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Research Article

Binary Representations of Regular Graphs

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For any 2-distance set X in the n-dimensional binary Hamming space H_n, let Γ_X be the graph with X as the vertex set and with two vertices adjacent if and only if the distance between them is the smaller of the two nonzero distances in X. The binary spherical representation number of a graph Γ, or $\mathrm{bsr}(\Gamma)$, is the least n such that Γ is isomorphic to Γ_X, where X is a 2-distance set lying on a sphere in H_n. It is shown that if Γ is a connected regular graph, then $\mathrm{bsr}(\Gamma) \geq b - m$, where b is the order of Γ and m is the multiplicity of the least eigenvalue of Γ, and the case of equality is characterized. In particular, if Γ is a connected strongly regular graph, then $\mathrm{bsr}(\Gamma) = b - m$ if and only if Γ is the block graph of a quasisymmetric 2-design. It is also shown that if a connected regular graph is cospectral with a line graph and has the same binary spherical representation number as this line graph, then it is a line graph.

1. Introduction

The subject of this paper is mutual relations between regular and strongly regular graphs, 2-distance sets in binary Hamming spaces, and quasisymmetric 1- and 2-designs.

The following relation between strongly regular graphs and 2-distance sets in Euclidean spaces is well known (cf. [1, Theorem 2.23]): *if m is the multiplicity of the least eigenvalue of a connected strongly regular graph Γ of order n, then the vertex set of Γ can be represented as a set of points, lying on a sphere in $\mathbb{R}^{n-m-1}$, so that there exist positive real numbers $h_1 < h_2$ such that the distance between any two distinct vertices is equal to h_1 if they are adjacent as vertices of Γ and it is equal to h_2 otherwise.* This result was recently generalized to all connected regular graphs in [2]. It has also been proved in [2] that, given n and m, such a representation of a connected regular graph in $\mathbb{R}^{n-m-2}$ is not possible.

The notion of a 2-distance set representing a graph makes sense for any metric space, and the spaces of choice in this paper are the binary Hamming spaces. We will show (Theorem 3.3) that the dimension of a binary Hamming space, in which a connected regular graph Γ can be represented, is at least $n-m$, where n and m have the same meaning as in the previous paragraph.

It is also well known that the block graph of a quasisymmetric 2-design is strongly regular. However, many strongly regular graphs are not block graphs, and there is no good characterization of the graphs that are block graphs of quasisymmetric 2-designs. The situation changes if we consider the representation of graphs in binary Hamming spaces. We will show (Theorem 4.6) that a connected strongly regular graph admits a representation in the binary Hamming space of the minimal dimension $n-m$ if and only if it is the block graph of a quasisymmetric 2-design.

At the dawn of graph theory there was a short-lived conjecture that a graph is determined by the spectrum of its adjacency matrix. Of course, it is not true (see a very interesting discussion in [3]). However, some classes of graphs can be described by their spectra. In particular, if a connected regular graph has the same spectrum as a line graph, then it is almost always a line graph itself (all exceptions are known). We will show (Corollary 5.7) that if a connected regular graph Γ is cospectral with a line graph $L(G)$ of a graph G and, beside that, the minimal dimension of a binary Hamming space, in which either graph can be represented, is the same for Γ and $L(G)$, then Γ is a line graph.

2. Preliminaries

All graphs in this paper are finite and simple, and all incidence structures are without repeated blocks. For a graph Γ, $|\Gamma|$ denotes the *order of* Γ, that is, the number of vertices. If x and y are vertices of a graph Γ, then $x \sim y$ means that x and y are adjacent, while $x \not\sim y$ means that x and y are distinct and nonadjacent. Two graphs are said to be *cospectral* if their adjacency matrices have the same characteristic polynomial.

Throughout the paper we use I to denote identity matrices and J to denote square matrices with every entry equal to 1. The order of I and J will be always apparent from the context. We denote as $\mathbf{0}$ and $\mathbf{1}$ vectors (columns, rows, points) with all entries (coordinates) equal to 0 or all equal to 1, respectively. In examples throughout the paper we will use digits and letters to denote elements of a small set and omit braces and commas when a subset of such a set is presented; for example, we will write $1350b$ instead of $\{1,3,5,0,b\}$.

If n is a positive integer, then $[n]$ denotes the set $\{1,2,\dots,n\}$.

Definition 2.1. The *binary Hamming space* H_n consists of all n-tuples $\mathbf{a} = (a_1, a_2, \dots, a_n)$ with each a_i equal to 0 or 1. When it is convenient, one identifies $\mathbf{a}$ with the set $\{i \in [n] : a_i = 1\}$. The distance $h(\mathbf{a}, \mathbf{b})$ between $\mathbf{a}$ and $\mathbf{b} = (b_1, b_2, \dots, b_n) \in H_n$ is the number of indices i for which $a_i \neq b_i$. The Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is denoted as $\|\mathbf{x}\|$, so, for $\mathbf{a}, \mathbf{b} \in H_n$, $h(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|^2$.

A set $X \subset H_n$ is called a *2-distance set* if $|\{h(\mathbf{a}, \mathbf{b}) : \mathbf{a}, \mathbf{b} \in X, \mathbf{a} \neq \mathbf{b}\}| = 2$.

A *sphere* with center $\mathbf{c} \in H_n$ and integer radius k, $1 \le k \le n-1$, is the set of all points $\mathbf{x} \in H_n$ such that $h(\mathbf{c}, \mathbf{x}) = k$. Any subset of a sphere (of radius k) is called a *spherical set* (*of radius* k).

Remark 2.2. The sphere of radius k in H_n, centered at $\mathbf{a}$, coincides (as a set) with the sphere of radius $n-k$ centered at the opposite point $\mathbf{b} = \mathbf{1} - \mathbf{a}$. This allows us to assume, when needed, that the radius of a sphere does not exceed $n/2$. A sphere of radius k in H_n centered at $\mathbf{0}$, regarded as a subset of $\mathbb{R}^n$, is the intersection of the unit cube and the hyperplane $x_1 + x_2 + \cdots + x_n = k$.

Remark 2.3. For $n \geq 2$, the distance between any two points of a spherical set in H_n is even.

Definition 2.4. An *incidence structure* (without repeated blocks) is a pair $\mathbf{D} = (V, \mathcal{B})$, where V is a nonempty finite set (of *points*) and $\mathcal{B}$ is a nonempty set of subsets of V (*blocks*). The cardinality of the intersection of two distinct blocks is called an *intersection number* of $\mathbf{D}$. An incidence structure is said to be *quasisymmetric* if it has exactly two distinct intersection numbers. For a nonnegative integer t, an incidence structure $\mathbf{D}$ is called a *t-design* if all blocks of $\mathbf{D}$ have the same cardinality and every set of t points is contained in the same number of blocks. A t-design $\mathbf{D}$ with an (points versus blocks) incidence matrix N is called *nonsquare* if N is not a square matrix, and it is called *nonsingular* if $\det(NN^\top) \neq 0$. A 2-design is also called a (v, b, r, k, λ)-design, where v is the number of points, b is the number of blocks, r is the *replication number*, that is, the number of blocks containing any given point, k is the block size, and λ is the number of blocks containing any given pair of points.

