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Research Article

Global Integrable Solution for a Nonlinear Functional Integral Inclusion

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We study the global existence of positive integrable solution for the functional integral inclusion of fractional order $x(t) \in p(t) + F_1(t, I^\alpha f_2(t, x(\varphi(t))))$, $t \in (0,1)$, $\alpha \geq 0$, where $F_1(t, x(t))$ is a set-valued function defined on $(0,1) \times R^+$.

1. Introduction

Consider the functional integral equation

$$x(t) = g(t) + f\left(t, \int_0^1 k(t,s)x(\varphi(t))\right)ds, \quad t \in (0,1). \tag{1}$$

The authors (see [1]) proved the existence of monotonic integrable solution of (1), where the the function $f(t,x)$ is assumed to be monotonic in both its arguments and satisfies the Caratheodory and growth conditions.

In [2], the author discusses the existence of solution of the functional integral equation

$$x(t) = u(t) + \int_0^t k(t,s) f(s, x(\varphi(s)))ds, \quad t \in (0,1). \tag{2}$$

In [3], the author omitted the condition of monotonicity of $f(t,x)$ and proved the existence of integrable solution of (1).

Also he proved (see [4]) the existence of integrable solution of the functional integral equation

$$x(t) = f_1\left(t, r\int_0^1 k(t,s) f_2(s, x(s))\, ds\right), \quad t \in (0,1), \tag{3}$$

where $f(t,x)$ is assumed be satisfy the Caratheodory and growth conditions.

Consider the functional integral equation

$$x(t) = \int_0^t k_1(t,s) f\left(s, \int_0^s k_2(t,\theta)x(\varphi(\theta))d\theta\right), \quad t \in (0,1). \tag{4}$$

The existence of the nonincreasing integrable solution of (4) was studied in [5], where $f(t,x)$ is nondecreasing in both its arguments and satisfies the Caratheodory and growth conditions.

Here we are concerned with the functional integral equation of fractional order

$$x(t) = p(t) + f_1(t, I^\alpha f_2(t, x(\varphi(t)))), \quad t \in (0,1), \quad \alpha \geq 0, \tag{5}$$

we prove the global existence of positive integrable solution of (5), where $f_1(t,x)$ and $f_2(t,x)$ satisfy the Caratheodory and growth conditions. As a generalization of our results we study the global existence of positive integrable solution of the nonlinear functional integral inclusion of fractional order

$$x(t) \in p(t) + F_1(t, I^\alpha f_2(t, x(\varphi(t)))), \quad t \in (0,1), \quad \alpha \geq 0, \tag{6}$$

where the set-valued map $F_1 : (0,1) \times R^+ \to 2^{R^+}$ has nonempty closed values.

2. Preliminaries

Let $L^1 = L^1(I)$ be the class of Lebesgue integrable functions on the interval $I = [a,b]$, $0 \leq a < b < \infty$, and let $\Gamma(\cdot)$ be the gamma function.

Definition 1. The fractional integral of the function $f(\cdot) \in L^1(I)$ of order $\alpha \in R^+$ is defined by (cf. [6–9])

$$I_a^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds. \quad (7)$$

Theorem 1 (nonlinear alternative of Leray-Shauder type [10]). *Let U be an open subset of a convex set D in a Banach space X. Assume $0 \in U$ and $T \in C(\overline{U}, D)$. Then either*

(A_1) *T has a fixed point in $\overline{U}$ or*

(A_2) *there exists $\gamma \in (0,1)$ and $x \in \partial U$ such that $x = \gamma T x$.*

Theorem 2 (Kolmogorov compactness criterion [11]). *Let $\Omega \subseteq L^p(0,1), 1 \le P \le \infty$. If*

(i) *Ω is bounded in $L^p(0,1)$*

(ii) *$x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^p(0,1)$, where*

$$x_h(t) = \frac{1}{h}\int_0^{t+h} x(s)\,ds. \quad (8)$$

3. Main Results

In this section we present our main result by proving the global existence of positive solution $x \in L^1$ for the functional integral equatoin (1).

To facilitate our discussion, let us first state the following assumptions.

(i) $p \in L^1$.

