Research Article

On Symmetric Transversal Designs $\text{STD}_\lambda[24; 3]$'s

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The existence of a class regular symmetric transversal design $\text{STD}_\lambda[3\lambda; 3]$ is equivalent to a generalized Hadamard matrix of order $3u$ over $\text{GF}(3)$. Let n_λ be the number of nonisomorphic $\text{STD}_\lambda[3\lambda; 3]$'s. It is known that $n_1 = 1$, $n_2 = 1$, $n_3 = 4$, $n_4 = 1$, $n_5 = 0$, $n_6 \geq 20$, and $n_7 \geq 5$. In this paper, it is shown that $n_8 \geq 24$.

1. Introduction

A *symmetric transversal design* $\text{STD}_\lambda[k; u]$ (STD) is an incidence structure $\mathfrak{D} = (\mathcal{P}, \mathcal{B}, I)$ satisfying the following three conditions, where $k \geq 2$, $u \geq 2$, and $\lambda \geq 1$.

- (i) Each block contains exactly k points.
- (ii) The point set $\mathcal{P}$ is partitioned into k point sets $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{k-1}$ of equal size u such that any two distinct points are incident with exactly λ blocks or no block according as they are contained in different $\mathcal{P}_i$'s or not. $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{k-1}$ are said to be the *point classes* of $\mathfrak{D}$.
- (iii) The dual structure of $\mathfrak{D}$ also satisfies the above conditions (i) and (ii). The point classes of the dual structure of $\mathfrak{D}$ are said to be the *block classes* of $\mathfrak{D}$.

Let $\mathfrak{D} = (\mathcal{P}, \mathcal{B}, I)$ be an STD with the set of point classes Ω and the set of block classes Δ . Let G be an automorphism group. Then, by definition of STD, G induces a permutation group on $\Omega \cup \Delta$. If G fixes any element of $\Omega \cup \Delta$, then G is said to be an *elation group* and any element of G is said to be an *elation*. In this case, it is known that G acts semiregularly on each point class and on each block class. Especially, if G is an elation group of order u , then $\mathfrak{D}$ is said to be *class regular with respect to G* .

If $\mathfrak{D}$ is a class regular $\text{STD}_\lambda[k; u]$ with respect to a group U of order u , we can construct a generalized Hadamard matrix H of order k over U ($\text{GH}(k, U)$) from the incidence matrix of $\mathfrak{D}$, and vice versa.

The existence of an $\text{STD}_\lambda[k; 2]$ ($k = 2\lambda$) is equivalent to the existence of a Hadamard matrix of order k . Therefore, in this case $k \equiv 0 \pmod{4}$. It is easily showed that the number of nonisomorphic $\text{STD}_\lambda[k; 2]$'s is the same as the number of inequivalent Hadamard matrices of order k . Thus, the numbers of nonisomorphic $\text{STD}_\lambda[k; 2]$'s are now known for k 's up through 28 [1]. We are interested in the numbers of nonisomorphic $\text{STD}_\lambda[k; 3]$'s ($k = 3\lambda$) with the next class size. Let n_λ be the number of nonisomorphic $\text{STD}_\lambda[k; 3]$'s. Then, $n_1 = 1$, $n_2 = 1$, $n_3 = 4$, $n_4 = 1$, and $n_5 = 0$ [2–4]. We remark that $n_5 = 0$, that is, the nonexistence of $\text{STD}_5[15; 3]$ is the special case of a theorem proved by Haemers [4]. But, for $\lambda \geq 6$ the value n_λ is not determined except λ with known $n_\lambda = 0$. Recently de Launey informed us that one of his colleagues showed that $n_6 \geq 40$ using a computer [5]. The combinatorial constructions of $\text{GH}(24, \text{GF}(3))$'s by de Launey [6] and Zhang et al. [7] are known. Therefore, $\text{STD}_8[24; 3]$'s exist. It is difficult to know how many nonisomorphic $\text{STD}_8[24; 3]$'s were constructed in [7], because their construction is very indirect. We calculated the full automorphism group of the $\text{STD}_8[24; 3]$ corresponding to only one $\text{GH}(24, \text{GF}(3))$ constructed in [7]. This STD is not self-dual. On the other hand, since we were not able to secure de Launey's Ph.D. thesis [6], we did not calculate the full automorphism groups of $\text{STD}_8[24; 3]$'s corresponding to $\text{GH}(24, \text{GF}(3))$'s constructed by de Launey.