With any quasisymmetric incidence structure we associate its *block graph*.

Definition 2.5. If $\mathbf{D}$ is a quasisymmetric incidence structure with intersection numbers $\alpha < \beta$, then the *block graph* of $\mathbf{D}$ is the graph whose vertices are the blocks of $\mathbf{D}$ and two vertices are adjacent if and only if the corresponding blocks meet in β points.

Remark 2.6. If a regular graph, other than a complete graph, is connected, then it has at least three distinct eigenvalues. It is strongly regular if and only if it has exactly three distinct eigenvalues. If $\mathbf{D}$ is a quasisymmetric 2-design, then it is nonsquare and its block graph is strongly regular. If $\mathbf{D}$ is a quasisymmetric t-design with block size k and intersection numbers $\alpha < \beta$, then $N^\top N = (k - \alpha)I + (\beta - \alpha)A + \alpha J$, where N is an incidence matrix of $\mathbf{D}$ and A is an adjacency matrix of the block graph of $\mathbf{D}$. If $\mathbf{D}$ is a (v, b, r, k, λ)-design, then $NN^\top = (r - \lambda)I + \lambda J$. Therefore, $\det(NN^\top) = rk(r - \lambda)^{v-1} \neq 0$, so $\mathbf{D}$ is nonsingular. For these and other basic results on designs and regular graphs, see [1] or [4].

Definition 2.7. Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_b\}$ be a 2-distance set of cardinality b in H_n, and let $h_1 < h_2$ be the nonzero distances in X. One denotes as Γ_X the graph whose vertex set is X and the edge set is the set of all pairs $\{\mathbf{x}_i, \mathbf{x}_j\}$ with $h(\mathbf{x}_i, \mathbf{x}_j) = h_1$. For $i = 1, 2, \ldots, b$, let $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ and $B_i = \{j \in [n] : x_{ij} = 1\}$, so $\mathbf{x}_i$ is the characteristic vector of B_i. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$. One denotes as $\mathbf{D}_X$ the incidence structure $([n], \mathcal{B})$.

Remark 2.8. If X is a spherical 2-distance set centered at $\mathbf{0}$, then the incidence structure $\mathbf{D}_X$ is a quasisymmetric 0-design and Γ_X is its block graph.

Proposition 2.9. *Let X be a 2-distance set in H_n, and let $h_1 < h_2$ be the nonzero distances in X. If the graph Γ_X is connected, then $h_2 \leq 2h_1$.*

Proof. Suppose $h_2 > 2h_1$. If x, y, and z are distinct vertices of Γ_X such that $x \sim y$ and $x \sim z$, then the triangle inequality implies that $y \sim z$. Therefore, all neighbors of x form a connected component of Γ_X. Since Γ_X is not a complete graph, it is not connected; a contradiction. □

Definition 2.10. One will say that a spherical 2-distance set $X \subset H_n$ *represents a graph* Γ *in* H_n if Γ is isomorphic to Γ_X. The least n for which such a set X exists is called the *binary spherical representation number of* Γ and is denoted as $\mathrm{bsr}(\Gamma)$.

Proposition 2.11. *Every simple graph Γ, except null graphs and complete graphs, admits a spherical representation in H_n if n is sufficiently large.*

Proof. Let Γ be a noncomplete graph of order b with $e \geq 1$ edges, and let $N = [n_{ij}]$ be an incidence matrix of Γ. For $i = 1, 2, \ldots, b$, let $X_i = \{j \in [e] : n_{ij} = 1\}$. Let $k = \max\{|X_i| : 1 \leq i \leq b\}$, and let $Y_1, Y_2, \ldots, Y_b$ be pairwise disjoint subsets of $\{e+1, e+2, \ldots, e+bk\}$ such that $|Y_i| = k - |X_i|$. For $i = 1, 2, \ldots, b$, let $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{i,e+bk}) \in H_{e+bk}$, where $x_{ij} = 1$ if and only if $j \in X_i \cup Y_i$. Then, for $1 \leq i < j \leq b$, the distance between points $\mathbf{x}_i$ and $\mathbf{x}_j$ is equal to $2(k-2)$ if the ith and jth vertices of Γ are adjacent, and it is equal to $2k$ otherwise. Since Γ is not a complete graph, the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_b\}$ is a 2-distance set representing Γ in H_{e+bk}, and this set lies on a sphere of radius k centered at $\mathbf{0}$. □

If the graph Γ in the above proof is regular, we do not need to add columns to its incidence matrix N.

Proposition 2.12. *If Γ is a noncomplete regular graph with $e \geq 1$ edges, then* $\operatorname{bsr}(\Gamma) \leq e$.

Theorem 5.1 implies that if Γ is a cycle, then its binary spherical representation number equals the number of edges.

For any graph G, the *line graph of* G, denoted as $L(G)$, is the graph whose vertex set is the edge set of G; two distinct vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share a vertex. Line graphs are precisely the graphs representable by spherical 2-distance sets of radius 2.

Proposition 2.13. *A graph Γ can be represented in H_n by a spherical 2-distance sets of radius 2 if and only if Γ is isomorphic to the line graph of a graph of order n.*

Proof. If $\Gamma = L(G)$, where G is a graph of order n, then the columns of an incidence matrix of G form a 2-distance subset of H_n of radius 2 representing Γ. Conversely, let X be a 2-distance subset of H_n of radius 2 centered at $\mathbf{0}$ and representing a graph Γ. Let G be a graph whose incidence matrix coincides with an incidence matrix of $\mathbf{D}_X$. Then $|G| = n$ and Γ is isomorphic to $L(G)$. □

Remark 2.14. Let G be a regular graph of degree r, and let X be the set of columns of an incidence matrix N of G. Then $\mathbf{D}_X$ is a quasisymmetric 1-design (with block size 2 and replication number r) and N is its incidence matrix. If $r \geq 3$, this design is non-square. The next result (Proposition 2.3 in [5]) yields a necessary and sufficient condition for this 1-design to be nonsingular.

Proposition 2.15. *If N is an incidence matrix of a graph Γ of order n and c is the number of connected components of Γ, then*

$$\operatorname{rank}\left(NN^{\top}\right) = \begin{cases} n, & \text{if } \Gamma \text{ is not a bipartite graph,} \\ n - c, & \text{if } \Gamma \text{ is a bipartite graph.} \end{cases} \tag{2.1}$$

3. Lower Bounds

The main tool in obtaining a lower bound on $\operatorname{bsr}(\Gamma)$ is the following classical theorem of distance geometry.

