Research Article

The Elman-Lam-Krüskemper Theorem

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For a (formally) real field K , the vanishing of a certain power of the fundamental ideal in the Witt ring of $K(\sqrt{-1})$ implies that the same power of the fundamental ideal in the Witt ring of K is torsion free. The proof of this statement involves a fact on the structure of the torsion part of powers of the fundamental ideal in the Witt ring of K . This fact is very difficult to prove in general, but has an elementary proof under an assumption on the stability index of K . We present an exposition of these results.

1. Introduction

In the study of sums of squares and quadratic forms over fields, it has often been fruitful to consider the quadratic extension obtained by adjoining $\sqrt{-1}$. Consider, for example, an extension K/R of transcendence degree n of a real closed field R . Tsen-Lang theory yields that quadratic forms over $K(\sqrt{-1})$ of dimension greater than 2^n are isotropic. From this, one can derive that in K any totally positive element is a sum of 2^n squares and that torsion quadratic forms over K of dimension greater than 2^{n+2} are isotropic. The proof of the statement on sums of squares in K uses the observation that a certain field property (A_{n+1}) goes down a quadratic extension.

We assume that the reader is familiar with the classical theory of quadratic forms over fields as presented in [1]. We use standard notation and terminology. When we speak about quadratic forms we always assume them to be regular, and we often just call them *forms*.

Let K be a field of characteristic different from 2. We denote by $K^\times$ the multiplicative group of K and by $\sum K^2$ the subgroup of the nonzero sums of squares in K . By the Artin-Schreier Theorem, K admits a field ordering if and only if $-1 \notin \sum K^2$; in this case, we say that K is *real*, otherwise *nonreal*.

Let WK denote the Witt ring of K and IK its fundamental ideal. Let $W_t K$ be the torsion part of WK . If K is real, then $W_t K \subseteq IK$, otherwise $W_t K = WK$.

Let n be a positive integer. We put $I^n K = (IK)^n$ and $I_t^n K = W_t K \cap I^n K$. Following [2] and [1, Chapter XI, Section 4], we denote by (A_n) the condition that every torsion n -fold Pfister form over a given field is hyperbolic. Recall that $I^n K$ is generated (as a group) by the n -fold Pfister forms over K . Hence, if K is a nonreal field, then K satisfies (A_n) if and only if $I^n K = 0$. It was conjectured in [2] that, without assuming that K is nonreal, the following holds.

Conjecture 1.1. *The field K satisfies (A_n) if and only if $I_t^n K = 0$.*

This is obviously implied by the following stronger conjecture, which was proved in [3, (2.8)] as a consequence of Voevodsky's proof of the Milnor Conjecture and the deep results in [4].

Conjecture 1.2. *One has $I_t^n K = I_t K \cdot I^{n-1} K$.*

Nevertheless, one may wish to have more elementary proofs of these conjectures, even for special cases. For $n \leq 3$, such a proof for Conjecture 1.1 can be found in [1, Chapter XI, (4.1) and Chapter XII, (3.1)]. For an elementary proof of Conjecture 1.2 for $n = 2$, see [1, Chapter XI, (4.2)] or Proposition 3.2 below. For arbitrary n , Kruskemper showed in [5, Theorem 3] by elementary arguments that a sum of up to three scaled n -fold Pfister forms lies in $I_t^n K$ if and only if it lies in $I_t K \cdot I^{n-1} K$; in Theorem 3.8, we give a self-contained exposition of this result.

There is another situation where an elementary proof of Conjecture 1.2 is possible, depending on the stability index of the field K . Following [6], the (reduced) stability index of K is defined as

$$\text{st}(K) = \min \left\{ r \in \mathbb{N} \mid I^{r+1} K = 2I^r K + I_t^{r+1} K \right\} \in \mathbb{N} \cup \{\infty\}. \quad (1.1)$$

Note that $\text{st}(K) = 0$ if and only if K is either nonreal or uniquely ordered.