In this paper we construct 22 class regular $\text{STD}_8[24; 3]$'s which have a noncyclic automorphism group of order 9 containing an elation of order 3. Here, G acts semiregularly on blocks (points), but does not act semiregularly on points (blocks) for any STD of those. We used orbit theorems [8, 9] of STD's to determine such action of G on points and blocks. Also any $\text{STD}_8[24; 3]$ which we constructed is not isomorphic to anyone of two STD's constructed by Zhang et al. stated above. Thus, we have $n_8 \geq 24$.

It is an interesting problem to construct a class regular $\text{STD}_8[8u; u]$ or $\text{GH}(8u, \text{GF}(u))$ for a prime power u with $u \leq 19$, because the existence of $\text{GH}(8u, \text{GF}(u))$'s is known for a prime power u with $19 < u < 200$ or for a prime u with $u > 19$. For $u \in \{5, 7, 11, 13, 17, 19\}$, this problem is open. We expect that our construction is useful to solve this problem.

For general notation and concepts in design theory, we refer the reader to basic textbooks in the subject such as [10–12] or [13].

2. Isomorphisms and Automorphisms of $\text{STD}_\lambda[3\lambda; 3]$'s

Let $\mathfrak{D} = (\mathcal{P}, \mathcal{B}, I)$ be an $\text{STD}_\lambda[k; 3]$, where $k = 3\lambda$. Let $\Omega = \{\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{k-1}\}$ be the set of point classes of $\mathfrak{D}$ and $\Delta = \{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{k-1}\}$ the set of block classes of $\mathfrak{D}$. Let $\mathcal{P}_i = \{p_{3i}, p_{3i+1}, \dots, p_{3(i+1)-1}\}$ and $\mathcal{B}_j = \{B_{3j}, B_{3j+1}, \dots, B_{3(j+1)-1}\}$ ($0 \leq i, j \leq k-1$).

On the other hand let $\mathfrak{D}' = (\mathcal{P}', \mathcal{B}', I')$ be an $\text{STD}_\lambda[k; 3]$. Let $\Omega' = \{\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{k-1}\}$ be the set of point classes of $\mathfrak{D}'$ and $\Delta' = \{\mathcal{B}'_0, \mathcal{B}'_1, \dots, \mathcal{B}'_{k-1}\}$ the set of block classes of $\mathfrak{D}'$. Let $\mathcal{P}'_i = \{p'_{3i}, p'_{3i+1}, \dots, p'_{3(i+1)-1}\}$ and $\mathcal{B}'_j = \{B'_{3j}, B'_{3j+1}, \dots, B'_{3(j+1)-1}\}$ ($0 \leq i, j \leq k-1$).

Let Λ be the set of permutation matrices of degree 3. Let

$$L = \begin{pmatrix} L_0 & 0 & \cdots & L_0 & k-1 \\ \vdots & & & \vdots & \\ L_{k-1} & 0 & \cdots & L_{k-1} & k-1 \end{pmatrix}, \quad L' = \begin{pmatrix} L'_0 & 0 & \cdots & L'_0 & k-1 \\ \vdots & & & \vdots & \\ L'_{k-1} & 0 & \cdots & L'_{k-1} & k-1 \end{pmatrix} \quad (2.1)$$