Definition 3.1. Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_b\}$ be a set of b points in $\mathbb{R}^n$. The *Schoenberg matrix of X with respect to a point* $\mathbf{z} \in \mathbb{R}^n$ is the matrix $S_{\mathbf{z}}(X) = [s_{ij}]$ of order b with

$$s_{ij} = \|\mathbf{z} - \mathbf{x}_i\|^2 + \|\mathbf{z} - \mathbf{x}_j\|^2 - \|\mathbf{x}_i - \mathbf{x}_j\|^2. \tag{3.1}$$

Theorem 3.2 (see [6, 7]). *If X is a finite set in $\mathbb{R}^n$, then, for any $\mathbf{z} \in \mathbb{R}^n$, the Schoenberg matrix $S_{\mathbf{z}}(X)$ is positive semidefinite and* $\operatorname{rank}(S_{\mathbf{z}}(X)) \le n$.

We will now derive a sharp lower bound on the binary spherical representation number of a connected regular graph.

Theorem 3.3. *Let Γ be a connected regular graph, and let m be the multiplicity of the least eigenvalue of Γ. Then* $\operatorname{bsr}(\Gamma) \ge |\Gamma| - m$. *Moreover, $bsr(\Gamma) = |\Gamma| - m$ if and only if Γ is the block graph of a nonsquare nonsingular quasisymmetric 1-design.*

Proof. Let $\operatorname{bsr}(\Gamma) = n$, and let Γ be isomorphic to Γ_X, where X is a spherical 2-subset of H_n. Let $h_1 < h_2$ be the nonzero distances in X and k the radius of a sphere in H_n containing X. Without loss of generality, we assume that this sphere is centered at $\mathbf{0}$. Then $\mathbf{z} = (k/n, k/n, \ldots, k/n)$ is the center of an Euclidean sphere containing X. The radius of this sphere is equal to $\sqrt{k(n-k)/n}$. Let A be an adjacency matrix of Γ_X. Then the matrix

$$S = S_{\mathbf{z}}(X) = h_2 I + (h_2 - h_1)A + \left(\frac{2k(n-k)}{n} - h_2\right)J \tag{3.2}$$

is the Schoenberg matrix of the set X with respect to $\mathbf{z}$.

Let d be the degree of Γ, $\rho_0 = d > \rho_1 > \cdots > \rho_s = \rho$ all distinct eigenvalues of A, and $m_0 = 1, m_1, \ldots, m_s = m$ their respective multiplicities. Then the eigenvalues of S are

$$\sigma_0 = \frac{2|\Gamma|k(n-k)}{n} - dh_1 - (|\Gamma| - d - 1)h_2, \tag{3.3}$$

with $\mathbf{1}$ as an eigenvector, and, for $i = 1, 2, \ldots, s$,

$$\sigma_i = h_2 + (h_2 - h_1)\rho_i \tag{3.4}$$

(with eigenvectors orthogonal to $\mathbf{1}$). For $0 \le i \le s$, the multiplicity of σ_i is at least m_i. (It is greater than m_i if $\sigma_i = \sigma_0$, $i \ne 0$.)

Theorem 3.2 implies that all eigenvalues of S are nonnegative, so $\sigma_i > 0$ for $1 \le i \le s-1$. Therefore, $\operatorname{rank}(S) \ge \sum_{i=1}^{s-1} m_i = |\Gamma| - m - 1$. On the other hand, since both X and $\mathbf{z}$ lie in the hyperplane $x_1 + x_2 + \cdots + x_n = k$, Theorem 3.2 implies that $\operatorname{rank}(S) \le n - 1$, so $n \ge |\Gamma| - m$.

Suppose now that $n = |\Gamma| - m$. Then $\operatorname{rank}(S) = |\Gamma| - m - 1$, and therefore $\sigma_s = \sigma_0 = 0$. From $\sigma_0 = 0$ we derive

$$2|\Gamma|k(n-k) = n(dh_1 + (|\Gamma| - d - 1)h_2). \tag{3.5}$$

The incidence structure $\mathbf{D}_X = ([n], \mathcal{B})$ has n points, $|\Gamma|$ blocks, all of cardinality k, and two intersection numbers, $\alpha = k - h_2/2 < \beta = k - h_1/2$. The graph Γ is the block graph of $\mathbf{D}_X$. Using $h_1 = 2(k-\beta)$ and $h_2 = 2(k-\alpha)$, we transform (3.5) into

$$(\beta - \alpha)d = k\left(\frac{|\Gamma|k}{n} - 1\right) - \alpha(|\Gamma| - 1). \tag{3.6}$$

For each $i \in [n]$, let r_i denote the number of blocks of $\mathbf{D}_X$ containing i. Fix a block C and count in two ways pairs (B, i), where $B \in \mathcal{B}$, $B \neq C$, and $i \in C \cap B$:

$$d\beta + (|\Gamma| - d - 1)\alpha = \sum_{i \in C}(r_i - 1). \tag{3.7}$$

Using this equation and (3.6), we derive

$$\sum_{i \in C} r_i = d\beta + (|\Gamma| - d - 1)\alpha + k = \frac{|\Gamma|k^2}{n}. \tag{3.8}$$

Therefore,

$$\sum_{C \in \mathcal{B}} \sum_{i \in C} r_i = \frac{|\Gamma|^2 k^2}{n}. \tag{3.9}$$

Since each $i \in [n]$ contributes r_i^2 into the left-hand side of this equation, we obtain that

$$\sum_{i=1}^{n} r_i^2 = \frac{|\Gamma|^2 k^2}{n}. \tag{3.10}$$

On the other hand, counting in two ways pairs (i, B) with $B \in \mathcal{B}$ and $i \in B$ yields

$$\sum_{i=1}^{n} r_i = \sum_{B \in \mathcal{B}} |B| = |\Gamma| k. \tag{3.11}$$

Thus,

$$\left(\frac{1}{n}\sum_{i=1}^{n} r_i\right)^2 = \frac{1}{n}\sum_{i=1}^{n} r_i^2. \tag{3.12}$$

Therefore, $r_i = r = |\Gamma|k/n$ for $i = 1, 2, \ldots, n$. Thus, $\mathbf{D}_X$ is a quasisymmetric 1-design. (Note that we have derived this result from (3.5) rather than from a stronger equation $\operatorname{rank}(S) = |\Gamma| - m - 1$.) Since $n < |\Gamma|$, the 1-design $\mathbf{D}_X$ is non-square, so we have to show that it is nonsingular.The incidence matrix N of $\mathbf{D}_X$ satisfies the equation

$$N^\top N = (k - \alpha)I + (\beta - \alpha)A + \alpha J. \tag{3.13}$$

Therefore, the eigenvalues of $N^\top N$ are

$$\tau_0 = k - \alpha + (\beta - \alpha)\rho_0 + \alpha|\Gamma|, \qquad \tau_i = k - \alpha + (\beta - \alpha)\rho_i \quad (1 \leq i \leq s). \tag{3.14}$$

Since $\tau_0 > \tau_1 > \cdots > \tau_s$ and since $\operatorname{rank}(N^\top N) \leq n$, we obtain that $\tau_s = 0$ and $\tau_i > 0$ for $0 \leq i \leq s-1$. Since the multiplicity of τ_s is the same as the multiplicity of ρ_s, we have $\operatorname{rank}(N^\top N) = n$. Therefore, $\operatorname{rank}(NN^\top) = n$, and then $\det(NN^\top) \neq 0$, that is, $\mathbf{D}_X$ is nonsingular.