In [7], Kruskemper outlined an elementary proof of Conjecture 1.2 for the case where $n \geq \text{st}(K)$, using several facts stated in [8] without a detailed proof.

Our efforts to understand the mentioned results from [5, 7] led us to a more detailed and structured exposition, which we want to make available to a wider audience, though we do not claim any originality of the ideas. Shortly after completing this note, we learned that another exposition of Kruskemper's proof that Conjectures 1.2 and 1.1 hold for $n \geq \text{st}(K)$ appeared in [9, Section 35.B]. Here, following [7], we work within the more general framework of preorderings and the reduced theory of quadratic forms.

Let us point out that having Conjecture 1.2 for $n > \text{st}(K)$ leads to a proof of the following statement, as observed in [5, Proposition 1].

Theorem 1.3 (Elman-Lam-Kruskemper). *For $n \geq 1$, the following statements are equivalent:*

- (i) $I^n K(\sqrt{-1}) = 0$.
- (ii) $I_t^n K = 0$ and $I^n K = 2I^{n-1} K$.
- (iii) $I_t^n K = 0$ and $\text{st}(K) < n$.

Assuming that $I_t^n K = 0$, it is clear that $I^n K = 2I^{n-1} K$ holds if and only if $\text{st}(K) < n$. Therefore, (ii) and (iii) are equivalent. Elman and Lam showed in [2, (4.4), (4.5), (4.7)] that

$I^n K(\sqrt{-1}) = 0$ if and only if K satisfies (A_n) and $I^n K = 2I^{n-1}K$ (see also [1, Chapter XI, (4.7), (4.14), (4.18)]); this fact was rediscovered by Elman and Prestel in [10, (3.3)]. Hence, to complete the proof of Theorem 1.3, it is sufficient to know that Conjecture 1.1 holds when $n > \text{st}(K)$.

Finally, note that the equivalent conditions in Theorem 1.3 hold in the case where K is an extension of a real closed field of transcendence degree strictly smaller than n . In fact, since any quadratic form over $K(\sqrt{-1})$ of dimension exceeding 2^{n-1} is isotropic, it is clear that $I^n K(\sqrt{-1}) = 0$, and [2, (6.1)] gives a more direct way to show that $I_t^n K = 0$ in this case.

2. Preorderings

We recall some facts from real algebra and the reduced theory of quadratic forms modulo a preordering. For details, we refer to [11]. Let $X(K)$ denote the set of all field orderings of K . For a quadratic form φ over K and $P \in X(K)$, we denote by $\text{sign}_P(\varphi)$ the signature of φ at P .

A *preordering* of K is a proper subset $T \subsetneq K$ that contains the squares in K and that is closed under addition and multiplication in K . Note that these conditions entail that $\{0\} \cup \sum K^2 \subseteq T$ and $-1 \notin T$, in particular K is real. Conversely, if K is real, then K has preorderings and the smallest one is $\sum K^2 \cup \{0\}$.

Let T be a fixed preordering of K . We denote $T^\times = T \cap K^\times = T \setminus \{0\}$. We denote by $X_T(K)$ the set of field orderings of K that contain T . By [1, Chapter VIII, (9.6)], we have $T = \bigcap_{P \in X_T(K)} P$. The set $X_T(K)$ is considered as a topological space with the *Harrison topology*, which is generated by the sets $H(a) = \{P \in X_T(K) \mid a \in P\}$ where $a \in K^\times$. For $r \in \mathbb{N}$ and $a_1, \dots, a_r \in K^\times$, we put

$$H(a_1, \dots, a_r) = H(a_1) \cap \dots \cap H(a_r) = \{P \in X(K) \mid a_1, \dots, a_r \in P\}; \quad (2.1)$$

the sets of this shape are called *Harrison sets*, and they form a base of the Harrison topology on $X_T(K)$. For a nontrivial Harrison set H of $X_T(K)$, the smallest $r \in \mathbb{N}$ such that $H = H(a_1, \dots, a_r)$ with $a_1, \dots, a_r \in K^\times$ is called the *degree of H* ; the trivial Harrison sets $\emptyset$ and $X_T(K)$ are given the degree 0.