be the incidence matrices of $\mathfrak{D}$ and $\mathfrak{D}'$ corresponding to these numberings of the point sets and the block sets, where $L_{ij}, L'_{ij} \in \Lambda$ ($0 \leq i, j \leq k-1$), respectively. Let E be the identity matrix of degree 3. Then, we may assume that $L_{i \ 0} = L'_{i \ 0} = E$ ($0 \leq i \leq k-1$) and $L_{0 \ j} = L'_{0 \ j} = E$ ($0 \leq j \leq k-1$) after interchanging some rows of $(3r+1)$ th row, $(3r+2)$ th row, $\dots$, $(3(r+1)-1)$ th row and interchanging some columns of $(3s)$ th column, $(3s+1)$ th column, $\dots$, $(3(s+1)-1)$ th column of L and L' for $0 \leq r, s \leq k-1$.

Definition 2.1. Let $S = \{0, 1, \dots, k-1\}$. We denote the symmetric group on S by $\text{Sym } S$. Let $f = \begin{pmatrix} 0 & 1 & \dots & k-1 \\ f(0) & f(1) & \dots & f(k-1) \end{pmatrix} \in \text{Sym } S$ and $X_0, X_1, \dots, X_{k-1} \in \Lambda$.

(i) We define

$$(f, (X_0, X_1, \dots, X_{k-1})) = \begin{pmatrix} X_{0 \ 0} & \dots & X_{0 \ k-1} \\ \vdots & \dots & \vdots \\ X_{k-1 \ 0} & \dots & X_{k-1 \ k-1} \end{pmatrix} \quad (2.2)$$

by

$$X_{ij} = \begin{cases} X_i & \text{if } j = f(i), \\ O & \text{otherwise,} \end{cases} \quad (2.3)$$

where O is the 3×3 zero matrix.

(ii) We define

$$\left(f, \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{k-1} \end{pmatrix} \right) = \begin{pmatrix} X_{0 \ 0} & \dots & X_{0 \ k-1} \\ \vdots & \dots & \vdots \\ X_{k-1 \ 0} & \dots & X_{k-1 \ k-1} \end{pmatrix} \quad (2.4)$$

by

$$X_{ij} = \begin{cases} X_j & \text{if } i = f(j), \\ O & \text{otherwise,} \end{cases} \quad (2.5)$$

where O is the 3×3 zero matrix.

From [14, Lemma 3.2], it follows that an isomorphism from $\mathfrak{D}$ to $\mathfrak{D}'$ is given by $f, g \in \text{Sym } S$ and $X_0, X_1, \dots, X_{k-1}, Y_0, Y_1, \dots, Y_{k-1} \in \Lambda$ satisfying

$$(f, (X_0, X_1, \dots, X_{k-1})) L \left(g, \begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{k-1} \end{pmatrix} \right) = L'. \quad (2.6)$$

Set

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}. \quad (2.7)$$

Lemma 2.2 (see [15, Corollary 3.4]). *Let $L_{ij}, L_{ij}' \in \Gamma$ for $0 \leq i, j \leq k-1$. Then, two $\text{STD}_\lambda[3\lambda, 3]$'s $\mathfrak{D}$ and $\mathfrak{D}'$ are isomorphic if and only if there exists $(f, g) \in \text{Sym } S \times \text{Sym } S$ such that*

$$L_{f(i) \ g(0)}^{-1} L_{f(i) \ g(j)} L_{f(0) \ g(j)}^{-1} L_{f(0) \ g(0)} = L_{ij}' \quad (2.8)$$

for $0 \leq i \leq k-1$ and $1 \leq j \leq k-1$ or there exists $(f, g) \in \text{Sym } S \times \text{Sym } S$ such that

$$L_{f(i) \ g(0)}^{-1} L_{f(i) \ g(j)} L_{f(0) \ g(j)}^{-1} L_{f(0) \ g(0)} = L_{ij}'^{-1} \quad (2.9)$$

for $0 \leq i \leq k-1$ and $1 \leq j \leq k-1$.

