International Scholarly Research Network
ISRN Mathematical Analysis
Volume 2011, Article ID 150979, 11 pages
doi:10.5402/2011/150979

Research Article

The Cauchy Problem for the Laplace Equation and Application to Image Inpainting

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Received 4 August 2011; Accepted 11 September 2011

Academic Editors: G. Garcea and C. Zhu

The moment approach to solve the Cauchy problems is investigated. First, we consider the Cauchy problem for the Laplace equation, and we present a moment method for solving it in the case of a flat boundary. Second, we consider the reciprocity gap concept used to solve the problem of crack detection, as a stopping criterion and we study the case of noise data. Finally, we propose an application to the Cauchy problem for the Laplace equation, for the inpainting problem. Some numerical results showing the efficiency of the method proposed are also given.

1. Introduction

This paper deals with the inverse problem of crack detection defined by overdetermined boundary data, in the framework of the Laplace equation. The Cauchy problem for the Laplace equation is well known to be highly ill posed since Hadamard. That means the solution does not depend continuously on the given Cauchy data and any small change in the given data may cause large change to the solution [1–3]. In [4], the authors propose a method for data matching via a moment problem and give an application by using the Legendre polynomials. The idea to transform the problem into a moment problem has been initiated by Cheng et al. in the work [5]; then, several numerical methods have been proposed to solve this problem, such as Gram-Schmidt orthonormalization technique [6], the Backus-Gilbert algorithm [7], the conjugate gradient method [8], the fourth-order derivative method [9], the Fourier method [10], and the wavelets method [11], and we refer the reader to [12] for theoretical aspects of moment problems. Let us note here that the Talenti method is efficient but is restricted to the Hausdorff moment problem. In [4], a general character of the orthonormal basis chosen is presented: for any orthonormal basis, the authors can calculate the moments associated. Our motivation in this paper is based on three main features that make it according to our knowledge an efficient method. The first feature concerns its robustness: it compares very well with [5] at recovering processes in the sense that it allows the reconstruction of highly oscillating data. The second feature we would like to point out concerns the cost of the present method: since we have converted the data-matching problem into a moment one, then the recovering process turns out to be quasi-explicit; that is, it does not require any resolution of the forward problem. Finally, the third feature is shown in the general character of possible applications. In fact, for a potential application of present boundary data completion algorithm, one can consider the image processing area and more precisely restoration and inpainting problems. Our motivation is the following: image inpainting refers to image restoration techniques which can be based on crack detection. So, as a crack can be detected as it was presented in [4], then it will be easy to reconstruct a particularly damaged part of an image using the same idea. On the other part, as the result of the reconstruction depends on the number of projection used, so we propose to use the gap reciprocity function introduced by Andrieux and Ben Abda [13, 14], as a stopping criterion and study the error estimate.

This paper is organized as follows. The opening section is devoted to the Cauchy problem of the Laplace equation and its reformulation as a moment problem. In Section 3, the gap reciprocity function is presented in the framework of crack detection. We propose then to adapt the reciprocity gap concept, as a stopping criterion. In Section 4, we treat the inpainting problem using the Cauchy problem for the Laplace equation. The concluding section is devoted to some discussions and open problems.

2. Formulation of the Problem and Transformation to a Moment Problem

Let us consider a domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$. We suppose that Ω is a regular Lipschitz bounded domain. Let us denote by n the outward unit normal to the boundary $\Gamma = \partial\Omega$. We assume that Γ is partitioned into two parts Γ_1 and Γ_2. Given both the Dirichlet and Neumann conditions on Γ_1, one wants to reconstruct the corresponding conditions on the remaining part of the boundary Γ_2. This problem arises in electrostatic. One can suppose that u represents the electrostatic potential in a body Ω of which only the part Γ_1 is accessible to measurements. Hence, in this application, the corresponding inverse problem can be seen as the reconstruction of the boundary Γ_2 by determining the inaccessible measurements from those imposed on Γ_1. For different applications of this problem, we can refer the reader to [15–17]. This problem can be expressed as a Cauchy problem for the Laplace equation and therefore termed mathematically as follows.