We write

$$\text{sign}_T : WK \longrightarrow \mathbb{Z}^{X_T(K)}, \quad [\varphi] \longmapsto (\text{sign}_P(\varphi))_{P \in X_T(K)} \quad (2.2)$$

and denote the kernel of this homomorphism by $I_T K$. (Note that in [11] this notation is used for what is $IK/I_T K$ in our notation: the fundamental ideal in the reduced Witt ring $W_T K = WK/I_T K$.) For $n \in \mathbb{N}$, let $I_T^n K = I^n K \cap I_T K$.

Let φ be a quadratic form over K . We say that φ is *T-positive* if $D_K(\varphi) \subseteq T$. We say that φ is *T-isotropic* if $\vartheta \otimes \varphi$ is isotropic for some *T-positive* form ϑ over K ; by [11, (1.20)], this is equivalent to having $\text{sign}_T(\varphi) = \text{sign}_T(\psi)$ for some form ψ over K with $\dim(\psi) < \dim(\varphi)$. We write $D_T(\varphi)$ for the set of elements $a \in K^\times$ such that $a \in D_K(\vartheta \otimes \varphi)$ for some *T-positive* form ϑ over K . By [11, (1.16)], $\text{sign}_T(\varphi) = 0$ if and only if there is a *T-positive* Pfister form ϑ over K such that $\vartheta \otimes \varphi$ is hyperbolic. Moreover, the ideal $I_T K$ is generated by the binary forms $\langle 1, -t \rangle$ with $t \in T^\times$, by [11, (1.26)]. A form ψ over K with $\dim(\psi) = \dim(\varphi)$ and $\text{sign}_T(\psi) = \text{sign}_T(\varphi)$ is said to be *T-isometric to φ* ; by [11, (1.19)] this implies that $D_T(\psi) = D_T(\varphi)$. These facts will be used in the next section.

The *stability index* of T is defined as

$$\text{st}(T) = \sup\{\deg(H) \mid H \text{ is a Harrison set of } X_T(K)\} \in \mathbb{N} \cup \{\infty\}. \quad (2.3)$$

This is reconcilable with the above definition of the stability index of a field. In fact, if K is real then for the preordering $S = \sum K^2 \cup \{0\}$, we have $\text{st}(K) = \text{st}(S)$ by [11, (13.1)], and furthermore $I_S K = W_t K$ and $I_S^n K = I_t^n K$ for $n \geq 1$.

3. Kruskemper's Theorem

Let T be a preordering on K and $n \in \mathbb{N}$. A more general version of Conjecture 1.2 is the following conjecture.

Conjecture 3.1. *One has $I_T^n K = I_T K \cdot I^{n-1} K$.*

This conjecture is proved in [3, (2.7)], but the proof of the nontrivial inclusion $I_T^n K \subseteq I_T K \cdot I^{n-1} K$ uses the deep results in [4]. In [7], Kruskemper sketches an elementary proof of Conjecture 3.1 for $n \geq \text{st}(T)$. We want to provide a complete exposition of Kruskemper's result, which we achieve with Theorem 3.10. On the way, we shall obtain with Theorem 3.8 that, for any n , a sum of up to three scaled n -fold Pfister forms over K belongs to $I_T^n K$ if and only if it belongs to $I_T K \cdot I^{n-1} K$.

We begin with the elementary proof of Conjecture 3.1 for $n = 2$, adapted from [1, Chapter XI, (4.1)].

