Research Article

Combinatorial Methods for Detecting Surface Subgroups in Right-Angled Artin Groups

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We give a short proof of the following theorem of Sang-hyun Kim: if $A(\Gamma)$ is a right-angled Artin group with defining graph Γ , then $A(\Gamma)$ contains a hyperbolic surface subgroup if Γ contains an induced subgraph $\overline{C}_n$ for some $n \geq 5$, where $\overline{C}_n$ denotes the complement graph of an n -cycle. Furthermore, we give a new proof of Kim's cocontraction theorem.

1. Introduction and Definitions

Suppose that Γ is a simple finite graph with vertex set V_Γ and edge set E_Γ . We say that Γ is the defining graph of the right-angled Artin group defined by the presentation

$$A(\Gamma) = \langle V_\Gamma; [v, w] := v w v^{-1} w^{-1} = 1, \forall \{v, w\} \in E_\Gamma \rangle. \quad (1.1)$$

Right-angled Artin groups are also called graph groups or partially commutative groups in the literature. All graphs in this paper are assumed to be simple and finite.

Right-angled Artin groups have been studied using both combinatorial and geometric methods. In particular, it is well known that these groups have simple solutions to the word and conjugacy problems. Moreover, each right-angled Artin group can be geometrically represented as the fundamental group of a nonpositively curved cubical complex X_Γ called the Salvetti complex. For these and other fundamental results, we refer the reader to the survey article by Charney [1].

Let Γ be a graph, and suppose that $W \subset V_\Gamma$. The induced subgraph Γ_W is the maximal subgraph of Γ on the vertex set W . A subgraph $\Lambda \subset \Gamma$ is called an induced subgraph if $\Lambda = \Gamma_{V_\Lambda}$. In this case, the subgroup of $A(\Gamma)$ generated by V_Λ is canonically isomorphic to $A(\Lambda)$. This

follows from the fact that $f : A(\Gamma) \rightarrow A(\Lambda)$ given by $f(v) = v$ if $v \in V_\Lambda$ and $f(v) = 1$ if $v \notin V_\Lambda$ defines a retraction. Therefore, we identify $A(\Lambda)$ with its image in $A(\Gamma)$.

In this paper, we study the following problem: find conditions on a graph Γ which imply or deny the existence of hyperbolic surface subgroup in $A(\Gamma)$. Herein, we say that a group is a hyperbolic surface group if it is the fundamental group of a closed orientable surface with negative Euler characteristic.

Servatius et al. proved that if Γ contains an induced n -cycle, that is, the underlying graph of a regular n -gon, for some $n \geq 5$, then $A(\Gamma)$ has a hyperbolic surface subgroup [2]. In fact, they construct an isometrically embedded closed surface of genus $1 + (n - 4)2^{n-3}$ in the cover of the Salvetti complex X_Γ corresponding to the commutator subgroup of $A(\Gamma)$.

Kim and, independently, Crisp et al. gave the first examples of graphs without induced n -cycles, $n \geq 5$, which define a right-angled Artin groups which, nonetheless, contain hyperbolic surface subgroups [3, 4]. We give one such example here to illustrate the main lemma of this paper.

Consider the graphs in Figure 1. The map $\phi : A(P') \rightarrow A(P)$ sending v_1 to v_1^2 , v_i to v_i , and v'_i to v_i for $i > 1$ defines an injective homomorphism onto an index two subgroup of $A(P)$ (see the discussion below). Since P' contains an induced circuit of length five (the induced subgraph on the vertices $v_2, \dots, v_5$, and v'_6), $A(P)$ contains a hyperbolic surface subgroup; however, P does not contain an n -cycle for any $n \geq 5$.

That the map ϕ is injective can be seen from several perspectives. The approach of Kim and Crisp et al. is to use dissection diagrams; these are collections of simple closed curves which are dual to van Kampen diagrams on a surface over the presentation $A(\Gamma)$. The method was introduced in this context by Crisp and Wiest [5] and used with much success by Kim [3] and Crisp et al. [4].