Find f and g on the remaining part of the boundary Γ_2 such that there exists a solution u satisfying

$$\begin{aligned} \Delta u &= 0, \Omega, \\ \nabla u \cdot n &= F, \Gamma_1, \\ u &= G, \Gamma_1. \end{aligned} \tag{2.1}$$

In [5], Cheng et al. give a numerical method to approximate the solution of (2.1) using Green's formula. In fact, by considering the system

$$\begin{aligned} \Delta v &= 0, \Omega, \\ \frac{\partial v}{\partial n} &= 0, \Gamma_2, \end{aligned} \tag{2.2}$$

the authors transform the original problem to a moment problem whose numerical solution can be obtained accurately and efficiently. We refer the reader to [5] for the details of the proof of the following proposition.

Proposition 2.1. *If the Cauchy problem* (2.1) *has a solution u such that $(\partial u/\partial n)/\Gamma_2 \in L^2(\Gamma_2)$, then $\beta = (\partial u/\partial n)/\Gamma_2$ satisfies the following moment problem:*

$$\int_{\Gamma_2} v\beta \, ds = \int_{\Gamma_1} \left(G \frac{\partial v}{\partial n} - Fv \right) ds, \tag{2.3}$$

where $v \in H_1 = \{v/\Gamma_2 : v$ verifies (2.2)$\}$.

Conversely if $\beta \in L^2(\Gamma_2)$ satisfies (2.3)*, then there exists a solution u of the Cauchy problem* (2.1) *such that $(\partial u/\partial n)/\Gamma_2 \in L^2(\Gamma_2)$.*

To approach the solution of (2.3), the authors in [5] proved that the Cauchy problem for the Laplace equation (2.1) is equivalent to the Hausdorff moment problem, which has been studied by many researchers. More precisely, to obtain the desired result, they used the method of Talenti [12], which is based on the Gram matrix of a given set of vectors, and discussed the error estimation. We note here that Hilbert matrices are well known to be extremely ill conditioned, which will disturb the convergence of the solution.

Let us first recall the moment problem as given by Talenti [12]. We consider a Hilbert space $\mathscr{H}$, and we consider given basis functions $v_j \in \mathscr{H}$ and real numbers μ_j, $j = 0, 1, 2, \ldots$.

The moment problem consists in determining an unknown function $u \in \mathscr{H}$ such that

$$(u, v_j)_{\mathscr{H}} = \int_{\Omega} v_j(x) u(x) dx = \mu_j, \quad j = 0, 1, 2, \ldots \tag{2.4}$$

As a trivial example of such a basis, we can take the polynomial basis $(x^j)_j$, $j = 0, 1, 2, \ldots$, defined in a one-dimensional space. Working in a two-dimensional space, one can take $\mathscr{H} = L^2(\Omega)$ and choose $v_j = e^{jx}\cos(jy)$, $j = 0, 1, 2, \ldots$.

We have $\overline{\{v_j, j = 0, 1, 2, \ldots\}} = L^2(\Omega)$ assuring the density of $\operatorname{span}\{v_0, v_1, \ldots\}$ in $L^2(\Omega)$.

In [4], the authors generalize the above result obtained in [5]. In fact, the authors propose to recover u and $\partial u/\partial n$ on Γ_2 from their moments on any orthonormal basis of $L^2(\Gamma_2)$. The idea is to make the components $\{u_j\}_j$ and $\{\partial u_j/\partial n\}_j$ appear directly according to a given orthonormal basis of $L^2(\Gamma_2)$.

For a numerical convenience, let us suppose that Ω is a bounded simply Lipschitz connected domain of $\mathbb{R}^2$ given by

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \ 0 \leq y \leq 1 \right\}, \tag{2.5}$$

where

$$\Gamma_2 = \left\{ (x,y) \in \mathbb{R}^2 : y = 0,\ 0 \le x \le 1 \right\}, \tag{2.6}$$

and Γ_1 is the Lipschitz curve in $\{(x,y) \in \mathbb{R}^2 : y \ge 0\}$ which connects the two points $(0,0)$ and $(1,0)$ such that $\{(x,y) \in \mathbb{R}^2 : y = 0,\ 0 \le x \le 1\} \cup \Gamma_1 = \partial\Omega$.