Suppose now that Γ is the block graph of a nonsquare nonsingular quasisymmetric 1-design $\mathbf{D}$ with intersection numbers $\alpha < \beta$. The design $\mathbf{D}$ has less points than blocks, so let b be the number of blocks and $b - m$ the number of points. We have to show that m is the multiplicity of the least eigenvalue of Γ and that $\operatorname{bsr}(\Gamma) = b - m$.

Let N be an incidence matrix of $\mathbf{D}$ and X the set of all columns of N regarded as points in H_{b-m}. Then X is a 2-distance set and $\mathbf{D}$ is $\mathbf{D}_X$. The set X lies on a sphere of radius k centered at $\mathbf{0}$, where k is the cardinality of each block of $\mathbf{D}$, and the nonzero distances in X are $h_1 = 2(k - \beta)$ and $h_2 = 2(k - \alpha)$.

Matrix N satisfies (3.13) with A being an adjacency matrix of Γ. Let $\rho_0 > \rho_1 > \cdots > \rho_s$ be all distinct eigenvalues of A. Then the eigenvalues of $N^\top N$ are given by (3.14). Since $N^\top$ has more rows than columns, we have $\tau_s = 0$. Since $\det(NN^\top) \neq 0$, the sum of the multiplicities of the nonzero eigenvalues of $N^\top N$ is $b - m$, so the multiplicity of τ_s is equal to m. Therefore, the multiplicity of ρ_s is equal to m, and then $\operatorname{bsr}(\Gamma) \geq b - m$. Since X is in H_{b-m}, we have $\operatorname{bsr}(\Gamma) = b - m$. □

It has been shown in the course of this proof that if $\operatorname{bsr}(\Gamma) = |\Gamma| - m$, then $\sigma_0 = 0$, which implies (3.5), and $\sigma_s = 0$, which implies

$$\frac{h_2}{h_1} = \frac{\rho}{\rho + 1}. \tag{3.15}$$

In fact, (3.15) must hold whenever $\operatorname{bsr}(\Gamma) < |\Gamma|$, because otherwise $\operatorname{rank}(A) \geq |\Gamma| - 1$ and then $\operatorname{bsr}(\Gamma) \geq |\Gamma|$. If $\operatorname{bsr}(\Gamma) = |\Gamma|$ and (3.15) does not hold, then $\sigma_0 = 0$. It has also been shown that if $\operatorname{bsr}(\Gamma) = |\Gamma| - m$, then the replication number of the corresponding 1-design is $|\Gamma|k/(b - m)$. We combine these observations in the following two theorems.

Theorem 3.4. *Let Γ be a connected regular graph of order b and degree d, and let m be the multiplicity of the least eigenvalue ρ of Γ. Let $\operatorname{bsr}(\Gamma) = b - m$, and let Γ be isomorphic to Γ_X, where X is a 2-subset of H_{b-m} lying on a sphere of radius k centered at $\mathbf{0}$. Let $h_1 < h_2$ be the nonzero distances in X. Then,*

(i) $2bk(b - m - k) = (b - m)(dh_1 + (b - d - 1)h_2)$;

(ii) $h_2/h_1 = \rho/(\rho + 1)$;

(iii) $\mathbf{D}_X$ *is a nonsquare nonsingular quasisymmetric 1-design with $b - m$ points, b blocks, block size k, replication number $bk/(b - m)$, and intersection numbers $k - h_1/2$ and $k - h_2/2$.*

Theorem 3.5. *Let Γ be a connected regular graph of order b and degree d, and let ρ be the least eigenvalue of Γ. Let $\operatorname{bsr}(\Gamma) = n$, and let Γ be isomorphic to Γ_X, where X is a 2-subset of H_n lying on* *a sphere of radius* k*. Let* $h_1 < h_2$ *be the nonzero distances in* X*:*

(i) *if* $2bk(n-k) = n(dh_1 + (b-d-1)h_2)$*, then* $\mathbf{D}_X$ *is a quasisymmetric 1-design;*

(ii) *if* $n < b$*, then* $h_2/h_1 = \rho/(\rho+1)$*;*

(iii) *if* $n = b$ *and* $h_2/h_1 \neq \rho/(\rho+1)$*, then* $2k(b-k) = dh_1 + (b-d-1)h_2$*.*

If $h_2/h_1 = \rho/(\rho+1)$, then ρ is rational, so (ii) implies that the following useful result.

Corollary 3.6. *If the least eigenvalue of a connected regular graph* Γ *is irrational, then* $\mathrm{bsr}(\Gamma) \geq |\Gamma|$*.*

An infinite family of regular graphs attaining the lower bound of Theorem 3.3 is given in the following example.

Example 3.7. Let $\mathbf{D}$ be a $(v, b, r, k, 1)$-design with $b \geq v+r$ and $k \geq 3$, and let $\mathbf{D}'$ be an incidence structure obtained by deleting from $\mathbf{D}$ one point and all blocks containing this point. Then $\mathbf{D}'$ is a 1-design with $v-1$ points, $b-r > v-1$ blocks of cardinality k, replication number $r-1$, and intersection numbers 0 and 1. Without loss of generality, we assume that the point set of $\mathbf{D}$ is $[v]$, the deleted point is v, and the deleted blocks are

$$\{1, 2, \ldots, k-1, v\}, \{k, k+1, \ldots, 2k-2, v\}, \ldots, \{v-k+1, v-k+2, \ldots, v\}. \tag{3.16}$$

Let N be the corresponding incidence matrix of $\mathbf{D}'$. Then $NN^\top$ is an $r \times r$ block matrix of $(k-1) \times (k-1)$ blocks with all diagonal blocks equal to $(r-1)I$ and all off-diagonal blocks equal J. The spectrum of $NN^\top$ consists of eigenvalues $(r-1)k$ of multiplicity 1, $r-1$ of multiplicity $(k-2)r$, and $r-k$ of multiplicity $r-1$. Therefore, $\det(NN^\top) \neq 0$; that is, the design $\mathbf{D}'$ is nonsingular. The spectrum of $N^\top N$ is obtained by adjoining the eigenvalue 0 of multiplicity $b-r-v+1$ to the spectrum of $NN^\top$. Since $N^\top N = kI + A$, where A is an adjacency matrix of the block graph Γ of $\mathbf{D}'$, we determine that the multiplicities of the largest and the smallest eigenvalues of A are 1 and $b-r-v+1$, respectively. Therefore, Γ is a connected regular graph and $\mathrm{bsr}(\Gamma) = v-1$.

4. Strongly Regular Graphs

For strongly regular graphs we first obtain a sharp upper bound for the binary spherical representation number.

