is usually called the coercivity condition of the dissipative term and the external body force. Assumption (A.3) is the monotonicity condition of dissipative term and the external body force. The first half of the inequality is used in the proof of the uniqueness in Section 5. The second half of the inequality is used in the proof of the existence in Section 4. Assumption (A.4) is the linear growth condition of the external body force.

Under above assumptions, we adjust systems (2.6) and (3.2) to the following two systems:

$$\begin{gathered} d\mathbf{u}_\varepsilon(t) + \nu\widehat{\mathbf{A}}\mathbf{u}_\varepsilon(t)dt + \widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(t))dt + \nabla p_\varepsilon(t)dt = \mathbf{F}(\mathbf{u}_\varepsilon(t))dt + \mathbf{Z}_\varepsilon(t)d\mathbb{W}(t), \\ \varepsilon dp_\varepsilon(t) + \nabla\cdot\mathbf{u}_\varepsilon(t)dt = Z_\varepsilon(t)dW(t), \\ \mathbf{u}_\varepsilon(\tau) = \xi, \qquad p_\varepsilon(\tau) = \eta, \end{gathered} \tag{4.12}$$

$$\begin{gathered} d\mathbf{u}_\varepsilon^N(t) = -\nu\widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(t)dt - \widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^N(t)\right)dt - \nabla p_\varepsilon^N(t)dt + \mathbf{F}^N\left(\mathbf{u}_\varepsilon^N(t)\right)dt + \mathbf{Z}_\varepsilon^N(t)d\mathbb{W}^N(t), \\ \varepsilon dp_\varepsilon^N(t) + \nabla\cdot\mathbf{u}_\varepsilon^N(t)dt = Z_\varepsilon^N(t)dW^N(t), \\ \mathbf{u}_\varepsilon^N(\tau) = \xi^N, \qquad p_\varepsilon^N(\tau) = \eta^N \end{gathered} \tag{4.13}$$

for $0 \le t \le \tau$. The existence and uniqueness of an adapted solution of (4.13) can be easily checked in the same fashion as in Proposition 4.3.

Lemma 4.5. *Assume* $\mathbf{u}$ *and* $\mathbf{v} \in \mathbb{L}^4$. *Then the following inequality is true:*

$$\left|\left\langle \widehat{\mathbf{B}}(\mathbf{u}) - \widehat{\mathbf{B}}(\mathbf{v}), \mathbf{u}-\mathbf{v}\right\rangle\right| \le (\kappa-\nu)\|\mathbf{u}-\mathbf{v}\|^2 + \frac{27}{16(\kappa-\nu)^3}|\mathbf{u}-\mathbf{v}|^2\|\mathbf{v}\|_{\mathbb{L}^4}^4. \tag{4.14}$$

Corollary 4.6. *Let* $\mathbf{u}$ *and* $\mathbf{v} \in \mathbb{L}^4$. *Define*

$$\begin{aligned} l_1(t) &\triangleq \int_t^T \left\{2\alpha + \frac{27}{8(\kappa-\nu)^3}\|\mathbf{u}(s)\|_{\mathbb{L}^4}^4\right\}ds, \\ l_2(t) &\triangleq \int_t^T \left\{2\alpha + \frac{27}{8(\kappa-\nu)^3}\|\mathbf{v}(s)\|_{\mathbb{L}^4}^4\right\}ds. \end{aligned} \tag{4.15}$$

Then

$$\left\langle \nu\widehat{\mathbf{A}}(\mathbf{u}-\mathbf{v}) + \widehat{\mathbf{B}}(\mathbf{u}) - \widehat{\mathbf{B}}(\mathbf{v}) - (\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{v})) + \frac{1}{2}l_i(t)(\mathbf{u}-\mathbf{v}), \mathbf{u}-\mathbf{v}\right\rangle \le 0, \qquad i = 1,2. \tag{4.16}$$

Remark 4.7. To prove Corollary 4.6, the monotonicity assumption (A.3) is used.

Proposition 4.8. (i) *Let $\xi \in L^{\infty}_{\mathcal{F}_\tau}(\Omega;\mathbb{L}^2)$ and $\eta \in L^{\infty}_{\mathcal{F}_\tau}(\Omega;L^2)$. Then for any solution of system* (4.13), *the following is true:*

$$\begin{gathered}\left(\mathbf{u}_\varepsilon^N,\mathbf{Z}_\varepsilon^N\right)\in\left\{L^{\infty}_{\mathcal{F}}\left([0,\tau]\times\Omega;\mathbb{L}^2\right)\cap L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,\tau;\mathbb{H}^1_0\right)\right)\right\}\times L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_{\mathbb{Q}})\right),\\ \left(p_\varepsilon^N,Z_\varepsilon^N\right)\in L^{\infty}_{\mathcal{F}}\left([0,\tau]\times\Omega;L^2\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_Q)\right).\end{gathered}\tag{4.17}$$

Moreover, there exists a constant K, independent of N, such that

$$\begin{gathered}\sup_{t\in[0,\tau]}\left|\mathbf{u}_\varepsilon^N(t)\right|^2+E\int_0^\tau\left\|\mathbf{u}_\varepsilon^N(s)\right\|^2ds+\varepsilon\sup_{t\in[0,\tau]}\left|p_\varepsilon^N(t)\right|^2\\ +E\int_0^\tau\left\|\mathbf{Z}_\varepsilon^N(s)\right\|^2_{L_{\mathbb{Q}}}ds+E\int_0^\tau\left\|Z_\varepsilon^N(s)\right\|^2_{L_Q}ds\le K,\end{gathered}\tag{4.18}$$

P-a.s.