We consider H_1 and H_2 as follows:

$$\begin{aligned} \mathrm{H}_1 &= \left\{ v : \Delta v = 0 \text{ in } \Omega \text{ and } \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_2 \right\}, \\ \mathrm{H}_2 &= \{ v : \Delta v = 0 \text{ in } \Omega \text{ and } v = 0 \text{ on } \Gamma_2 \}. \end{aligned} \tag{2.7}$$

Let $\{w_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2[0,1]$, where $[0,1]$ represents Γ_2. To determine the moments of the lacking data directly on $\{w_j\}_{j\in\mathbb{N}}$, we give the following results (see [4] for the proofs).

Theorem 2.2. *There exists a set $\{Q_j\}_{j\in\mathbb{N}} \in H_2$ such that*

$$u = \sum_{j=0}^{\infty} m_j^1 w_j, \tag{2.8}$$

where the moments m_j^1 are defined by

$$m_j^1 = \int_{\Gamma_1} \left(F Q_j - \frac{\partial Q_j}{\partial n} G \right) ds, \quad \forall j \in \mathbb{N}. \tag{2.9}$$

Theorem 2.3. *There exists a set $\{D_j\}_{j\in\mathbb{N}} \in H_1$ such that*

$$\frac{\partial u}{\partial n} = \sum_{j=0}^{\infty} m_j^2 w_j, \tag{2.10}$$

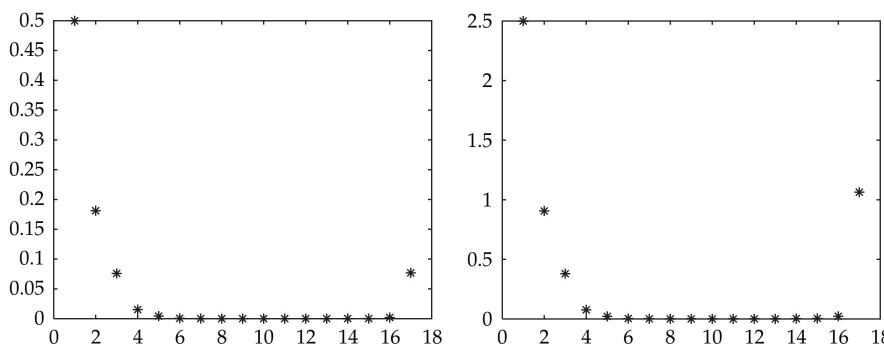
where the moments m_j^2 are defined by

$$m_j^2 = \int_{\Gamma_1} \left(F D_j - \frac{\partial D_j}{\partial n} G \right) ds, \quad \forall j \in \mathbb{N}. \tag{2.11}$$

We note that an approximation of the solution u on Γ_2 is given by the function $u_{h,N}$ such that $u_{h,N}(x) = \sum_{j=0}^{N} m_j^1 w_j(x)$, where m_j^1 are the moments of the function u and are given by relation (2.9), and an approximation of $\partial u/\partial n$ is given by the function $\partial u_{h,N}/\partial n$ such that $\partial u_{h,N}/\partial n = \sum_{j=0}^{N} m_j^2 w_j(x)$, where m_j^2 are the moments of the function $\partial u/\partial n$ and are determined by relation (2.11).

Table 1: The error in the reconstruction of u and $\partial u/\partial n$ is evaluated with respect to the number of moments calculated $N := 1, \ldots, 19$.