(ii) $f_i : (0,1) \times R^+ \to R^+, i = 1,2$, satisfy Caratheodory condition that is, f_i are measurabel in t for any $x \in R^+$ and continuous in x for almost all $t \in (0,1)$.

There exist four functions $t \to a_i(t), t \to b_i(t)$ such that

$$|f_i(t,x)| \le a_i(t) + b_i(t)|x|, \quad i = 1,2,\ \forall t \in (0,1),\ x \in R, \quad (9)$$

where $a_i(\cdot) \in L^1$ and $b_i(\cdot)$ are measurable and bounded.

(iii) $\phi : (0,1) \to (0,1)$ is absolutely continuous, and there exists a constant $M > 0$ such that $\varphi'(t) \ge M, \forall t \in (0,1)$.

(iv) Assume that every solution $x(\cdot) \in L^1$ to the equation

$$x(t) = \gamma\left(p(t) + f_1\left(t, I^\alpha f_2(t, x(\varphi(t)))\right)\right), \quad (10)$$

$t \in (0,1), 0 < \beta < 1, \gamma \in (0,1)$, satisfies $\|x\| \ne r$ ($r > 0$ is arbitrary but fixed).

Define the operator T,

$$Tx(t) = p(t) + f_1\left(t, I^\alpha f_2(t, x(\varphi(t)))\right), \quad t \in (0,1). \quad (11)$$

Now, we are in position to formulate and prove our main result.

Theorem 3. *Let the assumptions* (i)–(iv) *satisfied. Then* (1) *has at least one positive solution $x \in L_1$.*

Proof. Let x be an arbitrary element in the open set $B_r = \{x : \|x\|\langle r, r\rangle 0\}$.

Then from assumptions (i) and (ii), we have,

$$\begin{aligned}
\|Tx\| &= \int_0^1 |(Tx)(t)|\,dt \le \int_0^1 |p(t)|\,dt \\
&\quad + \int_0^1 \left| f_1\left(t, I^\alpha f_2(t, x(\varphi(t)))\right)\right| dt \\
&\le \int_0^1 |p(t)|\,dt + \int_0^1 a_1(t)dt \\
&\quad + \int_0^1 b(t)\left|I^\alpha f_2(t, x(\varphi(t)))\right| dt \\
&\le \int_0^1 |p(t)|\,dt + \int_0^1 |a_1(t)|dt \\
&\quad + \int_0^1 |b_1(t)| I^\alpha \left(a_2(t) + b_2(t)|x(\varphi(t))|\right) dt \\
&\le \|p\| + \|a_1\| + b_1 \int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} a_2(s)ds\,dt \\
&\quad + b_1 \int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |b_2(s)|\,|x(\varphi(s))|\,ds\,dt \\
&\le \|p\| + \|a_1\| + b_1 \int_0^1 |a_2(s)| \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt\,ds \\
&\quad + b_1 b_2 \int_0^1 |x(\varphi(s))| \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt\,ds \\
&\le \|p\| + \|a_1\| + b_1 \int_0^1 |a_2(s)| \frac{(t-1)^\alpha}{\Gamma(\alpha+1)} ds \\
&\quad + b_1 b_2 \int_0^1 |x(\varphi(s))| \frac{(t-1)^\alpha}{\Gamma(\alpha+1)} ds \\
&\le \|p\| + \|a_1\| + \frac{b_1\|a_2\|}{\Gamma(\alpha+1)} \\
&\quad + \frac{b_1 b_2}{\Gamma(\alpha+1)M} \int_0^1 |x(\varphi(s))|\,\varphi'(s)ds \\
&\le \|p\| + \|a_1\| + \frac{b_1\|a_2\|}{\Gamma(\alpha+1)} + \frac{b_1 b_2}{\Gamma(\alpha+1)M} \int_{\varphi(0)}^{\varphi(1)} |x(u)|du \\
&\le \|p\| + \|a_1\| + \frac{b_1\|a_2\|}{\Gamma(\alpha+1)} + \frac{b_1 b_2}{\Gamma(\alpha+1)M} \int_0^1 |x(u)|du \\
&\le \|p\| + \|a_1\| + \frac{b_1\|a_2\|}{\Gamma(\alpha+1)} + \frac{b_1 b_2\|x\|}{\Gamma(\alpha+1)M}.
\end{aligned} \quad (12)$$

Hence the previous inequality means that the operator T maps B_r into L^1.