(ii) *Let $\xi \in L^2_{\mathcal{F}_\tau}(\Omega;\mathbb{L}^2)$ and $\eta \in L^2_{\mathcal{F}_\tau}(\Omega;L^2)$. The following is true for any solution of system* (4.13)*:*

$$\begin{gathered}\left(\mathbf{u}_\varepsilon^N,\mathbf{Z}_\varepsilon^N\right)\in\left\{L^{\infty}\left(0,\tau;L^2_{\mathcal{F}}\left(\Omega;\mathbb{L}^2\right)\right)\cap L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,\tau;\mathbb{H}^1_0\right)\right)\right\}\times L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_{\mathbb{Q}})\right),\\ \left(p_\varepsilon^N,Z_\varepsilon^N\right)\in L^{\infty}\left(0,\tau;L^2_{\mathcal{F}}\left(\Omega;L^2\right)\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_Q)\right).\end{gathered}\tag{4.19}$$

Moreover, there exists a constant K, independent of N, such that

$$\begin{gathered}\sup_{t\in[0,\tau]}E\left|\mathbf{u}_\varepsilon^N(t)\right|^2+E\int_0^\tau\left\|\mathbf{u}_\varepsilon^N(s)\right\|^2ds+\varepsilon\sup_{t\in[0,\tau]}E\left|p_\varepsilon^N(t)\right|^2\\ +E\int_0^\tau\left\|\mathbf{Z}_\varepsilon^N(s)\right\|^2_{L_{\mathbb{Q}}}ds+E\int_0^\tau\left\|Z_\varepsilon^N(s)\right\|^2_{L_Q}ds\le K.\end{gathered}\tag{4.20}$$

Proof. (i) Similar to the proof of Proposition 3.1, utilizing Assumption (A.2), (3.6) becomes

$$\begin{aligned}&\left|\mathbf{u}_\varepsilon^N(t\wedge\tau)\right|^2\\ &\quad=\left|\xi^N\right|^2+2\int_{t\wedge\tau}^\tau\left\langle\nu\widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(s)+\widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^N(s)\right)+\nabla p_\varepsilon^N(s)-\mathbf{F}^N\left(\mathbf{u}_\varepsilon^N(s)\right),\mathbf{u}_\varepsilon^N(s)\right\rangle ds\\ &\qquad-2\int_{t\wedge\tau}^\tau\left\langle\mathbf{Z}_\varepsilon^N(s)d\mathbb{W}^N(s),\mathbf{u}_\varepsilon^N(s)\right\rangle-\int_{t\wedge\tau}^\tau\left\|\mathbf{Z}_\varepsilon^N(s)\right\|^2_{L_{\mathbb{Q}}}ds\\ &\quad\le\left|\xi^N\right|^2+2\int_{t\wedge\tau}^\tau\left\{\alpha\left|\mathbf{u}_\varepsilon^N(s)\right|^2-\beta\left\|\mathbf{u}_\varepsilon^N(s)\right\|^2+\left\langle\nabla p_\varepsilon^N(s),\mathbf{u}_\varepsilon^N(s)\right\rangle\right\}ds\\ &\qquad-2\int_{t\wedge\tau}^\tau\left\langle\mathbf{Z}_\varepsilon^N(s)d\mathbb{W}^N(s),\mathbf{u}_\varepsilon^N(s)\right\rangle-\int_{t\wedge\tau}^\tau\left\|\mathbf{Z}_\varepsilon^N(s)\right\|^2_{L_{\mathbb{Q}}}ds.\end{aligned}\tag{4.21}$$

For $0 < r \le t$, taking the conditional expectation with respect to $\mathcal{F}_{r\wedge\tau}$, one gets

$$\begin{aligned} &E^{\mathcal{F}_{r\wedge\tau}}\left|\mathbf{u}_\varepsilon^N(t\wedge\tau)\right|^2 + E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_{\mathbb{Q}}}^2 ds + 2\beta E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|\mathbf{u}_\varepsilon^N(s)\right\|^2 ds \\ &\quad\le E^{\mathcal{F}_{r\wedge\tau}}\left|\xi^N\right|^2 + 2\alpha E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left|\mathbf{u}_\varepsilon^N(s\wedge\tau)\right|^2 ds + \varepsilon E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau} d\left|p_\varepsilon^N(s)\right|^2 \\ &\quad - \frac{1}{\varepsilon}E^{\mathcal{F}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau}\left\|Z_\varepsilon^N(s)\right\|_{L_{\mathbb{Q}}}^2 ds, \end{aligned} \tag{4.22}$$

P-a.s. By the backward Gronwall inequality, and letting $r = t$, we get (4.18).

(ii) The proof is similar to (i). □

Proposition 4.9. *Suppose that* $\xi \in L^\infty_{\mathcal{F}_\tau}(\Omega;\mathbb{H}_0^1)$ *and* $\eta \in L^\infty_{\mathcal{F}_\tau}(\Omega;H_0^1)$. *Then for any solution* $(\mathbf{u}_\varepsilon, \mathbf{Z}_\varepsilon, p_\varepsilon, Z_\varepsilon)$ *of system* (4.13), *there exists a constant* K_0, *such that*

$$\sup_{t\in[0,\tau]}\|\mathbf{u}_\varepsilon(t)\|^2 + \sup_{t\in[0,\tau]}\|p_\varepsilon\|^2 \le K_0. \tag{4.23}$$

Proof. The proof involves an application of the Itô formula to $\|\mathbf{u}_\varepsilon(t)\|^2$, and the second half of the coercivity assumption. We skip the proof since it is similar to Proposition 3.1. □

Theorem 4.10. *Let* $\xi \in L^\infty_{\mathcal{F}_\tau}(\Omega;\mathbb{H}_0^1)$ *and* $\eta \in L^\infty_{\mathcal{F}_\tau}(\Omega;H_0^1)$. *For system* (4.12), *there exists a solution* $(\mathbf{u}_\varepsilon, \mathbf{Z}_\varepsilon, p_\varepsilon, Z_\varepsilon)$ *in*

$$L^\infty_{\mathcal{F}}\left([0,\tau]\times\Omega;\mathbb{H}_0^1\right) \times L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_{\mathbb{Q}})\right) \times L^\infty_{\mathcal{F}}\left([0,\tau]\times\Omega;H_0^1\right) \times L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_Q)\right). \tag{4.24}$$

Proof. We have the following steps.