Proposition 3.2. *One has $I_T^2 K = I_T K \cdot IK$.*

Proof. For $a \in K^\times$ and $t \in T^\times$, one has $\langle a, -at \rangle \equiv \langle 1, -t \rangle \pmod{I_T K \cdot IK}$. For $t, t' \in T^\times$, one has $\langle 1, -t \rangle \perp \langle 1, -t' \rangle \equiv \langle 1, -tt' \rangle \pmod{I_T K \cdot IK}$. As the ideal $I_T K$ is generated by the forms $\langle 1, -t \rangle$ with $t \in T^\times$, any element of $I_T^2 K$ is given by a form $\varphi = \langle a_1, -a_1 t_1 \rangle \perp \cdots \perp \langle a_r, -a_r t_r \rangle$, where $r \in \mathbb{N}$, $a_1, \dots, a_r \in K^\times$, and $t_1, \dots, t_r \in T^\times$ with $t_1 \cdots t_r \in K^{\times 2}$. Using induction on r and the above congruences shows that $\varphi \in I_T K \cdot IK$. $\square$

Lemma 3.3. *Let ψ and ρ be Pfister forms over K , and let ρ' be the form over K such that $\rho = \langle 1 \rangle \perp \rho'$. Then, for $b \in D_T(\psi \otimes \rho')$, there exists a Pfister form μ over K with $\text{sign}_T(\psi \otimes \rho) = \text{sign}_T(\psi \otimes \langle 1, b \rangle \otimes \mu)$.*

Proof. There exists a T -positive Pfister form τ such that $b \in D_K(\tau \otimes \psi \otimes \rho')$. By [1, Chapter X, (1.10)], we have $\tau \otimes \psi \otimes \rho = \tau \otimes \psi \otimes \langle 1, b \rangle \otimes \mu$ for a Pfister form μ , and then $\text{sign}_T(\psi \otimes \rho) = \text{sign}_T(\psi \otimes \langle 1, b \rangle \otimes \mu)$ holds. $\square$

The next two statements are contained without proof in [8, Section 5].

Lemma 3.4 (Marshall). *Let π be an n -fold Pfister form over K where $n \geq 1$. If $a \in D_T(\pi')$, then there is an $(n-1)$ -fold Pfister form ρ over K such that $\pi \equiv \langle 1, a \rangle \otimes \rho \pmod{I_T K \cdot I^{n-1} K}$. If $b \in D_T(\pi)$, then $b\pi \equiv \pi \pmod{I_T K \cdot I^n K}$.*

Proof. We use induction on n . For $n = 1$, the first part is trivial. In the induction step, we will use both parts of the statement for $n - 1$ to prove the first part for n . Before we do so, we

explain how, for given π , the second part of the statement follows from the first part. This together will give the proof.

Let $b \in D_T(\pi)$. Then, $b \in D_T(\langle 1, a \rangle)$ for some $a \in D_T(\pi')$. By Proposition 3.2, we have $\langle 1, -b \rangle \otimes \langle 1, a \rangle \in I_T K \cdot IK$. If now $\pi \equiv \langle 1, a \rangle \otimes \rho \bmod I_T K \cdot I^{n-1} K$ for an $(n-1)$ -fold Pfister form ρ , then $\langle 1, -b \rangle \otimes \pi \equiv \langle 1, -b \rangle \otimes \langle 1, a \rangle \otimes \rho \equiv 0 \bmod I_T K \cdot I^n K$ and thus $b\pi \equiv \pi \bmod I_T K \cdot I^n K$.

For the induction step, assume now that $n > 1$ and write $\pi = \langle 1, c \rangle \otimes \sigma$ with $c \in K^\times$ and an $(n-1)$ -fold Pfister form σ . Let $a \in D_T(\pi')$. Since $\pi' = \sigma' \perp c\sigma$, there exist $u \in D_T(\sigma')$ and $v \in D_T(\sigma)$ such that $a \in D_K(\langle u, cv \rangle)$ and thus $\langle u, cv \rangle = \langle a, acuv \rangle$. By the induction hypothesis, we then have $\sigma \equiv \langle 1, u \rangle \otimes \tau \bmod I_T K \cdot I^{n-2} K$ for some $(n-2)$ -fold Pfister form τ , and $\sigma \equiv v\sigma \bmod I_T K \cdot I^{n-1} K$. We obtain