Now, we will show that T is compact. To achieve this goal we will apply Theorem 2. So, let Ω be a bounded subset of B_r. Then $T(\Omega)$ is bounded in L^1; that is, condition (i) of Theorem 2 is satisfied.

It remains to show that $(Tx)_h \to Tx$ in L^1 as $h \to 0$ uniformly with respect to $Tx \in \Omega$. We have the following:

$$\begin{aligned} &\|(Tx)_h - (Tx)\| \\ &\quad = \int_0^1 |(Tx)_h(t) - (Tx)(t)|\,dt \\ &\quad = \int_0^1 \left| \frac{1}{h}\int_t^{t+h} (Tx)_h(\tau)d\tau - (Tx)(t) \right| dt \\ &\quad = \int_0^1 \left| \frac{1}{h}\int_t^{t+h} ((Tx)_h(\tau) - (Tx)(t))d\tau \right| dt \\ &\quad \le \int_0^1 \frac{1}{h}\int_t^{t+h} |p(\tau) - p(t)|\,d\tau\,dt \\ &\qquad + \int_0^1 \frac{1}{h}\int_t^{t+h} \Big| f_1\left(\tau, I^\beta f_2(\tau, x(\varphi(\tau)))\right) \\ &\qquad\qquad - f_1\left(t, I^\beta f_2(t, x(\varphi(t)))\right) \Big|\,d\tau\,dt. \end{aligned} \tag{13}$$

Now $f_1, f_2 \in L^1$ and $I^\beta f_2 \in L^1$, then (cf. [12])

$$\begin{aligned} &\frac{1}{h}\int_t^{t+h} \Big| f_1\left(\tau, I^\beta f_2(\tau, x(\varphi(\tau)))\right) \\ &\qquad - f_1\left(t, I^\beta f_2(t, x(\varphi(t)))\right) \Big|\,d\tau \longrightarrow 0. \end{aligned} \tag{14}$$

Moreover, $p(\cdot) \in L^1$. So, we have

$$\frac{1}{h}\int_t^{t+h} |p(\tau) - p(t)|\,d\tau \longrightarrow 0 \tag{15}$$

for a.e $t \in L^1$. Therefore, by Theorem 2 we deduce that $T(\Omega)$ is relatively compact; that is, T is compact operator.

Set $U = B_r$ and $D = X = L^1(0,1)$. Then in the view of assumption (iv) condition (A_2) of Theorem 1 does not hold. Theorem 1 implies that T has a fixed point. This completes the proof. □

4. Integral Inclusion

Consider now the integral inclusion (2), where $F_1 : [0,1] \times R^+ \to 2^{R^+}$ has nonempty closed convex values.

As an important consequence of the main result we can present the following

Theorem 4. *Let the assumptions of Theorem 3 be satisfied. The multifunction F_1 satisfies the following assumptions:*

(1) *$F_1(t,x)$ are nonempty, closed and convex for all $(t,x) \in (0,1) \times R^+$,*

(2) *$F_1(t,\cdot)$ is lower semicontinuous from R^+ into R^+,*

(3) *$F_1(\cdot,\cdot)$ is measurable,*

(4) *there exist a function $a \in L_1$ and a measurable and bounded function b such that*

$$|F_1(t,x)| \le a(t) + b(t)|x| \quad \forall t \in (0,1), \tag{16}$$

Then there exists at least one positive solution $x \in L^1$ of the integral inclusion (2).

Proof. By conditions (1)–(4) (see [13–16]) we can find a selection function f_1 (Caratheodory function) $f_1 : (0,1) \times R^+ \to R^+$ such that $f_1(t,x) \in F_1(t,x)$ for all $(t,x) \in (0,1) \times R^+$, this function satisfies condition (ii) of Theorem 3.

Clearly all assumptions of Theorem 3 are hold, then there exists a positive solution $x \in L^1$ such that

$$x(t) - p(t) = f_1\left(t, I^\beta f_2(t, x(\varphi(t)))\right) \in F_1\left(t, I^\beta f_2(t, x(\varphi(t)))\right). \tag{17}$$

□

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