Lemma 2.3 (see [15, Corollary 3.5]). *Let $L_{ij} \in \Gamma$ for $0 \leq i, j \leq k-1$. Then, any automorphism of $\mathfrak{D}$ is given $(f, g, Y) \in \text{Sym } S \times \text{Sym } S \times \Gamma$ such that*

$$L_{f(i) \ g(0)}^{-1} L_{f(i) \ g(j)} L_{f(0) \ g(j)}^{-1} L_{f(0) \ g(0)} = L_{ij} \quad (2.10)$$

for $0 \leq i \leq k-1$ and $1 \leq j \leq k-1$ or $(f, g, Y) \in \text{Sym } S \times \text{Sym } S \times (\Lambda - \Gamma)$ such that

$$L_{f(i) \ g(0)}^{-1} L_{f(i) \ g(j)} L_{f(0) \ g(j)}^{-1} L_{f(0) \ g(0)} = L_{ij}^{-1} \quad (2.11)$$

for $0 \leq i \leq k-1$ and $1 \leq j \leq k-1$. Here $X_i = Y_0^{-1} L_{f(i) \ g(0)}^{-1}$ ($0 \leq i \leq k-1$) and $Y_j = L_{f(0) \ g(j)}^{-1} L_{f(0) \ g(0)} Y_0$ ($1 \leq j \leq k-1$).

Remark 2.4. Let $H = (h_{ij})_{0 \leq i, j \leq k}$ be a generalized Hadamard matrix over $\text{GF}(3)$ of order $k = 3\lambda$. Set

$$d(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d(1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad d(2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2.12)$$

Then, the incidence matrix of $\text{STD}_\lambda[3\lambda; 3] \ \mathfrak{D}(H)$ corresponding to H is given by $d(H) = (d(h_{ij}))_{0 \leq i, j \leq k-1}$.

3. Class Regular $\text{STD}_6[24; 3]$'s and $\text{GH}(24, \text{GF}(3))$'s

The constructions by de Launey [6] and Zhang et al. [7] are known about $\text{GH}(24, \text{GF}(3))$'s. We give one of $\text{GH}(24, \text{GF}(3))$'s constructed by Zhang et al.

Lemma 3.1 (see [7]). *Consider the following:*

[illegible]

is a $\text{GH}(24, \text{GF}(3))$ and $|\text{Aut } \mathfrak{D}(H_1)| = 32 \times 3$.

Proof. Let $H_1 = (h_{ij})_{0 \leq i, j \leq 23}$ and we normalize H_1 . That is, set $H_1' = (h_{ij} - h_{0j})_{0 \leq i, j \leq 23} = (h_{ij}')_{0 \leq i, j \leq 23}$ and then set $H_1'' = (h_{ij}' - h_{i0}')_{0 \leq i, j \leq 23} = (h_{ij}'')_{0 \leq i, j \leq 23}$. Then, $\mathfrak{D}(H_1) \cong \mathfrak{D}(H_1'')$. We calculate the full automorphism group of $\mathfrak{D}(H_1'')$ by Lemma 2.2 using a computer. Then, we have $|\mathfrak{D}(H_1)| = |\mathfrak{D}(H_1'')| = 32 \times 3$. $\square$

Example 3.2. Let H_1'' be the $\text{GH}(24, \text{GF}(3))$ stated in the proof of Lemma 3.1. Then,

$$H_1'' = \left(\begin{array}{cccc|cccc|cccc|cccc|cccc} 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \\ \hline 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 2 & 2 & 1 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ \hline 0 & 1 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 & 1 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 2 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\ \hline 0 & 1 & 0 & 2 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & 0 & 2 & 2 & 1 & 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 0 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 0 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\ \hline 0 & 1 & 0 & 2 & 0 & 2 & 1 & 2 & 2 & 0 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 2 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 2 & 2 & 0 & 1 & 2 & 2 & 0 & 1 & 2 & 2 & 0 & 1 & 2 & 2 & 0 \end{array} \right). \quad (3.2)$$