Step 1 (The limits). Clearly, by Proposition 4.8, there exist $\mathbf{u}_\varepsilon$, p_ε, $\mathbf{Z}_\varepsilon$, and Z_ε, such that

$$\begin{aligned} \mathbf{u}_\varepsilon^{N_k} &\xrightarrow{w} \mathbf{u}_\varepsilon \quad \text{in } L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,\tau;\mathbb{H}_0^1\right)\right), \\ p_\varepsilon^{N_k} &\xrightarrow{w} p_\varepsilon \quad \text{in } L^\infty_{\mathcal{F}}\left([0,\tau]\times\Omega;L^2\right), \\ \mathbf{Z}_\varepsilon^{N_k} &\xrightarrow{w} \mathbf{Z}_\varepsilon \quad \text{in } L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_{\mathbb{Q}})\right), \\ Z_\varepsilon^{N_k} &\xrightarrow{w} Z_\varepsilon \quad \text{in } L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_Q)\right), \end{aligned} \tag{4.25}$$

for a subsequence N_k. Since $\widehat{\mathbf{A}}$ is a continuous map from $\mathbb{H}_0^1$ to $\mathbb{H}^{-1}$,

$$\left\|\widehat{\mathbf{A}}\mathbf{u}_\varepsilon\right\|_{\mathbb{H}^{-1}} \le C\|\mathbf{u}_\varepsilon\| \tag{4.26}$$

for all $\mathbf{u}_\varepsilon \in \mathbb{H}_0^1$ and some constant C. Thus combined with the assumptions on $\mathbf{F}$, one knows that

$$\nu\widehat{\mathbf{A}}^{N_k}\mathbf{u}_\varepsilon^{N_k} - \mathbf{F}^{N_k}\left(\mathbf{u}_\varepsilon^{N_k}\right) \xrightarrow{w} \mathbf{A} \quad \text{in } L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,\tau;\mathbb{H}^{-1}\right)\right) \tag{4.27}$$

for some function $\mathbf{A}$ and some subsequence N_k. By Lemma 4.1,

$$\begin{aligned}\left\|\widehat{\mathbf{B}}^N(\mathbf{u}_\varepsilon^N(t))\right\|_{\mathbb{H}^{-1}} &= \sup_{\|\mathbf{w}\|=1}\left|\left\langle\widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^N(t)\right),\mathbf{w}\right\rangle\right| \\ &\le \sup_{\|\mathbf{w}\|=1} 2\left\|\mathbf{u}_\varepsilon^N(t)\right\|^{3/2}\left|\mathbf{u}_\varepsilon^N(t)\right|^{1/2}\|\mathbf{w}\|_{\mathbb{L}^4} \\ &\le K\left\|\mathbf{u}_\varepsilon^N(t)\right\|^{3/2}.\end{aligned} \tag{4.28}$$

Thus

$$\widehat{\mathbf{B}}^{N_k}\left(\mathbf{u}_\varepsilon^{N_k}\right) \xrightarrow{w} \mathbf{B} \quad \text{in } L^{4/3}_{\mathcal{F}}\left(\Omega; L^{4/3}\left(0,\tau;\mathbb{H}^{-1}\right)\right) \tag{4.29}$$

for some function $\mathbf{B}$ and some subsequence N_k. For every t, we define

$$\begin{aligned}\mathcal{L}_t : L^2_{\mathcal{F}}\left(\Omega; L^2(0,\tau;L_{\mathbb{Q}})\right) &\longrightarrow L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,\tau;\mathbb{H}^{-1}\right)\right) \\ M &\longmapsto \int_{t\wedge\tau}^{\tau} M(s)d\mathbb{W}(s).\end{aligned} \tag{4.30}$$

It can be shown that $\mathcal{L}_t$ is a bounded linear operator. Hence

$$\int_{t\wedge\tau}^{\tau}\mathbf{Z}_\varepsilon^{N_k}(s)d\mathbb{W}^{N_k}(s) \xrightarrow{w} \int_{t\wedge\tau}^{\tau}\mathbf{Z}_\varepsilon(s)d\mathbb{W}(s) \quad \text{in } L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,\tau;\mathbb{H}^{-1}\right)\right). \tag{4.31}$$

Similarly, one can prove that

$$\int_{t\wedge\tau}^{\tau}\left\{\nu\widehat{\mathbf{A}}^{N_k}\mathbf{u}_\varepsilon^{N_k}(s) - \mathbf{F}^{N_k}\left(\mathbf{u}_\varepsilon^{N_k}(s)\right) + \widehat{\mathbf{B}}^{N_k}\left(\mathbf{u}_\varepsilon^{N_k}(s)\right)\right\}ds \xrightarrow{w} \int_{t\wedge\tau}^{\tau}\{\mathbf{A}(s)+\mathbf{B}(s)\}ds \tag{4.32}$$

in $L^{4/3}_{\mathcal{F}}(\Omega; L^{4/3}(0,\tau;\mathbb{H}^{-1}))$ and

$$\int_{t\wedge\tau}^{\tau}Z_\varepsilon^{N_k}(s)dW^{N_k}(s) \xrightarrow{w} \int_{t\wedge\tau}^{\tau}Z_\varepsilon(s)dW(s) \quad \text{in } L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,\tau;H^{-1}\right)\right). \tag{4.33}$$

Also

$$\begin{aligned} \mathcal{L}_p : L^\infty_{\mathcal{F}}\left([0,\tau]\times\Omega; L^2\right) &\longrightarrow L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,\tau; H^{-1}\right)\right), \\ p &\longmapsto \int_{t\wedge\tau}^{\tau} \nabla p(s)ds \end{aligned} \tag{4.34}$$

is a bounded linear operator. Since $p_\varepsilon^{N_k} \in L^\infty_{\mathcal{F}}([0,\tau]\times\Omega; L^2)$, we have

$$\int_{t\wedge\tau}^{\tau} \nabla p_\varepsilon^{N_k}(s)ds \xrightarrow{w} \int_{t\wedge\tau}^{\tau} \nabla p_\varepsilon(s)ds \quad \text{in } L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,\tau; H^{-1}\right)\right). \tag{4.35}$$

Similarly,

$$\int_{t\wedge\tau}^{\tau} \nabla\cdot \mathbf{u}_\varepsilon^{N_k}(s)ds \xrightarrow{w} \int_{t\wedge\tau}^{\tau} \nabla\cdot \mathbf{u}_\varepsilon(s)ds \quad \text{in } L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,\tau; \mathbb{H}^{-1}\right)\right). \tag{4.36}$$

To sum up,

$$\begin{aligned} \mathbf{u}_\varepsilon(t\wedge\tau) &= \xi + \int_{t\wedge\tau}^{\tau} \left\{\mathbf{A}(s) + \mathbf{B}(s) + \nabla p_\varepsilon(s)\right\}ds - \int_{t\wedge\tau}^{\tau} \mathbf{Z}_\varepsilon(t)d\mathbb{W}(s), \\ \varepsilon p_\varepsilon(t\wedge\tau) &= \varepsilon\eta + \int_{t\wedge\tau}^{\tau} \nabla\cdot\mathbf{u}_\varepsilon(s)ds - \int_{t\wedge\tau}^{\tau} Z_\varepsilon(s)dW(s) \end{aligned} \tag{4.37}$$

hold P-a.s.