$$\pi \equiv \langle 1, cv \rangle \otimes \sigma \equiv \langle 1, u, cv, cuv \rangle \otimes \tau \equiv \langle 1, a, acuv, cuv \rangle \otimes \tau \bmod I_T K \cdot I^{n-1} K \quad (3.1)$$

and then choose $\rho = \langle 1, cuv \rangle \otimes \tau$. □

Lemma 3.5 (Marshall). *Assume that $n \geq 1$, and let π_1 and π_2 be two n -fold Pfister forms over K . If $\text{sign}_T(\pi_1) = \text{sign}_T(\pi_2)$, then $\pi_1 \equiv \pi_2 \bmod I_T K \cdot I^{n-1} K$.*

Proof. The proof is by induction on n . The claim is trivial for $n = 1$. Write $\pi_1 = \langle 1, a \rangle \otimes \sigma_1$ with $a \in K^\times$ and an $(n-1)$ -fold Pfister form σ_1 . As π'_1 and π'_2 are T -isometric, we have $a \in D_T(\pi'_1) = D_T(\pi'_2)$, and thus Lemma 3.4 yields that $\pi_2 \equiv \langle 1, a \rangle \otimes \sigma_2 \bmod I_T K \cdot I^{n-1} K$ for some $(n-1)$ -fold Pfister form σ_2 . If $a \in -T$, then $\pi_i \equiv \langle 1, a \rangle \otimes \sigma_i \equiv 0 \bmod I_T K \cdot I^{n-1} K$ for $i = 1, 2$. Assume now that $a \notin -T$. Then $T' = T + aT$ is a preordering. For any $r \in T'^\times$, there are $s, t \in T^\times$ with $r \in D_K(\langle s, at \rangle)$, and then

$$\langle 1, a \rangle \otimes \langle 1, -r \rangle \equiv \langle 1, at, -r, -art \rangle \equiv \langle 1, -s, arst, -art \rangle \equiv 0 \bmod I_T K \cdot IK. \quad (3.2)$$

This shows that $\langle 1, a \rangle \cdot I_T K \subseteq I_T K \cdot IK$. Since $\text{sign}_{T'}(\sigma_1) = \text{sign}_{T'}(\sigma_2)$, we have $\sigma_1 \equiv \sigma_2 \bmod I_T K \cdot I^{n-2} K$ by the induction hypothesis. Multiplying with $\langle 1, a \rangle$, we now obtain that $\pi_1 \equiv \pi_2 \bmod I_T K \cdot I^{n-1} K$. □

Corollary 3.6. *Let π be an n -fold Pfister form over K , and ψ a form over K of even dimension. Then, $\psi \otimes \pi \in I_T^{n+1} K$ if and only if $\psi \otimes \pi \in I_T K \cdot I^n K$.*

Proof. One implication is clear as trivially $I_T K \cdot I^n K \subseteq I_T^{n+1} K$. Assume that $\psi \otimes \pi \in I_T^{n+1} K$. Then, there exists a T -positive Pfister form τ such that $\psi \otimes \tau \otimes \pi$ is hyperbolic. For $s \in D_K(\tau \otimes \pi)$, the $(n+1)$ -fold Pfister form $\langle 1, -s \rangle \otimes \pi$ lies in the annihilator of τ , thus $\text{sign}_T(\langle 1, -s \rangle \otimes \pi) = 0$, and by Lemma 3.5 then $\langle 1, -s \rangle \otimes \pi \in I_T K \cdot I^n K$. By [1, Chapter XI, (3.1)], the binary forms $\langle 1, -s \rangle$ with $s \in D_K(\tau \otimes \pi)$ generate the subideal of IK annihilated by $\tau \otimes \pi$, and since ψ belongs to this ideal, we conclude that $\psi \otimes \pi \in I_T K \cdot I^n K$. □