Set $H_1'' = (w_{ij})_{0 \leq i, j \leq 23}$. H_1'' is a normalized $\text{GH}(24, \text{GF}(3))$. We use the notations used in Lemma 2.2. Set $S = \{0, 1, 2, \dots, 23\}$. Let

$$\begin{aligned} f &= \left(\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \\ 3 & 2 & 1 & 0 & 5 & 4 & 7 & 6 & 19 & 18 & 17 & 16 & 20 & 21 & 22 & 23 & 9 & 8 & 11 & 10 & 14 & 15 & 12 & 13 \end{array} \right) \\ &= (0, 3)(1, 2)(4, 5)(6, 7)(8, 19, 10, 17)(9, 18, 11, 16)(12, 20, 14, 22)(13, 21, 15, 23), \\ g &= \left(\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \\ 6 & 7 & 5 & 4 & 2 & 3 & 1 & 0 & 11 & 10 & 9 & 8 & 14 & 15 & 12 & 13 & 21 & 20 & 22 & 23 & 16 & 17 & 19 & 18 \end{array} \right) \\ &= (8, 11)(9, 10)(12, 14)(13, 15)(0, 6, 1, 7)(2, 5, 3, 4)(16, 21, 17, 20)(18, 22, 19, 23) \in \text{Sym } S. \end{aligned} \quad (3.3)$$

Then, it follows that $-w_{f(i) \ g(0)} + w_{f(i) \ g(j)} - w_{f(0) \ g(j)} + w_{f(0) \ g(0)} = w_{ij}$ for $0 \leq i \leq 23$, $1 \leq j \leq 23$ on $\text{GF}(3)$. Therefore, by Lemma 2.2, for any $Y_0 \in \Gamma(f, g, Y_0)$ one yields an automorphism of $\mathfrak{D}(H_1'')$. Here $X_i = Y_0^{-1} d(w_{f(i) \ g(0)})^{-1}$ ($0 \leq i \leq 23$), $Y_i = d(w_{f(0) \ g(j)})^{-1} d(w_{f(0) \ g(0)}) Y_0$ ($1 \leq j \leq 23$).

In the rest of this section we deal with $\text{GH}(24, \text{GF}(3))$'s with the following form, where for $\text{GF}(3) = \{0, 1, 2\}$ set $0' = 1$, $1' = 2$, and $2' = 0$:

$H =$

$$\left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} & a_{22} & a_{23} \\ a_2 & a_0 & a_1 & a_5 & a_3 & a_4 & a_8 & a_6 & a_7 & a_{11} & a_9 & a_{10} & a_{14} & a_{12} & a_{13} & a_{17} & a_{15} & a_{16} & a_{20} & a_{18} & a_{19} & a_{23} & a_{21} & a_{22} & \\ a_1 & a_2 & a_0 & a_4 & a_5 & a_3 & a_7 & a_8 & a_6 & a_{10} & a_{11} & a_9 & a_{13} & a_{14} & a_{12} & a_{16} & a_{17} & a_{15} & a_{19} & a_{20} & a_{18} & a_{22} & a_{23} & a_{21} \\ \hline b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} & b_{19} & b_{20} & b_{21} & b_{22} & b_{23} \\ b_2 & b_0 & b_1 & b_5 & b_3 & b_4 & b_8 & b_6 & b_7 & b_{11} & b_9 & b_{10} & b_{14} & b_{12} & b_{13} & b_{17} & b_{15} & b_{16} & b_{20} & b_{18} & b_{19} & b_{23} & b_{21} & b_{22} \\ b_1 & b_2 & b_0 & b_4 & b_5 & b_3 & b_7 & b_8 & b_6 & b_{10} & b_{11} & b_9 & b_{13} & b_{14} & b_{12} & b_{16} & b_{17} & b_{15} & b_{19} & b_{20} & b_{18} & b_{22} & b_{23} & b_{21} \\ \hline c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} & c_{18} & c_{19} & c_{20} & c_{21} & c_{22} & c_{23} \\ c_2 & c_0 & c_1 & c_5 & c_3 & c_4 & c_8 & c_6 & c_7 & c_{11} & c_9 & c_{10} & c_{14} & c_{12} & c_{13} & c_{17} & c_{15} & c_{16} & c_{20} & c_{18} & c_{19} & c_{23} & c_{21} & c_{22} \\ c_1 & c_2 & c_0 & c_4 & c_5 & c_3 & c_7 & c_8 & c_6 & c_{10} & c_{11} & c_9 & c_{13} & c_{14} & c_{12} & c_{16} & c_{17} & c_{15} & c_{19} & c_{20} & c_{18} & c_{22} & c_{23} & c_{21} \\ \hline d_0 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & d_9 & d_{10} & d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} & d_{17} & d_{18} & d_{19} & d_{20} & d_{21} & d_{22} & d_{23} \\ d_2 & d_0 & d_1 & d_5 & d_3 & d_4 & d_8 & d_6 & d_7 & d_{11} & d_9 & d_{10} & d_{14} & d_{12} & d_{13} & d_{17} & d_{15} & d_{16} & d_{20} & d_{18} & d_{19} & d_{23} & d_{21} & d_{22} \\ d_1 & d_2 & d_0 & d_4 & d_5 & d_3 & d_7 & d_8 & d_6 & d_{10} & d_{11} & d_9 & d_{13} & d_{14} & d_{12} & d_{16} & d_{17} & d_{15} & d_{19} & d_{20} & d_{18} & d_{22} & d_{23} & d_{21} \\ \hline e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} & e_{18} & e_{19} & e_{20} & e_{21} & e_{22} & e_{23} \\ e_2 & e_0 & e_1 & e_5 & e_3 & e_4 & e_8 & e_6 & e_7 & e_{11} & e_9 & e_{10} & e_{14} & e_{12} & e_{13} & e_{17} & e_{15} & e_{16} & e_{20} & e_{18} & e_{19} & e_{23} & e_{21} & e_{22} \\ e_1 & e_2 & e_0 & e_4 & e_5 & e_3 & e_7 & e_8 & e_6 & e_{10} & e_{11} & e_9 & e_{13} & e_{14} & e_{12} & e_{16} & e_{17} & e_{15} & e_{19} & e_{20} & e_{18} & e_{22} & e_{23} & e_{21} \\ \hline f_0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{17} & f_{18} & f_{19} & f_{20} & f_{21} & f_{22} & f_{23} \\ f_2 & f_0 & f_1 & f_5 & f_3 & f_4 & f_8 & f_6 & f_7 & f_{11} & f_9 & f_{10} & f_{14} & f_{12} & f_{13} & f_{17} & f_{15} & f_{16} & f_{20} & f_{18} & f_{19} & f_{23} & f_{21} & f_{22} \\ f_1 & f_2 & f_0 & f_4 & f_5 & f_3 & f_7 & f_8 & f_6 & f_{10} & f_{11} & f_9 & f_{13} & f_{14} & f_{12} & f_{16} & f_{17} & f_{15} & f_{19} & f_{20} & f_{18} & f_{22} & f_{23} & f_{21} \\ \hline g_0 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 & g_8 & g_9 & g_{10} & g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} & g_{19} & g_{20} & g_{21} & g_{22} & g_{23} \\ g_2 & g_0 & g_1 & g_5 & g_3 & g_4 & g_8 & g_6 & g_7 & g_{11} & g_9 & g_{10} & g_{14} & g_{12} & g_{13} & g_{17} & g_{15} & g_{16} & g_{20} & g_{18} & g_{19} & g_{23} & g_{21} & g_{22} \\ g_1 & g_2 & g_0 & g_4 & g_5 & g_3 & g_7 & g_8 & g_6 & g_{10} & g_{11} & g_9 & g_{13} & g_{14} & g_{12} & g_{16} & g_{17} & g_{15} & g_{19} & g_{20} & g_{18} & g_{22} & g_{23} & g_{21} \\ \hline h_0 & h_0' & h_0'' & h_1 & h_1' & h_1'' & h_2 & h_2' & h_2'' & h_3 & h_3' & h_3'' & h_4 & h_4' & h_4'' & h_5 & h_5' & h_5'' & h_6 & h_6' & h_6'' & h_7 & h_7' & h_7'' \\ k_0 & k_0' & k_0'' & k_1 & k_1' & k_1'' & k_2 & h_2' & k_2'' & k_3 & k_3' & k_3'' & k_4 & k_4' & k_4'' & k_5 & k_5' & k_5'' & k_6 & h_6' & k_6'' & k_7 & k_7' & k_7'' \\ l_0 & l_0' & l_0'' & l_1 & l_1' & l_1'' & l_2 & l_2' & l_2'' & l_3 & l_3' & l_3'' & l_4 & l_4' & l_4'' & l_5 & l_5' & l_5'' & l_6 & l_6' & l_6'' & l_7 & l_7' & l_7'' \end{array} \right) \quad (3.4)$$