Step 2 (The Itô formula). For convenience, let us denote N_k by N again. Let $M \le N$ and $(\mathbb{H}_0^1)_M = P_M(\mathbb{H}_0^1)$. For any $\mathbf{v} \in L^\infty_{\mathcal{F}}([0,\tau]\times\Omega; (\mathbb{H}_0^1)_M)$ and some constant K, such that $\|\mathbf{v}\| \le K$ uniformly, define

$$r(t) = \int_t^T \left\{2\alpha + \frac{27}{8(\kappa-\nu)^3}K^4\right\}ds. \tag{4.38}$$

Applying the Itô formula to $e^{-r(t)}|\mathbf{u}_\varepsilon^N(t)|^2$, we get

$$\begin{aligned} \left|\xi^N\right|^2 - e^{-r(0)}\left|\mathbf{u}_\varepsilon^N(0)\right|^2 &= \int_0^\tau -\dot{r}(s)e^{-r(s)}\left|\mathbf{u}_\varepsilon^N(s)\right|^2 ds \\ &\quad + 2\int_0^\tau e^{-r(s)}\Big\langle -\nu\widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(s) - \widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^N(s)\right) - \nabla p_\varepsilon^N(s) \\ &\qquad + \mathbf{F}^N\left(\mathbf{u}_\varepsilon^N(s)\right), \mathbf{u}_\varepsilon^N(s)\Big\rangle ds \\ &\quad + 2\int_0^\tau e^{-r(s)}\left\langle \mathbf{Z}_\varepsilon^N(s)d\mathbb{W}^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle + \int_0^\tau e^{-r(s)}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_{\mathbb{Q}}}^2 ds. \end{aligned} \tag{4.39}$$

By taking the expectation, we get

$$\begin{aligned}
&E\left|\xi^N\right|^2 - Ee^{-r(0)}\left|\mathbf{u}_\varepsilon^N(0)\right|^2 + 2E\int_0^\tau e^{-r(s)}\left\langle \nabla p_\varepsilon^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle ds \\
&\quad = E\int_0^\tau e^{-r(s)}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_\mathbb{Q}}^2 ds \\
&\qquad - 2E\int_0^\tau e^{-r(s)}\left\langle \nu\widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(s) - \mathbf{F}^N\left(\mathbf{u}_\varepsilon^N(s)\right) + \widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^N(s)\right) + \frac{1}{2}\dot{r}(s)\mathbf{u}_\varepsilon^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle ds.
\end{aligned} \tag{4.40}$$

Clearly, $\lim_{N\to\infty} E|\xi^N|^2 = E|\xi|^2$. By (3.5), it is clear that

$$2E\int_0^\tau e^{-r(s)}\left\langle \nabla p_\varepsilon^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle ds = \varepsilon E\left|\eta^N\right|^2 - \varepsilon Ee^{-r(0)}\left|p_\varepsilon^N(0)\right|^2 - \frac{1}{\varepsilon}E\int_0^\tau e^{-r(s)}\left\|Z_\varepsilon^N(s)\right\|_{L_Q}^2 ds. \tag{4.41}$$

Because of (4.40) and (4.41), one gets the following:

$$\begin{aligned}
&\varliminf_{N\to\infty} 2E\int_0^\tau e^{-r(s)}\left\langle \nu\widehat{\mathbf{A}}^N\mathbf{u}_\varepsilon^N(s) - \mathbf{F}^N\left(\mathbf{u}_\varepsilon^N(s)\right) + \widehat{\mathbf{B}}^N\left(\mathbf{u}_\varepsilon^N(s)\right) + \frac{1}{2}\dot{r}(s)\mathbf{u}_\varepsilon^N(s), \mathbf{u}_\varepsilon^N(s)\right\rangle ds \\
&\quad = -E|\xi|^2 + \varliminf_{N\to\infty} Ee^{-r(0)}\left|\mathbf{u}_\varepsilon^N(0)\right|^2 + \varliminf_{N\to\infty} E\int_0^\tau e^{-r(s)}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_\mathbb{Q}}^2 ds \\
&\qquad - \varepsilon E\left|\eta\right|^2 + \varepsilon \varliminf_{N\to\infty} Ee^{-r(0)}\left|p_\varepsilon^N(0)\right|^2 + \frac{1}{\varepsilon}\varliminf_{N\to\infty} E\int_0^\tau e^{-r(s)}\left\|Z_\varepsilon^N(s)\right\|_{L_Q}^2 ds \\
&\quad \geq 2E\int_0^\tau e^{-r(s)}\left\langle \mathbf{A}(s) + \mathbf{B}(s) + \frac{1}{2}\dot{r}(s)\mathbf{u}_\varepsilon(s), \mathbf{u}_\varepsilon(s)\right\rangle ds.
\end{aligned} \tag{4.42}$$

Note that one gets the last inequality by applications of the Itô formula to (4.37), and the fact that

$$\begin{aligned}
\varliminf_{N\to\infty} Ee^{-r(0)}\left|\mathbf{u}_\varepsilon^N(0)\right|^2 &\geq Ee^{-r(0)}|\mathbf{u}_\varepsilon(0)|^2, \\
\varliminf_{N\to\infty} Ee^{-r(0)}\left|p_\varepsilon^N(0)\right|^2 &\geq Ee^{-r(0)}\left|p_\varepsilon(0)\right|^2, \\
\varliminf_{N\to\infty} E\int_0^\tau e^{-r(s)}\left\|\mathbf{Z}_\varepsilon^N(s)\right\|_{L_\mathbb{Q}}^2 ds &\geq E\int_0^\tau e^{-r(s)}\|\mathbf{Z}_\varepsilon(s)\|_{L_\mathbb{Q}}^2 ds, \\
\varliminf_{N\to\infty} E\int_0^\tau e^{-r(s)}\left\|Z_\varepsilon^N(s)\right\|_{L_Q}^2 ds &\geq E\int_0^\tau e^{-r(s)}\|Z_\varepsilon(s)\|_{L_Q}^2 ds.
\end{aligned} \tag{4.43}$$