Proposition 4.1. *If* Γ *is a connected strongly regular graph of order* n*, then* $\mathrm{bsr}(\Gamma) \leq n$*.*

Proof. Let Γ be an $\mathrm{srg}(n, d, \lambda, \mu)$, and let A be an adjacency matrix of Γ. Then $A^2 = (d-\mu)I + (\lambda-\mu)A + \mu J$. Therefore, $(A+I)^2 = (d-\mu+1)I + (\lambda-\mu+2)A + \mu J$. Let X be the set of rows of $A+I$ regarded as points in H_n. Then the distance between two distinct points of X is equal to $2(d-\lambda-1)$ if the points correspond to adjacent vertices of Γ; otherwise, it is equal to $2(d-\mu+1)$. Thus, X is a 2-distance set in H_n, lying on a sphere of radius $d+1$ centered at $\mathbf{0}$, and, if $\lambda \geq \mu-1$, then Γ is isomorphic to Γ_X.

If $\lambda \leq \mu-1$, then let Y be the set of rows of the matrix $J-A$. The distance between two distinct points of Y is equal to $2(d-\mu)$ if the points correspond to adjacent vertices of Γ; otherwise, it is equal to $2(d-\lambda)$. Therefore, Y is a 2-distance set in H_n, lying on a sphere of radius $n-d$ centered at O, and Γ is isomorphic to Γ_Y. □

This proposition and Corollary 3.6 imply the next result.

Corollary 4.2. *If the least eigenvalue of a strongly regular graph* Γ *of order* n *is irrational, then* $\mathrm{bsr}(\Gamma) = n$.

Remark 4.3. The least eigenvalue of a strongly regular graph is irrational if and only if it is an $\mathrm{srg}(n, (n-1)/2, (n-5)/4, (n-1)/4)$, where $n \equiv 1 \pmod 4$ is not a square. A graph with these parameters exists if and only if there exists a conference matrix of order $n+1$.

Example 4.4. Let Γ be the complement of the cycle C_7. The least eigenvalue of Γ is irrational, so $\mathrm{bsr}(\Gamma) \geq 7$. Suppose $\mathrm{bsr}(\Gamma) = 7$, and let Γ be isomorphic to Γ_X, where X is a 2-subset of H_7 with nonzero distances $h_1 < h_2$, lying on a sphere of radius k centered at $\mathbf{0}$. Since h_1 and h_2 are even, $h_2 \leq 7$ and $h_2 \leq 2h_1$ (Proposition 2.9), we have $h_1 = 2$, $h_2 = 4$ or $h_1 = 4$, $h_2 = 6$. In either case, Theorem 3.5(iii) yields an equation without integer solutions. Thus, $\mathrm{bsr}(\Gamma) \geq 8$, so the strong regularity in Proposition 4.1 is essential.

Remark 4.5. There are 167 nonisomorphic strongly regular graphs with parameters $(64, 18, 2, 6)$ [8]. The least eigenvalue of these graphs is -6 of multiplicity 18. Theorem 3.3 and Proposition 4.1 imply that if Γ is any of these 167 graphs, then $46 \leq \mathrm{bsr}(\Gamma) \leq 64$. Therefore, there are nonisomorphic graphs with these parameters having the same binary spherical representation number.

Also, there are 41 nonisomorphic strongly regular graphs with parameters $(29, 14, 6, 7)$ [8]. The least eigenvalue of these graphs is irrational, so by Corollary 4.2 the binary spherical representation number of all these graphs is 29.

Theorem 3.3 for regular graphs can be rectified if the graph is strongly regular.

Theorem 4.6. *Let* Γ *be a connected strongly regular graph of order* b, *and let* m *be the multiplicity of the least eigenvalue of* Γ. *Then* $\mathrm{bsr}(\Gamma) = b - m$ *if and only if* Γ *is the block graph of a quasisymmetric 2-design.*

Proof. If Γ is the block graph of a quasisymmetric 2-design $\mathbf{D}$, then Remark 2.6 and Theorem 3.3 imply that $\mathrm{bsr}(\Gamma) = b - m$.

Suppose now that $\mathrm{bsr}(\Gamma) = b - m$, and let X be a spherical 2-distance subset of H_{b-m} representing Γ. Let $h_1 < h_2$ be the nonzero distances in X and k the radius of the sphere centered at $\mathbf{0}$ and containing X. Every block of the incidence structure $\mathbf{D}_X = ([b-m], \mathcal{B})$ is of cardinality k, the intersection numbers of $\mathbf{D}$ are $\alpha = k - h_2/2 < \beta = k - h_1/2$, and the replication number of $\mathbf{D}$ is $r = bk/(b-m)$ (Theorem 3.4). The graph Γ is the block graph of $\mathbf{D}$. Let $\rho_0 = d > \rho_1 > \rho_2$ be the eigenvalues of Γ. Since Γ is connected, the multiplicity of ρ_0 is 1. Since the multiplicity of ρ_2 is m, the multiplicity of ρ_1 is $b - m - 1$.

Let A be an adjacency matrix of Γ. Theorem 3.4(ii) implies that $(\beta - \alpha)\rho_2 = (1/2)(h_1 - h_2)\rho_2 = -(k - \alpha)$. Since $\mathrm{Tr}(A) = d + (b - m - 1)\rho_1 + m\rho_2 = 0$, we use Theorem 3.4(i) to derive that

$$(\beta - \alpha)\rho_1 = \alpha - k + r - \lambda, \tag{4.1}$$

where $\lambda = r(k-1)/(b-m-1)$. Since $k < b - m$, we have $\lambda < r$.

Let N be an incidence matrix of $\mathbf{D}_X$. Then

$$N^\top N J = N N^\top J = krJ,$$
$$N^\top N = (k-\alpha)I + (\beta-\alpha)A + \alpha J. \tag{4.2}$$

From these equations we determine the eigenvalues of $N^\top N$: $\tau_0 = kr$, $\tau_1 = k-\alpha+(\beta-\alpha)\rho_1 = r-\lambda$, and $\tau_2 = k-\alpha+(\beta-\alpha)\rho_2 = 0$. Their respective multiplicities are 1, $b-m-1$, and m. Therefore, the eigenvalues of $NN^\top$ are τ_0 of multiplicity 1 and τ_1 of multiplicity $b-m-1$. Since $NN^\top J = krJ$, the eigenspace E_0 of $NN^\top$ corresponding to the eigenvalue τ_0 is generated by $\mathbf{1}$. Therefore, $E_1 = E_0^\perp$ is the eigenspace corresponding to the eigenvalue τ_1. On the other hand, the matrix $M = (r-\lambda)I + \lambda J$ has the same eigenvalues with the same respective eigenspaces. Thus, $NN^\top = M$, and therefore $\mathbf{D}_X$ is a quasisymmetric 2-design with intersection numbers α and β. The graph Γ is the block graph of this design. □

Example 4.7. The *Cocktail Party graph* $\mathrm{CP}(n)$ has $2n$ vertices split into n pairs with two vertices adjacent if and only if they are not in the same pair. It is the block graph of a quasisymmetric 2-design if and only if the design is a *Hadamard 3-design* with $n+1$ points (cf. [4, Theorem 8.2.23]). The least eigenvalue of $\mathrm{CP}(n)$ is -2 of multiplicity $n-1$. By Theorem 4.6, $\mathrm{bsr}(\mathrm{CP}(n)) \geq n+1$ and $\mathrm{bsr}(\mathrm{CP}(n)) = n+1$ if and only there exists a Hadamard matrix of order $n+1$. This example shows that it is hard to expect a simple general method for computing the binary spherical representation number of a strongly regular graph.