Lemma 3.7. *Assume that $n \geq 1$, and let π_1, π_2, π_3 be n -fold Pfister forms over K and $b \in K^\times$ such that $\text{sign}_T(\pi_1) = \text{sign}_T(\pi_2) + \text{sign}_T(b\pi_3)$. Then, there exists an $(n-1)$ -fold Pfister form ψ over K and $c_1, c_2, c_3 \in K^\times$ with $c_3 = c_1 c_2$ such that $\pi_i \equiv \langle 1, -c_i \rangle \otimes \psi \bmod I_T K \cdot I^{n-1} K$ for $i = 1, 2, 3$.*

Proof. Choose an m -fold Pfister form ψ and $(n-m)$ -fold Pfister forms ρ_1, ρ_2 such that $\text{sign}_T(\pi_i) = \text{sign}_T(\psi \otimes \rho_i)$ for $i = 1, 2$, where $0 \leq m \leq n-1$ and where m is as large as possible in this range.

We claim that $m = n - 1$. Suppose on the contrary that $m < n - 1$. Then, the form $\vartheta = \psi \otimes (\rho'_1 \perp -\rho'_2)$ is T -isotropic, given that $\text{sign}_T(\vartheta) = \text{sign}_T(b\pi_3)$ and $\dim(\vartheta) = 2^{n+1} - 2^{m+1} > 2^n = \dim(b\pi_3)$. So, there exists an element $b \in D_T(\psi \otimes \rho'_1) \cap D_T(\psi \otimes \rho'_2)$, and Lemma 3.3 now gives a contradiction to the maximality of m . Hence, ρ_1 and ρ_2 are 1-fold Pfister forms.

We choose $c_1, c_2 \in K^\times$ such that $\rho_i = \langle 1, -c_i \rangle$ for $i = 1, 2$. Then, we have $\text{sign}_T(\pi_i) = \text{sign}_T(\langle 1, -c_i \rangle \otimes \psi)$ for $i = 1, 2$. Let $c_3 = c_1 c_2$. It follows that

$$\text{sign}_T(\pi_3) = \text{sign}_T(b\pi_1 \perp -b\pi_2) = \text{sign}_T(bc_2 \langle 1, -c_3 \rangle \otimes \psi). \quad (3.3)$$

Since the signatures of the Pfister forms π_3 and $\langle 1, -c_3 \rangle \otimes \psi$ lie in $\{0, 2^n\}$, we thus have $\text{sign}_T(\pi_i) = \text{sign}_T(\langle 1, -c_i \rangle \otimes \psi)$ for $i = 1, 2, 3$. Now, Lemma 3.5 yields that $\pi_i \equiv \langle 1, -c_i \rangle \otimes \psi \pmod{I_T K \cdot I^{n-1} K}$ for $i = 1, 2, 3$. $\square$

The following result appears in [5, Theorem 3].

Theorem 3.8 (Krüskemper). *Let $n \geq 1$. Let $\varphi = a_1\pi_1 \perp a_2\pi_2 \perp a_3\pi_3$ for n -fold Pfister forms π_1, π_2, π_3 over K and $a_1, a_2, a_3 \in K^\times$. Then, $\text{sign}_T(\varphi) = 0$ if and only if $\varphi \in I_T K \cdot I^{n-1} K$.*

Proof. If $\varphi \in I_T K \cdot I^{n-1} K$, then obviously $\text{sign}_T(\varphi) = 0$. Assume now that $\text{sign}_T(\varphi) = 0$. Then, $\text{sign}_T(a_1\pi_1 \perp a_2\pi_2) = \text{sign}_T(-a_3\pi_3)$, and comparing dimensions we conclude that $a_1\pi_1 \perp a_2\pi_2$ is T -isotropic, so there is an element $a \in D_T(a_1\pi_1) \cap D_T(-a_2\pi_2)$. Using Lemma 3.4, we obtain that $(-1)^{i-1} a a_i \pi_i \equiv \pi_i \pmod{I_T K \cdot I^n K}$ for $i = 1, 2$. Hence, with $d = a_3$, we have

$$\varphi \equiv a\pi_1 \perp -a\pi_2 \perp d\pi_3 \pmod{I_T K \cdot I^{n-1} K}. \quad (3.4)$$