The purpose of this paper is to demonstrate that classical methods from combinatorial group theory offer another perspective and simplify some of the arguments in the aforementioned articles. We will use the Reidemeister-Schreier rewriting process to give a direct proof that the map ϕ , above, is injective, and we also indicate how this can be proven using normal forms for splittings of groups. This in turn will lead to a short proof of Kim's theorem on cocontractions (in [3, Theorem 4.2]) alluded to in the abstract; see Theorem 3.5 in this paper.

Lemma 1.1. *Suppose that $A(\Gamma)$ is a right-angled Artin group, and let n be a positive integer. Choose a vertex $z \in V(\Gamma)$, and define $\phi : A(\Gamma) \rightarrow \langle x; x^n = 1 \rangle \cong \mathbb{Z}/n\mathbb{Z}$ by $\phi(v) = 1$ if $v \neq z$ and $\phi(z) = x$. Then $\ker \phi$ is a right-angled Artin group with defining graph Γ' obtained by gluing n copies of $\Gamma \setminus st(z)$ to $st(z)$ along $lk(z)$, where st and lk are the star and link, respectively. Moreover, the vertices of Γ' naturally correspond to the following generating set:*

$$\{z^n\} \cup lk(z) \cup \{u : u \notin st(z)\} \cup \{zuz^{-1} : u \notin st(z)\} \cup \dots \cup \{z^{n-1}uz^{1-n} : u \notin st(z)\}. \quad (1.2)$$

The proof is a straightforward computation using the Reidemeister-Schreier method. The details are given in Section 2. Applying Lemma 1.1 to the graphs in Figure 1 proves that $A(P')$ injects into $A(P)$: if $\phi : A(P) \rightarrow \mathbb{Z}/2\mathbb{Z}$ maps $z = v_1$ to 1 mod 2, then $A(P') = \ker \phi$.

Another way to prove Lemma 1.1 is to take advantage of “visual” splittings of the groups $A(\Gamma)$ and $A(\Gamma')$ as an HNN extension or amalgamated free product. This second approach is stated in the article by Crisp et al. (see [4, Remark 4.1]).

We illustrate the utility of Lemma 1.1 by giving a short proof of the following theorem of Kim.

Figure 1: The group given by the graph P on (b) injects into the group given by the graph P' on (a). The vertex labeled by i is referred to as v_i in the discussion below.

Theorem 1.2 (see Kim [3, Corollary 4.3(2)]). *Let $\overline{C}_n$ denote the complement graph of an n -cycle. For each $n \geq 5$, the group $A(\overline{C}_n)$ contains a hyperbolic surface subgroup.*

In fact, we give a new proof of Kim's more general theorem (in [3, Theorem 4.2]) on cocontractions of right-angled Artin groups in Section 3. Kim's proof used the method of dissection diagrams. Kim has also discovered a short proof using visual splittings (personal communication).

In preparing this paper, we found that the Reidemeister-Schreier method has been used previously to study certain Bestvina-Brady subgroups of right-angled Artin groups (see [6, 7]).

This work was inspired by a desire to better understand Crisp et al.'s very interesting classification of the graphs on fewer than nine vertices which define right-angled Artin groups with hyperbolic surface subgroups. We hope that this paper will help to clarify some aspects of the general problem.

2. The Reidemeister-Schreier Method and Proof of Lemma 1.1

The Reidemeister-Schreier method solves the following problem: suppose that G is a group given by the presentation $\langle X; R \rangle$, and suppose that $H \subset G$ is a subgroup; find a presentation for H . The treatment below is brisk; see [8] for details and complete proofs.