The existence of such generalized Hadamard matrix H is equivalent to the existence of $\text{STD}_8[24; 3] \mathfrak{D}(H)$ with the following automorphism group G .

- (i) G is an elementary abelian group of order 9.
- (ii) G contains an elation of order 3.
- (iii) G acts semiregularly on blocks.
- (iv) G invaries only one point class.

Theorem 3.3. *All matrices H_i ($i = 2, 3, \dots, 12$) in the appendix are $\text{GH}(24, \text{GF}(3))$'s. Then, $\text{GH}(24, \text{GF}(3))$'s $\mathfrak{D}(H_i)$'s ($i = 1, 2, \dots, 12$) are not isomorphic to each other and are not self dual. Moreover one also has the following table, where Ω_i and Δ_i are a set of the point classes and a set of the block class of $\mathfrak{D}(H_i)$, respectively.*

Corollary 3.4. *Let n_λ be the number of nonisomorphic $\text{STD}_\lambda[3\lambda; 3]$'s. Then, $n_8 \geq 24$.*

Table 1

i	$ \text{Aut } \mathfrak{D}(H_i) $	Sizes of orbits on Ω_i	Sizes of orbits on Δ_i
1	32×3	(4,4,16)	(8,8,8)
2	6×3	(1,2,3,3,3,6,6)	(3,3,3,3,6,6)
3	6×3	(1,2,3,3,3,6,6)	(3,3,3,3,6,6)
4	6×3	(1,2,3,3,3,6,6)	(3,3,3,3,6,6)
5	6×3	(1,2,3,3,3,6,6)	(3,3,3,3,6,6)
6	9×3	(3,3,9,9)	(3,3,9,9)
7	18×3	(3,3,9,9)	(3,3,9,9)
8	36×3	(6,9,9)	(6,9,9)
9	96×3	(8,16)	(24)
10	96×3	(8,16)	(24)
11	96×3	(8,16)	(24)
12	96×3	(8,16)	(24)

Appendix

Consider the following:

$$H_2 = \left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 2 & 1 & 0 \\ \hline 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 2 & 0 \\ \hline 0 & 0 & 1 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 2 & 2 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 1 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 2 & 2 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 2 & 2 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ \hline 0 & 2 & 0 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 & 0 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 0 & 0 \\ 2 & 0 & 0 & 1 & 2 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 1 & 2 & 2 & 1 & 2 & 2 & 0 & 2 & 2 & 2 \\ \hline 0 & 2 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 & 2 & 2 & 2 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \end{array} \right), \quad (\text{A.1})$$

$$H_3 = \left(\begin{array}{c|c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 2 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 & 0 & 0 & 2 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ \hline 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 0 & 2 \\ \hline 0 & 0 & 1 & 2 & 2 & 0 & 1 & 2 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 2 & 2 & 1 & 1 & 1 & 2 & 2 \\ \hline 0 & 1 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 1 & 2 \\ 1 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 2 & 2 & 1 \\ \hline 0 & 2 & 0 & 0 & 2 & 0 & 1 & 2 & 2 & 2 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 0 \\ \hline 0 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 2 & 0 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 2 & 2 & 0 & 2 \\ \hline 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \end{array} \right), \quad (A.2)$$