By Lemma 3.7 applied with $b = -ad$, there is an $(n-1)$ -fold Pfister form ψ over K and $c_1, c_2, c_3 \in K^\times$ with $c_3 = c_1 c_2$ such that $\pi_i \equiv \langle 1, -c_i \rangle \otimes \psi \pmod{I_T K \cdot I^{n-1} K}$ for $i = 1, 2, 3$. It follows that

$$\varphi \equiv d \langle 1, -c_1 c_2, -adc_1, adc_2 \rangle \otimes \psi \pmod{I_T K \cdot I^{n-1} K}. \quad (3.5)$$

As $\text{sign}_T(\varphi) = 0$, we obtain that $\text{sign}_T(\langle 1, -c_1 c_2, -adc_1, adc_2 \rangle \otimes \psi) = 0$. By Lemma 3.5, then $\langle 1, -c_1 c_2, -adc_1, adc_2 \rangle \otimes \psi \in I_T K \cdot I^n K$, so $\varphi \in I_T K \cdot I^{n-1} K$. $\square$

Given a Pfister form π over K , we put $H(\pi) = \{P \in X_T(K) \mid \pi \text{ is } P\text{-positive}\}$. Hence, for $r \geq 1$ and $a_1, \dots, a_r \in K^\times$, we have $H(a_1, \dots, a_r) = H(\pi)$ with $\pi = \langle 1, a_1 \rangle \otimes \dots \otimes \langle 1, a_r \rangle$.

Proposition 3.9. *Assume that $n \geq \text{st}(T)$. Every class of $I^n K / I_T K \cdot I^{n-1} K$ is given by a form $\varphi_1 \otimes \pi_1 \perp \dots \perp \varphi_r \otimes \pi_r$ where $r \in \mathbb{N}$, $\varphi_1, \dots, \varphi_r$ are forms over K , and $\pi_1, \dots, \pi_r$ are n -fold Pfister forms over K such that $H(\pi_1), \dots, H(\pi_r)$ are nonempty and pairwise disjoint.*

Proof. Consider $a \in K^\times$ and an n -fold Pfister form π_1 over K . As $\text{st}(T) \leq n$, there exist n -fold Pfister forms π_2 and π_3 over K such that

$$H(\pi_2) = H(\pi_1) \cap H(a), \quad H(\pi_3) = H(\pi_1) \cap H(-a). \quad (3.6)$$

It follows that $\text{sign}_T(\pi_1) = \text{sign}_T(\pi_2) + \text{sign}_T(\pi_3)$. Applying Theorem 3.8 with $a_1 = 1$ and $a_2 = a_3 = -1$ yields that

$$\pi_1 \equiv \pi_2 \perp \pi_3 \pmod{I_T K \cdot I^{n-1} K}. \quad (3.7)$$

Consider now a finite set $F \subseteq K^\times$. For $S \subseteq F$, let H_S denote the Harrison set consisting of the orderings $P \in X_T(K)$ with $S \subseteq P$ and $F \setminus S \subseteq -P$. Let $\mathcal{P} = \mathcal{P}(F)$, the power set of F . Then, $(H_S)_{S \in \mathcal{P}}$ is a finite partition of $X_T(K)$. (However, H_S can be empty for many $S \in \mathcal{P}$.) Induction on the cardinality of F , using the above argument in the induction step, yields that, given an arbitrary n -fold Pfister form ρ over K , there exists a family of n -fold Pfister forms $(\rho_S)_{S \in \mathcal{P}}$ such that $H(\rho_S) = H(\rho) \cap H_S$ for every $S \in \mathcal{P}$ and

$$\rho \equiv \sum_{S \in \mathcal{P}} \rho_S \pmod{I_T K \cdot I^{n-1} K}. \quad (3.8)$$

Moreover, if $\rho = \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$ for $a_1, \dots, a_n \in F$ and $S \in \mathcal{P}$, then $H(\rho_S) = H_S$ in case $a_1, \dots, a_n$ all belong to S and $H(\rho_S) = \emptyset$ otherwise.