Let $F = F(X)$ be free with basis X , and let $\pi : F \rightarrow G$ extend the identity map on X . Consider the preimage $P = \pi^{-1}(H)$. Let $T \subset F$ be a right Schreier transversal for P in F , that is, T is a complete set of right coset representatives that is closed under the operation of taking initial subwords (of freely reduced words over X). Given $w \in F$, let $[w]$ be the unique element of T such that $Pw = P[w]$. For each $t \in T$ and $x \in X$, let $s(t, x) = tx[tx]^{-1}$. Define $S = \{s(t, x) : t \in T, x \in X, \text{ and } s(t, x) \neq 1\}$. Then S is a basis for the free group P . Define a rewriting process $\tau : F \rightarrow P$ on freely reduced words over X by

$$\tau(y_1 y_2 \cdots y_n) = s(1, y_1) s([y_1], y_2) \cdots s([y_1 \cdots y_{n-1}], y_n), \quad (2.1)$$

where $y \in X \cup X^{-1}$. Then $\tau(w) = w[w]^{-1}$ for every reduced word $w \in F$, and

$$H = \langle S; \tau(t^{-1}rt) = 1, \forall t \in T, r \in R \rangle. \quad (2.2)$$

This rewriting process together with the resulting presentation for the given subgroup H of $G = \langle X; R \rangle$ is called the Reidemeister-Schreier Method.

Proof of Lemma 1.1. Let Γ be a graph, $G = A(\Gamma)$ the corresponding right-angled Artin group, and z a distinguished vertex of Γ . Let $\phi : G \rightarrow \langle x; x^n \rangle$ be given by $\phi(v) = 1$ if $v \neq z$ and $\phi(z) = x$.

Let F be free on $X = V_\Gamma$, and let R be the set of defining relators corresponding to E_Γ . Let $H = \ker \phi$, and let P be the inverse image of H in F under the natural map $F \rightarrow G$. The set $T = \{1, z, \dots, z^{n-1}\}$ is a right Schreier transversal for $P < F$. One verifies (directly) that the following equations hold:

$$\begin{aligned} s(z^k, v) &= z^k v z^{-k}, \quad \text{if } v \neq z, \quad k = 0, \dots, n-1, \\ s(z^k, z) &= 1, \quad \text{if } k = 0, \dots, n-2, \\ s(z^k, z) &= z^n, \quad \text{if } k = n-1. \end{aligned} \tag{2.3}$$

Thus, we have a set S of generators for $\ker \phi$; however, many of these generators are redundant. Again, one verifies (using $\tau(w) = w[w]^{-1}$) that the following equations hold:

$$\begin{aligned} \tau(z^k[u, v]z^{-k}) &= [s(z^k, u), s(z^k, v)], \quad \text{if } u, v \neq z, \quad k = 0, \dots, n-1, \\ \tau(z^k[z, v]z^{-k}) &= s(z^{k+1}, v) \cdot s(z^k, v)^{-1}, \quad \text{if } v \neq z, \quad k = 0, \dots, n-2, \\ \tau(z^k[z, v]z^{-k}) &= z^n \cdot s(1, v) \cdot z^{-n} \cdot s(z^{n-1}, v)^{-1}, \quad \text{if } v \neq z, \quad k = n-1. \end{aligned} \tag{2.4}$$

Therefore, if $[z, v] = 1$ is a relation in $A(\Gamma)$, then

$$v = s(1, v) = s(z, v) = \dots = s(z^{n-1}, v), \quad [z^n, v] = 1 \tag{2.5}$$

hold in $\ker \phi$. It follows that $\ker \phi$ is generated by z^n , the vertices adjacent to z in Γ , and n copies $(u, zuz^{-1}, \dots, z^{n-1}uz^{1-n})$ of each vertex $u \neq z$ and not adjacent to z in Γ . Moreover, the relations are such that $\ker \phi$ is presented as a right-angled Artin group where the defining graph is obtained from Γ by taking the star of z and n copies of the complement of the star of z and gluing these copies along the link of z . This completes the proof of Lemma 1.1. $\square$

3. A Short Proof of Two Theorems of Kim

Suppose that Γ is a graph. The complement graph $\bar{\Gamma}$ is the graph having the same vertices as Γ but which has edges complementary to the edges of Γ . Recall that an n -cycle C_n is the underlying graph of a regular n -gon.