Step 3 (Monotonicity). By Corollary 4.6, we get

$$\begin{aligned} E\int_0^\tau e^{-r(s)}\Big\langle \nu\widehat{\mathbf{A}}\mathbf{u}_\varepsilon^N(s)+\widehat{\mathbf{B}}\Big(\mathbf{u}_\varepsilon^N(s)\Big)-\mathbf{F}\Big(\mathbf{u}_\varepsilon^N(s)\Big)+\frac{1}{2}\dot{r}(s)\mathbf{u}_\varepsilon^N(s),\mathbf{u}_\varepsilon^N(s)-\mathbf{v}(s)\Big\rangle ds \\ \le E\int_0^\tau e^{-r(s)}\Big\langle \nu\widehat{\mathbf{A}}\mathbf{v}(s)+\widehat{\mathbf{B}}(\mathbf{v}(s))-\mathbf{F}(\mathbf{v}(s))+\frac{1}{2}\dot{r}(s)\mathbf{v}(s),\mathbf{u}_\varepsilon^N(s)-\mathbf{v}(s)\Big\rangle ds. \end{aligned} \tag{4.44}$$

Note that $\mathbf{v}\in L^\infty_{\mathcal{F}}([0,\tau]\times\Omega;(\mathbb{H}_0^1)_M)$ where $M\le N$. An application of (4.42) yields

$$\begin{aligned} E\int_0^\tau e^{-r(s)}\langle \mathbf{A}(s)+\mathbf{B}(s)+\frac{1}{2}\dot{r}(s)\mathbf{u}_\varepsilon(s),\mathbf{u}_\varepsilon(s)-\mathbf{v}(s)\rangle ds \\ \le E\int_0^\tau e^{-r(s)}\langle \nu\widehat{\mathbf{A}}\mathbf{v}(s)+\widehat{\mathbf{B}}(\mathbf{v}(s))-\mathbf{F}(\mathbf{v}(s))+\frac{1}{2}\dot{r}(s)\mathbf{v}(s),\mathbf{u}_\varepsilon(s)-\mathbf{v}(s)\rangle ds. \end{aligned} \tag{4.45}$$

Since the above inequality is true for all $M\in\mathbb{N}$ and $K>0$, it remains true for all $\mathbf{v}\in L^\infty_{\mathcal{F}}([0,\tau]\times\Omega;\mathbb{H}_0^1)$. Thus let $\mathbf{v}=\mathbf{u}_\varepsilon+\lambda\mathbf{w}$ where $\mathbf{w}\in L^\infty_{\mathcal{F}}([0,\tau]\times\Omega;\mathbb{H}_0^1)$ and $\lambda>0$, and

$$\begin{aligned} E\int_0^\tau e^{-r(s)}\langle \mathbf{A}(s)+\mathbf{B}(s)-\nu\widehat{\mathbf{A}}\mathbf{u}_\varepsilon(s)-\widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(s)+\lambda\mathbf{w}(s))+\mathbf{F}(\mathbf{u}_\varepsilon(s)+\lambda\mathbf{w}(s)),\lambda\mathbf{w}(s)\rangle ds \\ \ge E\int_0^\tau e^{-r(s)}\langle \lambda\nu\widehat{\mathbf{A}}\mathbf{w}(s)+\frac{\lambda}{2}\dot{r}(s)\mathbf{w}(s),\lambda\mathbf{w}(s)\rangle ds. \end{aligned} \tag{4.46}$$

By the fact that

$$\begin{aligned} &\langle \widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(t)+\lambda\mathbf{w}(t)),\mathbf{w}(t)\rangle \\ &\quad= -\langle \widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(t)+\lambda\mathbf{w}(t),\mathbf{w}(t)),\mathbf{u}_\varepsilon(t)+\lambda\mathbf{w}(t)\rangle \\ &\quad= -\langle \widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(t)+\lambda\mathbf{w}(t),\mathbf{w}(t)),\mathbf{u}_\varepsilon(t)\rangle \\ &\quad= -\langle \widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(t),\mathbf{w}(t)),\mathbf{u}_\varepsilon(t)\rangle-\lambda\Big\langle \widehat{\mathbf{B}}(\mathbf{w}(t),\mathbf{w}(t)),\mathbf{u}_\varepsilon(t)\Big\rangle \\ &\quad= \langle \widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(t)),\mathbf{w}(t)\rangle+\lambda\Big\langle \widehat{\mathbf{B}}(\mathbf{w}(t),\mathbf{u}_\varepsilon(t)),\mathbf{w}(t)\Big\rangle, \end{aligned} \tag{4.47}$$

we have

$$\begin{aligned} E\int_0^\tau e^{-r(s)}\Big\langle \mathbf{A}(s)+\mathbf{B}(s)-\nu\widehat{\mathbf{A}}\mathbf{u}_\varepsilon(s)-\widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(s))+\mathbf{F}(\mathbf{u}_\varepsilon(s)+\lambda\mathbf{w}(s)),\mathbf{w}(s)\Big\rangle ds \\ \ge \lambda E\int_0^\tau e^{-r(s)}\Big\langle \nu\widehat{\mathbf{A}}\mathbf{w}(s)+\widehat{\mathbf{B}}(\mathbf{w}(s),\mathbf{u}_\varepsilon(s))+\frac{1}{2}\dot{r}(s)\mathbf{w}(s),\mathbf{w}(s)\Big\rangle ds. \end{aligned} \tag{4.48}$$

Letting $\lambda \to 0$, and by the arbitrariness of $\mathbf{w}$ and the fact that $\mathbf{F}$ is continuous, we know that

$$\mathbf{A}(s) + \mathbf{B}(s) = \nu\widehat{\mathbf{A}}\mathbf{u}_\varepsilon(s) + \widehat{\mathbf{B}}(\mathbf{u}_\varepsilon(s)) - \mathbf{F}(\mathbf{u}_\varepsilon(s)) \quad \mathrm{P\text{-}a.s.}, \tag{4.49}$$

and this completes the proof. □

5. Uniqueness, Continuity and Convergence of Solutions

5.1. Uniqueness and Continuity

The backward Navier-Stokes equation is well-posed if the regularity of the terminal condition in Proposition 4.9 is imposed. Only the uniqueness and continuity are left to check. Let us first prove the following lemma.