An arbitrary element of $I^n K$ is given by a form

$$a_1 \rho_1 \perp \cdots \perp a_m \rho_m, \quad (3.9)$$

where $m \geq 1$, $a_1, \dots, a_m \in K^\times$ and where $\rho_1, \dots, \rho_m$ are n -fold Pfister forms over K . Choose a finite set $F \subseteq K^\times$ such that for $1 \leq i \leq m$ there exist $c_{i1}, \dots, c_{in} \in F$ such that $\rho_i = \langle 1, c_{i1} \rangle \otimes \cdots \otimes \langle 1, c_{in} \rangle$. Put $\mathcal{P} = \mathcal{P}(F)$. Using that $n \geq \text{st}(T)$, we choose a family of n -fold Pfister forms $(\pi_S)_{S \in \mathcal{P}}$ such that $H(\pi_S) = H_S$ for $S \in \mathcal{P}$.

By the above argument, we choose for $1 \leq i \leq m$ a family of n -fold Pfister forms $(\rho_{i,S})_{S \in \mathcal{P}}$ such that

$$\rho_i \equiv \sum_{S \in \mathcal{P}} \rho_{i,S} \pmod{I_T K \cdot I^{n-1} K} \quad (3.10)$$

and such that $H(\rho_{i,S})$ is either empty or equal to H_S for any $S \in \mathcal{P}$. Using Lemma 3.5, for $1 \leq i \leq m$ and $S \in \mathcal{P}$, it follows that $\rho_{i,S} \equiv 0 \pmod{I_T K \cdot I^{n-1} K}$ if $H(\rho_{i,S}) = \emptyset$ and $\rho_{i,S} \equiv \pi_S \pmod{I_T K \cdot I^{n-1} K}$ otherwise. Put $\mathcal{P}' = \{S \in \mathcal{P} \mid H_S \neq \emptyset\}$. For $S \in \mathcal{P}'$, let ψ_S be the orthogonal sum of those $\langle a_i \rangle$ with $1 \leq i \leq m$ such that $H(\rho_{i,S}) = H_S$. We conclude that

$$a_1 \rho_1 \perp \cdots \perp a_m \rho_m \equiv \sum_{S \in \mathcal{P}'} \psi_S \otimes \pi_S \pmod{I_T K \cdot I^{n-1} K}. \quad (3.11)$$

□

Theorem 3.10 (Krüskemper). *If $n \geq \text{st}(T)$, then $I_T^n K = I_T K \cdot I^{n-1} K$.*

Proof. We trivially have that $I_T K \cdot I^{n-1} K \subseteq I_T^n K$. To prove the converse inclusion, we assume that $n \geq \text{st}(T)$ and $\varphi \in I_T^n K$. By Proposition 3.9, we have

$$\varphi \equiv \psi_1 \otimes \pi_1 \perp \cdots \perp \psi_r \otimes \pi_r \pmod{I_T K \cdot I^{n-1} K}, \quad (3.12)$$

where $r \in \mathbb{N}$, $\psi_1, \dots, \psi_r$ are forms and where $\pi_1, \dots, \pi_r$ are n -fold Pfister forms such that $H(\pi_1), \dots, H(\pi_r)$ are nonempty and pairwise disjoint. Using this and the fact that $\text{sign}_T(\varphi) = 0$, we obtain for $1 \leq i \leq r$ that $\text{sign}_T(\psi_i \otimes \pi_i) = 0$ and that $\psi_i \in IK$ because $\text{sign}_T(\pi_i) \neq 0$, so that $\psi_i \otimes \pi_i \in I_T K \cdot I^n K$ by Corollary 3.6. Therefore, $\varphi \in I_T K \cdot I^{n-1} K$. $\square$

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