Theorem 1.2 follows from Kim's cocontraction theorem (see Theorem 3.5 below); however, we present a short independent proof here.

Proof of Theorem 1.2. Suppose that Γ is a graph which contains an induced C_5 . Then $A(\Gamma)$ contains a hyperbolic surface subgroup by [2]. Since $C_5 \cong \bar{C}_5$, Theorem 1.2 follows from Lemma 3.1 below. $\square$

Lemma 3.1 (see Kim [3, Corollary 4.3(1)]). *For each $n \geq 4$, $A(\bar{C}_{n-1}) < A(\bar{C}_n)$.*

Proof. Let $V_{C_n} = \{x_1, \dots, x_n\} = V_{\overline{C_n}}$. Define $\phi : A(\overline{C_n}) \rightarrow \langle a, a^2 \rangle$ by $\phi(x_n) = a$ and $\phi(x_i) = 1$ for $i \neq n$. By Lemma 1.1, the defining graph Γ of $\ker \phi$ has vertex set $V_\Gamma = \{z^2\} \cup \{x_1, \dots, x_{n-1}\} \cup \{y_1, y_{n-1}\}$, where $z = x_n$ and $y_i = zx_i z^{-1}$. Let $S = \{y_1, x_2, \dots, x_{n-1}\}$. Consider the induced subgraph Γ_S . The vertices y_1 and x_i are not adjacent if and only if $i \in \{2, n-1\}$. The vertices x_i and x_j are not adjacent if and only if $|i - j| \leq 1$. Therefore, $\Gamma_S \cong \overline{C_{n-1}}$. $\square$

Kim proved a more general theorem about subgroups of a right-angled Artin group $A(\Gamma)$ defined by “cocontractions.” Let $S \subset V_\Gamma$, and let $S' = V_\Gamma \setminus S$. If Γ_S is connected, then the contraction $\text{CO}(\Gamma, S)$ of Γ relative to S is defined by taking the induced subgraph $\Gamma_{S'}$ together with a vertex v_S and declaring v_S to be adjacent to $w \in S'$ if w is adjacent in Γ to some vertex in S . The cocontraction $\overline{\text{CO}}(\Gamma, S)$ is defined as follows:

$$\overline{\text{CO}}(\Gamma, S) = \overline{\text{CO}(\overline{\Gamma}, S)}. \quad (3.1)$$

Kim insists that Γ_S be connected whenever he considers the contraction $\text{CO}(\Gamma, S)$. This assumption is not necessary. Moreover, the following lemma shows that the structure of Γ_S is immaterial; the proof follows directly from the definitions.

Lemma 3.2. *Suppose that Γ is a graph and $S \subset V_\Gamma$. Let Γ' be the graph obtained from Γ by removing any edges joining two elements of S . Then $\text{CO}(\Gamma, S) = \text{CO}(\Gamma', S)$.*

Corollary 3.3. *Suppose that Γ is a graph and $S \subset V_\Gamma$. Let Γ' be a graph obtained from Γ by adding or deleting any collection of edges with both of their vertices belonging to S . Then $\text{CO}(\Gamma, S) = \text{CO}(\Gamma', S)$ and $\overline{\text{CO}}(\Gamma, S) = \overline{\text{CO}}(\Gamma', S)$.*

Lemma 3.4. *Suppose that Γ is a graph and $n \geq 2$. Suppose that $S = \{s_1, \dots, s_n\} \subset V_\Gamma$. Let $\Lambda = \overline{\text{CO}}(\Gamma, \{s_1, \dots, s_{n-1}\})$ and $S' = S \setminus \{s_n\}$. Then $\overline{\text{CO}}(\Gamma, S) = \overline{\text{CO}}(\Lambda, \{v_{S'}, s_n\})$.*

Proof. It suffices to compare the collection of vertices which are adjacent to v_S in $\Gamma_1 = \overline{\text{CO}}(\Gamma, S)$ and $\Gamma_2 = \overline{\text{CO}}(\Lambda, \{v_{S'}, s_n\})$; in the latter case, we are identifying the vertex $v_{S' \cup s_n}$ with v_S .