Lemma 5.1. *For any* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{H}_0^1$ *and* $\mathbf{w} \in \mathbb{L}^4$, *one has*

$$\left|\left\langle \widehat{\mathbf{B}}(\mathbf{u}) - \widehat{\mathbf{B}}(\mathbf{v}), \mathbf{w}\right\rangle\right| \le C(\|\mathbf{u}\|_{\mathbb{L}^4} + \|\mathbf{v}\|_{\mathbb{L}^4})\|\mathbf{u}-\mathbf{v}\|\|\mathbf{w}\|_{\mathbb{L}^4} + C(\|\mathbf{u}\| + \|\mathbf{v}\|)\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^4}\|\mathbf{w}\|_{\mathbb{L}^4}. \tag{5.1}$$

Proof. By Proposition 2.5,

$$\begin{aligned}
&\left|\left\langle \widehat{\mathbf{B}}(\mathbf{u}) - \widehat{\mathbf{B}}(\mathbf{v}), \mathbf{w}\right\rangle\right| \\
&\quad = \left|-\left\langle \widehat{\mathbf{B}}(\mathbf{u},\mathbf{w}), \mathbf{u}\right\rangle + \left\langle \widehat{\mathbf{B}}(\mathbf{v},\mathbf{w}), \mathbf{v}\right\rangle\right| \\
&\quad = \left|-\left\langle \widehat{\mathbf{B}}(\mathbf{u},\mathbf{w}), \mathbf{u}-\mathbf{v}\right\rangle - \left\langle \widehat{\mathbf{B}}(\mathbf{u},\mathbf{w}), \mathbf{v}\right\rangle + \left\langle \widehat{\mathbf{B}}(\mathbf{v},\mathbf{w}), \mathbf{v}\right\rangle\right| \\
&\quad = \left|-\left\langle \widehat{\mathbf{B}}(\mathbf{u},\mathbf{w}), \mathbf{u}-\mathbf{v}\right\rangle - \left\langle \widehat{\mathbf{B}}(\mathbf{u}-\mathbf{v},\mathbf{w}), \mathbf{v}\right\rangle\right| \\
&\quad = \left|\left\langle \widehat{\mathbf{B}}(\mathbf{u},\mathbf{u}-\mathbf{v}), \mathbf{w}\right\rangle + \left\langle \widehat{\mathbf{B}}(\mathbf{u}-\mathbf{v},\mathbf{v}), \mathbf{w}\right\rangle\right| \\
&\quad \le \|\mathbf{u}\|_{\mathbb{L}^4}\|\mathbf{u}-\mathbf{v}\|\|\mathbf{w}\|_{\mathbb{L}^4} + \frac{1}{2}\|\mathbf{u}\|\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^4}\|\mathbf{w}\|_{\mathbb{L}^4} + \|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^4}\|\mathbf{v}\|\|\mathbf{w}\|_{\mathbb{L}^4} \\
&\qquad + \frac{1}{2}\|\mathbf{u}-\mathbf{v}\|\|\mathbf{v}\|_{\mathbb{L}^4}\|\mathbf{w}\|_{\mathbb{L}^4} \\
&\quad \le C(\|\mathbf{u}\|_{\mathbb{L}^4} + \|\mathbf{v}\|_{\mathbb{L}^4})\|\mathbf{u}-\mathbf{v}\|\|\mathbf{w}\|_{\mathbb{L}^4} + C(\|\mathbf{u}\| + \|\mathbf{v}\|)\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^4}\|\mathbf{w}\|_{\mathbb{L}^4}.
\end{aligned} \tag{5.2}$$

□

Theorem 5.2. *Let* $\xi \in L^\infty_{\mathcal{F}_\tau}(\Omega;\mathbb{H}_0^1)$ *and* $\eta \in L^\infty_{\mathcal{F}_\tau}(\Omega;H_0^1)$. *System* (4.12) *admits a unique adapted solution in*

$$L^\infty_{\mathcal{F}}\left([0,\tau]\times\Omega;\mathbb{H}_0^1\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_{\mathbb{Q}})\right)\times L^\infty_{\mathcal{F}}\left([0,\tau]\times\Omega;H_0^1\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2(0,\tau;L_Q)\right). \tag{5.3}$$

Also the solution is continuous with respect to the terminal conditions in

$$L^{\infty}\left([0,\tau];L^{2}_{\mathcal{F}}\left(\Omega;\mathbb{L}^{2}\right)\right)\times L^{2}_{\mathcal{F}}\left(\Omega;L^{2}(0,\tau;L_{\mathbb{Q}})\right)\times L^{\infty}\left([0,\tau];L^{2}_{\mathcal{F}}\left(\Omega;L^{2}\right)\right)\times L^{2}_{\mathcal{F}}\left(\Omega;L^{2}(0,\tau;L_{Q})\right). \tag{5.4}$$

Proof. The existence of an adapted solution is shown in Theorem 4.10. Suppose that $(\mathbf{u}_{\varepsilon 1},\mathbf{Z}_{\varepsilon 1},p_{\varepsilon 1},Z_{\varepsilon 1})$ and $(\mathbf{u}_{\varepsilon 2},\mathbf{Z}_{\varepsilon 2},p_{\varepsilon 2},Z_{\varepsilon 2})$ are solutions of system (4.12) according to terminal conditions (ξ_1,η_1) and (ξ_2,η_2), respectively. The regularity of the solutions is guaranteed by Proposition 4.9. Denote

$$\begin{gathered}\overline{\mathbf{u}}_{\varepsilon}=\mathbf{u}_{\varepsilon 1}-\mathbf{u}_{\varepsilon 2},\quad \overline{\mathbf{Z}}_{\varepsilon}=\mathbf{Z}_{\varepsilon 1}-\mathbf{Z}_{\varepsilon 2},\quad \overline{Z}_{\varepsilon}=Z_{\varepsilon 1}-Z_{\varepsilon 2},\\ \overline{p}_{\varepsilon}=p_{\varepsilon 1}-p_{\varepsilon 2},\quad \overline{\xi}=\xi_1-\xi_2,\quad \overline{\eta}=\eta_1-\eta_2.\end{gathered} \tag{5.5}$$