5. Line Graphs

In this section we determine the binary spherical representation number for the line graphs of regular graphs. If N is an incidence matrix of a graph G, then $N^\top N = 2I + A$, where A is an adjacency matrix of the line graph $\Gamma = L(G)$. Let G be connected and have n vertices and e edges. If $e > n$, then the least eigenvalue of $N^\top N$ is 0, and therefore the least eigenvalue of Γ is $\rho = -2$. Since matrices $NN^\top$ and $N^\top N$ have the same positive eigenvalues, Proposition 2.15 implies that the multiplicity of ρ is equal to $e-n$ if the graph G is not bipartite, and it is equal to $e-n+1$ if G is a connected bipartite graph. If $e = n$, then G is a cycle, so $\Gamma = C_n$ is a cycle of order n too. If n is even, then the least eigenvalue of C_n is -2 of multiplicity 1; if $n \geq 5$ is odd, then the least eigenvalue of C_n is irrational. See [9] for details.

Theorem 5.1. *If* Γ *is the line graph of a connected regular graph of order* $n \geq 4$, *then* $\mathrm{bsr}(\Gamma) = n$.

Proof. Let Γ be the line graph of a connected regular graph G of order $n \geq 4$ and degree d. Then Γ is a connected regular graph of order $nd/2$ and degree $2d-2$. The columns of an incidence matrix of G form a spherical 2-distance set in H_n representing Γ, so $\mathrm{bsr}(\Gamma) \leq n$.

Suppose first that $d = 2$, that is, G is C_n, and that n is odd. Then the least eigenvalue of Γ is irrational. Therefore, $\mathrm{bsr}(\Gamma) \geq n$ by Corollary 3.6, so $\mathrm{bsr}(\Gamma) = n$.

Suppose now that $d \geq 3$ and the graph G is not bipartite. From Proposition 2.15, the multiplicity of the least eigenvalue of Γ is $nd/2-n$, and then Theorem 3.3 implies that $\mathrm{bsr}(\Gamma) \geq n$, so $\mathrm{bsr}(\Gamma) = n$.

Suppose finally that G is a bipartite graph (this includes the case $G = C_n$ with even n). Then the least eigenvalue of Γ is -2 and its multiplicity is $nd/2 - n + 1$. Therefore, by

Theorem 3.3, $\mathrm{bsr}(\Gamma) \geq n-1$. Suppose $\mathrm{bsr}(\Gamma) = n-1$. Theorem 3.4(ii) implies that $h_2 = 2h_1$ and then the condition (i) of Theorem 3.4 can be rewritten as

$$nk(n-1-k) = h_1(n-1)(n-2). \tag{5.1}$$

If n is odd, then $(n-1)(n-2)$ and n are relatively prime; if n is even, then $(n-1)(n-2)/2$ and $n/2$ are relatively prime. In either case, $(n-1)(n-2)/2$ divides $k(n-1-k)$. However, $k(n-1-k) \leq (n-1)^2/4 < (n-1)(n-2)/2$. Therefore, $\mathrm{bsr}(\Gamma) = n$. □

The graph $L_2(n)$ is the line graph of the bipartite graph with the bipartition sets of cardinality n. The following corollary generalizes the well-known result [10] that these graphs are not block graphs of quasisymmetric 2-designs.

Corollary 5.2. *The line graph of a connected regular graph G with more than three vertices is the block graph of a nonsquare nonsingular quasisymmetric 1-design if and only if G is not a cycle and is not a bipartite graph.*

Remark 5.3. If G is a semiregular connected bipartite graph of order n, then the graph $L(G)$ is regular and $\mathrm{bsr}(L(G)) = n$ or $n-1$. We do not know of any example when $\mathrm{bsr}(L(G)) = n-1$.

There exist regular graphs that are cospectral with a line graph but are not line graphs. The complete list of such graphs is given in the following theorem.

Theorem 5.4 (see [11]). *Let a regular graph Γ be cospectral with the line graph $L(G)$ of a connected graph G. If Γ is not a line graph, then G is a regular 3-connected graph of order 8 or $K_{3,6}$ or the semiregular bipartite graph of order 9 with 12 edges.*

Since $\mathrm{bsr}(L(G)) < 10$ for every graph G listed in Theorem 5.4, the next theorem implies that if a connected regular graph Γ is cospectral with a line graph $L(G)$ and if $\mathrm{bsr}(\Gamma) = \mathrm{bsr}(L(G))$, then Γ is a line graph. The proof is based on the following theorem according to Beineke [12].

Theorem 5.5. *A graph is a line graph if and only if it does not contain as an induced subgraph any of the nine graphs of Figure 1.*

Theorem 5.6. *Let the least eigenvalue of a connected regular graph Γ be equal to -2. If* $\mathrm{bsr}(\Gamma) < 10$*, then Γ is a line graph or the Petersen graph or $CP(n)$ with $4 \leq n \leq 7$.*

Proof. The Petersen graph P is the block graph the quasisymmetric $(6,10,5,3,2)$-design, so $\mathrm{bsr}(P) = 6$. We also have $\mathrm{bsr}(\mathrm{CP}(7)) = 8$ (Example 4.7). For $n = 4, 5$, and 6, $\mathrm{CP}(n)$ is an induced subgraph of $\mathrm{CP}(8)$, so $\mathrm{bsr}(\mathrm{CP}(n)) \leq 8$.

Let $\mathrm{bsr}(\Gamma) = n \leq 9$, and let a 2-distance set X represent Γ in H_n. Let $h_1 < h_2$ be the nonzero distances in X, and let X lie on a sphere of radius k centered at $\mathbf{0}$. Let f be an isomorphism from Γ to Γ_X. For each vertex x of Γ we regard $f(x)$ as a k-subset of $[n]$.

Suppose Γ is not a line graph. Since the least eigenvalue of Γ is -2, Theorem 3.4 implies that $h_2 = 2h_1$. Since $n \leq 9$, we assume that $k \leq 4$. Proposition 2.13 implies that $k \neq 2$, so $k = 3$ or 4. By Theorem 5.5, Γ contains one of the nine graphs of Figure 1 as an induced subgraph. All subgraphs of Γ considered throughout the proof are assumed to be induced subgraphs.