A vertex w in Γ_1 not belonging to S is adjacent to v_S if and only if w is adjacent to every s_i in Γ . A vertex w in Γ_2 not equal to $v_{S'}$ nor s_n is adjacent to v_S if and only if w is adjacent to $v_{S'}$ and s_n in Λ ; but this, in turn, means that w is adjacent to every s_i in Γ . (Note that the case of $n = 2$ is trivial since $\text{CO}(\Gamma, \{s_1\}) = \Gamma$.) $\square$

A collection of vertices $S \subset V_\Gamma$ is said to be anticonnected if $\overline{\Gamma}_S$ is connected. (Note that $(\overline{\Gamma})_S = \overline{(\Gamma_S)}$.)

Theorem 3.5 (see Kim [3, Theorem 4.2]). *Suppose that Γ is a graph and $S \subset V(\Gamma)$ is an anticonnected subset. Then $A(\overline{\text{CO}}(\Gamma, S))$ embeds in $A(\Gamma)$.*

Proof. First consider the case when S consists of two nonadjacent vertices $z, z' \in V_\Gamma$. Define $\phi : A(\Gamma) \rightarrow \langle x; x^2 \rangle$ by $\phi(z) = x$, and $\phi(v) = 1$ if $v \neq z$. Let $A(\Gamma') = \ker \phi$. Let

$$T = \left(V(\Gamma) \setminus \{z^2, z'\} \right) \cup \{zz'z^{-1}\} \subset V(\Gamma'). \quad (3.2)$$

We claim that $\Gamma'_T \cong \overline{\text{CO}}(\Gamma, S)$ via $v \mapsto v$ if $v \neq zz'z^{-1}$ and $zz'z^{-1} \mapsto v_S$.

If v and w are distinct from z and z' , then v and w are adjacent in $\overline{\text{CO}}(\Gamma, S)$ if and only if they are adjacent in Γ .

On the other hand, a vertex w is adjacent to v_S in $\overline{\text{CO}}(\Gamma, S)$ if and only if w is adjacent to z and z' , whereas a vertex w is adjacent to $zz'z^{-1}$ in Γ'_T if and only if w belongs to the link of z and to the link of z' , that is, w is adjacent to z and z' . Therefore, $\Gamma'_T \cong \overline{\text{CO}}(\Gamma, S)$ and, hence, $A(\overline{\text{CO}}(\Gamma, S))$ embeds in $A(\Gamma)$.

Now we prove the general statement by induction on $|S|$. Suppose that $S = \{s_1, \dots, s_n\}$ is anticonnected, and suppose that we have chosen the ordering so that $S' = \{s_1, \dots, s_{n-1}\}$ is also anticonnected. (This is always possible: choose s_n so that it is not a cut point of $\overline{\Gamma}_S$.) Let $\Lambda = \overline{\text{CO}}(\Gamma, S')$. Suppose that $A(\Lambda)$ embeds in $A(\Gamma)$. By the case of the two vertices above, $A(\overline{\text{CO}}(\Lambda, \{v_{S'}, s_n\}))$ embeds in $A(\Lambda)$. (Note that $v_{S'}$ and s_n are not adjacent in Λ for, otherwise, s_n would be adjacent to every s_i , $i = 1, \dots, n-1$, which would contradict the hypothesis that S is anticonnected.) This proves the inductive step. The proof of the theorem is completed by applying Lemma 3.4. $\square$

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