Then one has

$$\begin{aligned}&d\overline{\mathbf{u}}_{\varepsilon}(t)+\nu\widehat{\mathbf{A}}\overline{\mathbf{u}}_{\varepsilon}(t)dt+\left(\widehat{\mathbf{B}}(\mathbf{u}_{\varepsilon 1}(t))-\widehat{\mathbf{B}}(\mathbf{u}_{\varepsilon 2}(t))\right)dt+\nabla\overline{p}_{\varepsilon}(t)dt,\\ &\quad=(\mathbf{F}(\mathbf{u}_{\varepsilon 1}(t))-\mathbf{F}(\mathbf{u}_{\varepsilon 2}(t)))dt+\overline{\mathbf{Z}}_{\varepsilon}(t)d\mathbb{W}(t),\\ &\varepsilon d\overline{p}_{\varepsilon}(t)+\nabla\cdot\overline{\mathbf{u}}_{\varepsilon}(t)dt=\overline{Z}_{\varepsilon}(t)dW(t),\\ &\overline{\mathbf{u}}_{\varepsilon}(\tau)=\overline{\xi},\qquad \overline{p}_{\varepsilon}(\tau)=\overline{\eta}.\end{aligned} \tag{5.6}$$

Similar to Corollary 4.6, let us define

$$l(t)=\int_t^T\left\{2\alpha+\frac{27}{8(\kappa-\nu)^3}K_0^2\right\}ds, \tag{5.7}$$

where K_0 is the constant in Proposition 4.9. An application of the Itô formula to $e^{-l(t)}|\overline{\mathbf{u}}_{\varepsilon}(t)|^2$ and Corollary 4.6 imply

$$\begin{aligned}&e^{-l(t\wedge\tau)}|\overline{\mathbf{u}}_{\varepsilon}(t\wedge\tau)|^2+\int_{t\wedge\tau}^{\tau}e^{-l(s)}\left\|\overline{\mathbf{Z}}_{\varepsilon}(s)\right\|_{L_{\mathbb{Q}}}^2ds\\ &\quad=\left|\overline{\xi}\right|^2+2\int_{t\wedge\tau}^{\tau}e^{-l(s)}\Big\langle\nu\widehat{\mathbf{A}}\overline{\mathbf{u}}_{\varepsilon}(s)-\left(\widehat{\mathbf{B}}(\mathbf{u}_{\varepsilon 1}(s))-\widehat{\mathbf{B}}(\mathbf{u}_{\varepsilon 2}(s))\right)-(\mathbf{F}(\mathbf{u}_{\varepsilon 1}(s))-\mathbf{F}(\mathbf{u}_{\varepsilon 2}(s)))\\ &\qquad\qquad+\frac{1}{2}\dot{l}(s)\overline{\mathbf{u}}_{\varepsilon}(s),\overline{\mathbf{u}}_{\varepsilon}(s)\Big\rangle ds\\ &\quad+2\int_{t\wedge\tau}^{\tau}e^{-l(s)}\left\langle\nabla\overline{p}_{\varepsilon}(s),\overline{\mathbf{u}}_{\varepsilon}(s)\right\rangle ds-2\int_{t\wedge\tau}^{\tau}e^{-l(s)}\left\langle\overline{\mathbf{Z}}_{\varepsilon}(s)d\mathbb{W}(s),\overline{\mathbf{u}}_{\varepsilon}(s)\right\rangle\\ &\le\left|\overline{\xi}\right|^2+\varepsilon\int_{t\wedge\tau}^{\tau}e^{-l(s)}d\left|\overline{p}_{\varepsilon}(s)\right|^2-\frac{1}{\varepsilon}\int_{t\wedge\tau}^{\tau}e^{-l(s)}\left\|\overline{Z}_{\varepsilon}(s)\right\|_{L_Q}^2ds\\ &\quad-2\int_{t\wedge\tau}^{\tau}e^{-l(s)}\langle\overline{Z}_{\varepsilon}(t)dW(s),\overline{p}_{\varepsilon}(s)\rangle ds-2\int_{t\wedge\tau}^{\tau}e^{-l(s)}\langle\overline{\mathbf{Z}}_{\varepsilon}(s)d\mathbb{W}(s),\overline{\mathbf{u}}_{\varepsilon}(s)\rangle.\end{aligned} \tag{5.8}$$

Taking the expectation, the above inequality becomes

$$\begin{aligned} &E|\overline{\mathbf{u}}_\varepsilon(t\wedge\tau)|^2+\varepsilon E|\overline{p}_\varepsilon(t\wedge\tau)|^2+E\int_{t\wedge\tau}^{\tau}\left\|\overline{\mathbf{Z}}_\varepsilon(s)\right\|_{L_{\mathbb{Q}}}^2 ds+\frac{1}{\varepsilon}E\int_{t\wedge\tau}^{\tau}\left\|\overline{Z}_\varepsilon(s)\right\|_{L_Q}^2 ds \\ &\quad\le e^{-l(0)}\left\{E\left|\overline{\xi}\right|^2+\varepsilon E|\eta|^2\right\}. \end{aligned} \tag{5.9}$$

Thus we have proved the uniqueness and continuity of system (4.12). □

Remark 5.3. The uniqueness and continuity with weaker terminal conditions, such as when the terminal conditions are uniformly bounded in L^2 sense, are still open. The difficulty lies in the nonadaptiveness nature of the backward system. For instance, the function l_1 defined in Corollary 4.6 is not $\mathcal{F}_t$ adapted. This is why we defined another function $l(t)$ in the proof of the uniqueness based on the $\mathbb{H}_0^1$-bound of the solution. Fortunately, $l(t)$ is $\mathcal{F}_t$ adapted and has similar properties as $l_1(t)$. One can also show the uniqueness and continuity using Lemma 5.1, without introducing the function $l(t)$.

5.2. The Convergence of the Solution As ε Approaches Zero

It is very interesting to study the asymptotic behavior of stochastic Navier-Stokes system with artificial compressibility. We are going to show that as artificial compressibility vanishes, the limit of the solution becomes the solution of the corresponding Navier-Stokes system for a viscous incompressible flow given below:

$$\begin{gathered} d\mathbf{u}(t)=-\nu\mathbf{A}\mathbf{u}(t)dt-\mathbf{B}(\mathbf{u}(t))dt-\nabla p(t)+\mathbf{F}(t)dt+\mathbf{Z}(t)d\mathbf{W}(t), \\ \nabla\cdot\mathbf{u}(t)=0,\qquad \mathbf{u}(\tau)=\xi,\quad p(\tau)=\eta, \end{gathered} \tag{5.10}$$

where $\mathbf{A}\mathbf{u}\triangleq-(\Delta\mathbf{u})$ and $\mathbf{B}(\mathbf{u},\mathbf{v})\triangleq((\mathbf{u}\cdot\nabla)\mathbf{v})$ with the notation $\mathbf{B}(\mathbf{u})=\mathbf{B}(\mathbf{u},\mathbf{u})$ (see Temam [3]).