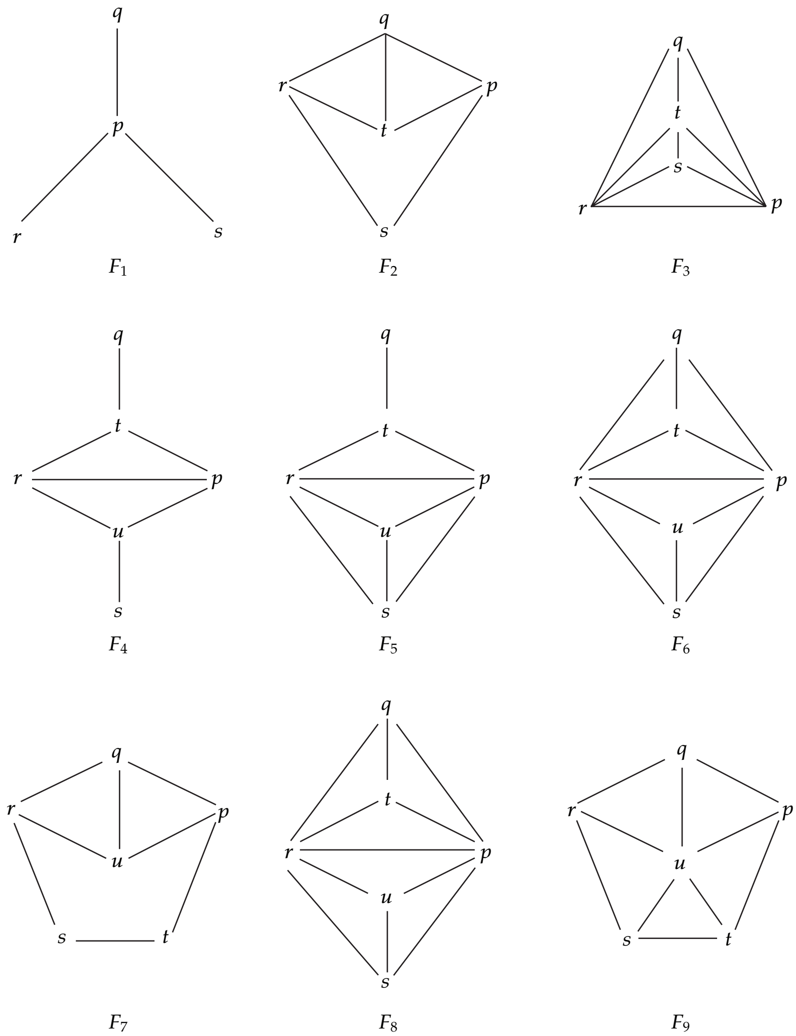

Figure 1

Case 1 ($h_1 = 4$. Then $h_2 = 8$, and therefore $k = 4$). If Γ contains a coclique xyz of size 3, then $|f(x) \cup f(y) \cup f(z)| = 12 > n$. This rules out subgraphs F_1 and F_4. If the subgraph induced by Γ on a set xyz of three vertices has only one edge, then $|f(x) \cup f(y) \cup f(z)| = 10 > n$. This rules out subgraphs F_2, F_5, F_6, F_7, F_8, and F_9.

Suppose Γ contains F_3 as a subgraph. Suppose also that every subgraph of order 3 of Γ has at least two edges. We assume without loss of generality that $f(q) = 1234$ and $f(s) = 5678$. Let x be a vertex of Γ, $x \neq q$ and $x \neq s$. Since the subgraph with the vertex set qsx has at least two edges, x is adjacent to both q and s. Therefore, $f(x)$ is a 4-subset of 12345678 for every vertex x of Γ. This implies that x is not adjacent to at most one other vertex. Since Γ is regular and not complete, Γ is a cocktail party graph $\mathrm{CP}(m)$. Therefore, $m + 1 \leq \mathrm{bsr}(\Gamma) \leq 9$ and $\mathrm{bsr}(\Gamma) \neq m + 1$ for $m = 8$ (Example 4.7). Since CP(2) and CP(3) are line graphs (of C_4 and K_4, resp.), we have $4 \leq n \leq 7$.

Case 2 ($h_1 = 2$ and $k = 3$). Suppose Γ contains F_1 as a subgraph. We let $f(p) = 123$, $f(q) = 124$, $f(r) = 135$, and $f(s) = 236$. Since the degree of q in F_1 is 1 and the degree of p is 3, Γ has vertices q_1 and q_2 adjacent to q but not to p. Then $f(q_1) = 146$ and $f(q_2) = 245$. Similarly, we find vertices r_1 and r_2 adjacent to r but not to p and vertices s_1 and s_2 adjacent to s but not to p and assume that $f(r_1) = 345$, $f(r_2) = 156$, $f(s_1) = 256$, and $f(s_2) = 346$. The set U of the 10 vertices that we have found is the vertex set of a Petersen subgraph of Γ. The set $f(U)$ consists of ten 3-subsets of 123456, no two of which are disjoint. Therefore, if Γ has a vertex $v \notin U$, then $f(v)$ is disjoint from at least one of the sets $f(x)$, $x \in U$; a contradiction. Thus, Γ is the Petersen graph.

If Γ contains F_2 as a subgraph, we let $f(p) = 123$ and $f(r) = 145$. Then $f(q), f(s), f(t) \in \{124, 125, 134, 135\}$. Since $|f(q) \cap f(s)| = |f(t) \cap f(s)| = 1$, there is no feasible choice for $f(s)$.

If Γ contains F_3 as a subgraph, we assume that $f(p) = 123$, $f(q) = 124$, and $f(s) = 135$. Then $f(r), f(t) \in \{125, 134\}$, and therefore $|f(r) \cap f(t)| \neq 2$.

Let Γ contains F_4 as a subgraph. Suppose first that $f(p) \cap f(q) \cap f(s) \neq \emptyset$. Then we assume that $f(p) = 123$, $f(q) = 145$, $f(s) = 167$, $f(t) = 124$, and $f(u) = 136$, and there is no feasible choice for $f(r)$. Suppose now that $f(p) \cap f(q) \cap f(s) = f(r) \cap f(q) \cap f(s) = \emptyset$. We assume that $f(p) = 123$, $f(q) = 145$, and $f(s) = 246$. Then $f(t) = 125$ or 134. If $f(t) = 125$, then $f(u) = 234$ and $f(r) = 235$. Γ has distinct vertices q_1 and q_2 adjacent to q but not to r, and we have $f(q_1) = f(q_2) = 156$; a contradiction. If $f(t) = 134$, then $f(u) = 126$ and $f(r) = 136$. Γ has distinct vertices q_1 and q_2 adjacent to q but not to t, and we have again $f(q_1) = f(q_2) = 156$.

If Γ contains F_5 as a subgraph, we let $f(p) = 123$, $f(q) = 145$, and $f(t) = 124$. Then $f(r) = 234$ or 126, and in either case $f(s) = f(u)$.