$$H_4 = \left(\begin{array}{c|c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 2 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 & 0 & 0 & 2 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ \hline 0 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 & 0 \\ \hline 0 & 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 & 2 & 2 & 1 & 2 \\ \hline 0 & 1 & 2 & 1 & 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 1 \\ \hline 0 & 2 & 0 & 1 & 2 & 1 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1 & 1 & 2 & 0 & 2 & 2 \\ 2 & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 2 \\ \hline 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 2 & 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 \\ \hline 0 & 1 & 2 & 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 \end{array} \right), \quad (A.3)$$

[illegible]

$$H_6 = \begin{pmatrix} \begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 1 & 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 2 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 2 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & 0 \\ \hline 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 2 & 2 & 2 & 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 2 & 2 & 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 2 & 0 & 2 & 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 2 & 2 & 2 \\ \hline 0 & 2 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 0 \\ 2 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 0 \\ \hline 0 & 2 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 0 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 0 & 0 & 2 & 1 & 2 & 2 & 1 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ \hline 0 & 1 & 2 & 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 2 \end{array} \end{pmatrix}, \quad (\text{A.5})$$

$$H_7 = \left(\begin{array}{c|c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 2 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 & 0 & 2 & 2 & 2 \\ 1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 0 & 0 & 2 & 2 \\ \hline 0 & 0 & 1 & 2 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 2 & 2 & 0 & 2 \\ \hline 0 & 1 & 2 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\ \hline 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 2 & 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 1 & 1 & 2 & 2 \\ 2 & 0 & 0 & 1 & 2 & 1 & 2 & 2 \\ \hline 0 & 2 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 2 & 2 & 1 & 1 & 2 & 2 \\ 2 & 0 & 0 & 1 & 2 & 1 & 2 & 2 \\ \hline 0 & 1 & 2 & 1 & 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 \end{array} \right), \quad (A.6)$$

$$H_8 = \left(\begin{array}{c|c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 & 1 & 2 & 2 \\ \hline 0 & 0 & 1 & 2 & 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\ \hline 0 & 1 & 2 & 1 & 0 & 0 & 2 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 2 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 & 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 1 & 1 & 0 & 2 & 1 \\ \hline 0 & 2 & 0 & 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 2 & 0 & 1 & 0 & 2 \\ \hline 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 \end{array} \right), \quad (A.7)$$

$$H_{11} = \left(\begin{array}{c|c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 & 0 & 2 & 1 & 2 \\ 1 & 0 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 2 & 0 & 2 & 1 & 2 \\ 1 & 0 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 & 0 & 1 & 2 \\ \hline 0 & 1 & 2 & 0 & 2 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 & 0 & 2 & 2 \\ \hline 0 & 2 & 0 & 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 2 & 2 & 1 & 1 & 2 & 2 \\ 2 & 0 & 0 & 1 & 2 & 1 & 2 & 2 \\ \hline 0 & 2 & 0 & 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 & 2 & 0 & 2 \\ \hline 0 & 1 & 2 & 0 & 1 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right), \quad (\text{A.10})$$

$$H_{12} = \left(\begin{array}{c|c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 & 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 2 & 2 & 2 & 0 \\ 0 & 1 & 0 & 2 & 1 & 2 & 0 & 2 \\ \hline 0 & 1 & 2 & 0 & 2 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 & 0 & 2 & 2 \\ \hline 0 & 2 & 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 2 & 2 & 1 & 1 & 0 \\ \hline 0 & 2 & 0 & 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 & 2 & 0 & 2 \\ \hline 0 & 1 & 2 & 0 & 1 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right). \quad (\text{A.11})$$

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