Theorem 5.4. *Assume the conditions in Theorem 4.10(ii). Then as ε approaches 0, the first three elements in the solution of* (4.12), $(\mathbf{u}_\varepsilon,\mathbf{Z}_\varepsilon,p_\varepsilon)$, *converge to* $(\mathbf{u},\mathbf{Z},p)$, *the solution of* (5.10).

Proof. Similar to Step 1 of the proof of Theorem 4.10, we know that there exist $\mathbf{u}$, p, $\mathbf{Z}$ and a sequence of positive numbers $\{\varepsilon_i\}_{i=1}^{\infty}$ such that $\varepsilon_i\to 0$, $\mathbf{u}_{\varepsilon_i}\xrightarrow{w}\mathbf{u}$, $p_{\varepsilon_i}\xrightarrow{w}p$ and $\mathbf{Z}_{\varepsilon_i}\xrightarrow{w}\mathbf{Z}$ in corresponding spaces.

From (4.18) and (4.20), one knows that along a subsequence,

$$E\left\langle\sqrt{\varepsilon_i}\frac{dp_{\varepsilon_i}}{dt},h\right\rangle\longrightarrow E\left\langle\frac{dD}{dt},h\right\rangle \tag{5.11}$$

for some $D\in L^2_{\mathcal{F}}(\Omega;L^2(0,\tau;L^2))$ and for all $h\in L^2_{\mathcal{F}}(\Omega;L^2(0,\tau;L^2))$. Thus we get

$$E\langle\varepsilon_i\frac{dp_{\varepsilon_i}}{dt},h\rangle\longrightarrow 0 \tag{5.12}$$

in the sense of distribution. Since

$$\left\langle \varepsilon dp_{\varepsilon_i}(t) + \nabla \cdot \mathbf{u}_{\varepsilon_i}(t)dt, h(t)\right\rangle = \langle Z_{\varepsilon_i}(t)dW(t), h(t)\rangle, \tag{5.13}$$

we know that

$$E\int_0^\tau \langle \nabla \cdot \mathbf{u}(t), h(t)\rangle dt = \lim_{i\to\infty} E\int_0^\tau \langle \nabla \cdot \mathbf{u}_{\varepsilon_i}(t), h(t)\rangle dt = -\lim_{i\to\infty} E\int_0^\tau \left\langle \varepsilon dp_{\varepsilon_i}(t), h(t)\right\rangle = 0 \tag{5.14}$$

for all $h \in L^2_{\mathcal{F}}(\Omega; L^2(0,\tau; L^2))$. So $\nabla \cdot \mathbf{u} = 0$ P-a.s. This shows that the limiting system is incompressible.

Similar to Steps 2 and 3 in the proof of Theorem 4.10, we are able to show that $(\mathbf{u}, \mathbf{Z}, p)$ solves (5.10). □

Acknowledgments

The author thanks Professor P. Sundar for his helpful discussion and insightful suggestions. The author also thanks the anonymous referee for offering valuable comments and suggestions on the earlier version of the paper.

References

[1] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach Science Publishers, New York, NY, USA, 1969.

[2] P. Constantin and C. Foias, *Navier-Stokes Equations*, The University of Chicago Press, Chicago, Ill, USA, 1988.

[3] R. Temam, *Navier-Stokes Equations*, North-Holland, New York, NY, USA, 1979.

[4] F. Flandoli and D. Gatarek, "Martingale and stationary solutions for stochastic Navier-Stokes equations," *Probability Theory and Related Fields*, vol. 102, no. 3, pp. 367–391, 1995.

[5] R. Mikulevicius and B. L. Rozovsky II, "Global L2-solutions of stochastic Navier-Stokes equations," *Annals of Probability*, vol. 33, no. 1, pp. 137–176, 2005.

[6] J.-L. Menaldi and S. S. Sritharan, "Stochastic 2-D Navier-Stokes equation," *Applied Mathematics and Optimization*, vol. 46, no. 1, pp. 31–53, 2002.

[7] J. L. Lions, "Sentinels and stealthy perturbations. Semicomplete set of sentinels," in *Mathematical and Numerical Aspects of Wave Propagation Phenomena*, pp. 239–251, SIAM, Philadelphia, Pa, USA, 1991.

[8] J.-L. Lions, "Distributed systems with incomplete data and problems of environment: some remarks," in *Mathematics, Climate and Environment*, vol. 27 of *RMA Research Notes in Applied Mathematics*, pp. 58–101, Masson, Paris, France, 1993.

[9] J.-M. Bismut, "Conjugate convex functions in optimal stochastic control," *Journal of Mathematical Analysis and Applications*, vol. 44, no. 2, pp. 384–404, 1973.

[10] E. Pardoux and S. G. Peng, "Adapted solution of a backward stochastic differential equation," *Systems and Control Letters*, vol. 14, no. 1, pp. 55–61, 1990.

[11] J. Ma and J. Yong, "On linear, degenerate backward stochastic partial differential equations," *Probability Theory and Related Fields*, vol. 113, no. 2, pp. 135–170, 1999.

[12] Y. Hu, J. Ma, and J. Yong, "On semi-linear degenerate backward stochastic partial differential equations," *Probability Theory and Related Fields*, vol. 123, no. 3, pp. 381–411, 2002.

[13] J. Ma and J. Yong, "Adapted solution of a degenerate backward spde, with applications," *Stochastic Processes and Their Applications*, vol. 70, no. 1, pp. 59–84, 1997.

[14] J. Yong and X. Y. Zhou, *Stochastic Controls*, Springer, New York, NY, USA, 1999.

[15] P. L. Chow, *Stochastic Partial Differential Equations*, Taylor & Francis, Boca Raton, Fla, USA, 2007.

[16] P. Sundar and H. Yin, "Existence and uniqueness of solutions to the backward 2D stochastic Navier-Stokes equations," *Stochastic Processes and Their Applications*, vol. 119, no. 4, pp. 1216–1234, 2009.

[17] G. Kallianpur and J. Xiong, *Stochastic Differential Equations in Infinite Dimensional Spaces*, vol. 26 of *IMS Lecture Notes-Monograph Series*, Institute of Mathematical Statistics, 1995.

[18] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Springer, Berlin, Germany, 2007.