If Γ contains F_6 as a subgraph, we let $f(p) = 123$, $f(q) = 124$, and $f(s) = 135$. Then we assume that $f(r) = 125$. This implies $f(u) = 235$ and $f(t) = 126$. Γ has distinct vertices q_1 and q_2 adjacent to q but not to r. Then $f(q_1) = 134$ and $f(q_2) = 234$, so both q_1 and q_2 are adjacent to p. Since $q_1 \not\sim u$ and $q_2 \sim u$, Γ has three distinct vertices u_i adjacent to u but not to p. However, $f(u_i) = 245$ for all these vertices.

If Γ contains F_8 as a subgraph, we let $f(p) = 123$ and $f(q) = 124$ and assume that $f(s) = 156$ or 345. If $f(s) = 156$, then $f(r) = 135$, $f(u) = 136$, and there is no feasible choice for $f(t)$. If $f(s) = 345$, then $f(r) = 135$, $f(u) = 235$, and again there is no feasible choice for $f(t)$.

Suppose Γ contains F_7 or F_9 as a subgraph. We let $f(p) = 123$, $f(q) = 124$, and $f(t) = 135$. Then $f(r) = 146$ or 245 and $f(s) = 156$ or 345, respectively. In either case, there is no feasible choice for $f(u)$.

Case 3 ($h_1 = 2$ and $k = 4$). Suppose Γ contains F_1 as a subgraph. We let $f(p) = 1234$, $f(q) = 1235$, $f(r) = 1246$, and $f(s) = 1347$. Γ has vertices q_1 and q_2 adjacent to q but not to p. Then $f(q_1) = 1257$ and $f(q_2) = 1356$. Similarly, we find vertices r_1 and r_2 adjacent to r but not to p and vertices s_1 and s_2 adjacent to s but not to p and assume that $f(r_1) = 1267$, $f(r_2) = 1456$, $f(s_1) = 1367$, and $f(s_2) = 1457$. The ten vertices that we have found form a Petersen subgraph of Γ. If $1 \in f(x)$ for every vertex x of Γ, then we delete 1 from each $f(x)$ and refer to Case 2. Suppose that there is a vertex x with $1 \notin f(x)$. Then the 4-set $f(x)$ must meet each of the sets $234, 235, 246, 347, 257, 356, 267, 456, 367$, and 457 in at least two points. Thus, there is no feasible choice for $f(x)$.

If Γ contains F_2 as a subgraph, we let $f(p) = 1234$, $f(q) = 1235$, and $f(r) = 1256$. Then $f(s) = 1246$ and there is no feasible choice for $f(t)$.

If Γ contains F_3 as a subgraph, we assume that $f(p) = 1234$, $f(q) = 1235$, and $f(s) = 1246$. Then $f(r), f(t) \in \{1236, 1245\}$, and therefore $|f(r) \cap f(t)| = 2$; a contradiction.

If Γ contains F_4 as a subgraph, we let $f(p) = 1234$, $f(t) = 1235$, $f(u) = 1246$, $f(r) = 1236$, and $f(q) = 1257$. Then $f(s) = 1456$. Let s_1 and s_2 be vertices of Γ adjacent to s but not to u. Then $f(s_1) = 1345$ and $f(s_2) = 1356$, and there is no feasible choice for $f(v)$, where $v \sim q$ and $v \not\sim t$.

If Γ contains F_5 as a subgraph, we let $f(p) = 1234$, $f(q) = 1256$, and $f(t) = 1235$. Then we may assume that either $f(s) = 1346$ and $f(u) = 2346$ or $f(s) = 1247$ and $f(u) = 1248$. In either case, there is no feasible choice for $f(r)$.

Suppose Γ contains F_6 as a subgraph. We let $f(p) = 1234$ and $f(r) = 1235$. Then $f(q), f(s), f(t), f(u) \in \{1245, 1345, 2345\} \cup \{123\alpha : \alpha \geq 6\}$. Since the subgraph induced on $qstu$ is triangle-free, we let $f(q) = 1236$, $f(t) = 1237$, $f(s) = 1245$, and $f(u) = 1345$. Let q_1 and q_2 be distinct vertices of Γ adjacent to q but not to p. Then $f(q_i) \in \{1256, 1356, 2356\}$, so both q_1 and q_2 are adjacent to r but not to t. Therefore, Γ has at least four vertices t_i adjacent to t but not to r. However, $f(t_i) \in \{1247, 1347, 2347\}$; a contradiction.

If Γ contains F_8 as a subgraph, we let $f(p) = 1234$, $f(q) = 1235$, and $f(r) = 1246$. Then $(f(t), f(u)) \in \{(1236, 1247), (1245, 1346), (1245, 2346)\}$, and there is no feasible choice for $f(s)$.

Suppose Γ contains F_7 or F_9 as a subgraph. We let $f(p) = 1234$, $f(q) = 1235$, and $f(t) = 1246$. Then $f(r) \in \{1356, 2356\} \cup \{125\alpha : \alpha \geq 7\}$ and $f(s) \in \{1456, 2456\} \cup \{126\alpha : \alpha \geq 7\}$, so we assume that $(f(r), f(s)) \in \{(1257, 1267), (1356, 1456), (2356, 2456)\}$. In each case, there is no feasible choice for $f(u)$. □

Corollary 5.7. *Let Γ be a connected regular graph cospectral with a line graph $L(G)$ of a connected graph G. If $\operatorname{bsr}(\Gamma) = \operatorname{bsr}(L(G))$, then Γ is a line graph.*

Proof. If G is not an exceptional graph from Theorem 5.4, then Γ is a line graph by that theorem. If G is one of the exceptional graphs, then $\operatorname{bsr}(L(G)) < 10$ and $L(G)$ has more edges than vertices. Therefore, the least eigenvalue of $L(G)$ is -2. Since the Petersen graph and graphs $\mathrm{CP}(n)$ are not exceptional, Theorem 5.6 implies that Γ is a line graph. □

Example 5.8. Let X be the set of all points of H_5 with even sum of coordinates. It is a 2-distance set and Γ_X is the complement of the *Clebsch graph*. The least eigenvalue of Γ_X is -2, and, since it is not a line graph, Theorem 5.6 implies that $\operatorname{bsr}(\Gamma_X) \geq 10$ (so X is not spherical). Let Y be the set of all points $(y_1, y_2, \ldots, y_{10}) \in H_{10}$ such that $\sum_{i=1}^{5} y_i$ is even and $y_i + y_{i+5} = 1$ for $i = 1, 2, 3, 4, 5$. Then Y is a spherical 2-distance set and Γ_Y is isomorphic to Γ_X. Thus, $\operatorname{bsr}(\Gamma_X) = 10$.

Example 5.9. The *Shrikhande graph* is cospectral with $L_2(4)$, and the three *Chang graphs* are cospectral with $T(8)$, the line graph of K_8, so we have examples of cospectral strongly regular graphs with distinct binary spherical representation numbers. It can be shown that the binary spherical representation number of the Shrikhande graph is 12.

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