N	$E(u)$	$E(\partial u/\partial n)$
1	0.4999	2.4994
2	0.1812	0.9058
3	0.0758	0.3792
4	0.0152	0.0761
5	0.0043	0.030
6	0.0006	0.0213
7	0.0001	0.0006
8	0.0	0.0001
9	0.0	0.0
10	0.0	0.0
11	0.0	0.0
12	0.0	0.0
13	0.0	0.0
14	0.0001	0.0026
15	0.0001	0.0221
16	0.0002	0.042
17	0.0019	1.0638
18	0.0769	3.46
19	0.11	52.2134
20	3.0485	1423.61



(a) (b)

Figure 1: L^2 reconstruction error for both u (a) and $\partial u/\partial n$ (b).

We present in Table 1 the L^2 error reconstruction with respect to the number of moment calculated for both the solution u and its derivative ∇u. We remark that this error is decreasing when the number of moments increases. The reconstruction of both the solution u and its derivative is excellent when the number of moments calculated varies between 9 and 13. Then, reconstruction error increases when the number of moments increases too, characterizing the fact that the problem (2.1) is ill posed and then implying a well-known property of a numerical instability.

Figure 1 shows the numerical results of the error reconstruction for both u and $\partial u/\partial n$.

In the rest of this work, we propose to use the reciprocity gap concept as a stopping criterion.

3. The Reciprocity Gap Concept

The reciprocity gap principle has been initially introduced in solid mechanics throughout the Maxwell-Betti theorem in 1872, relating the responses to different external and internal forcing terms. More precisely, let us consider given overdetermined boundary data (a flux $\partial u/\partial n = \Phi$ imposed on $\partial\Omega$ and a temperature $u = T$ measured on $\partial\Omega$). The Maxwell-Betti theorem establishes that the effect work of the flux $\partial u/\partial n$ in any field v in equilibrium in Ω with zero source term, is equal to the effect work of the temperature u in the flux $\partial v/\partial n$, implying that

$$\int_{\partial\Omega} v\frac{\partial u}{\partial n}ds = \int_{\partial\Omega} u\frac{\partial v}{\partial n}ds, \quad \forall v \in H(\Omega), \tag{3.1}$$

where $H(\Omega) = \{v \in H^1(\Omega), \Delta v = 0\}$. Let us note here that (3.1) is fundamentally derived from the symmetry of the operator involved in the modelisation of the phenomenon.

Using (3.1), we define the reciprocity gap functional as follows:

$$RG(v) = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds, \quad \forall v \in H(\Omega). \tag{3.2}$$

On the other part, from divergence theorem, Green's second identity, and working with an unperturbed domain Ω, we deduce that the reciprocity gap functional is identically zero and then

$$\int_{\Gamma_1} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds = -\int_{\Gamma_2} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds. \tag{3.3}$$

The reciprocity gap functional has been applied in different theories. For example, we can find its application to electromagnetic inverse scattering problems [18], or to groundwater flow problems [19]. Particularly, the nonidentical nullity of the reciprocity gap functional implies the presence of cracks in the domain, and this functional has been used to localize and determine the shape of one or several cracks in the initial domain [13, 14]. We propose in this work to apply this functional in order to recover missing boundary data on a part of the boundary domain from overdetermined ones existing on the remaining part, by applying the moment method. More precisely, since the left-hand side of (3.3) is not exact, then the idea is to use its approximation by the moment method applied in the previous section and then to apply the reciprocity condition $RG(v) = 0$ as a stopping criterion.

Let us consider the following equations:

$$\begin{aligned} \mu_1(v) &= -\int_{\Gamma_1} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds, \quad v \in H(\Omega), \\ \mu_2(v) &= \int_{\Gamma_2} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds, \quad v \in H(\Omega). \end{aligned} \tag{3.4}$$

Since the exact solution on the remaining part Γ_1 is not available, then we consider an approximation μ_{app} of μ_1 as follows:

$$\mu_{\text{app}}(v) = -\int_{\Gamma_1}\left(u_{\text{app}}\frac{\partial v}{\partial n} - v\frac{\partial u_{\text{app}}}{\partial n}\right)ds, \quad v \in H(\Omega), \tag{3.5}$$

where u_{app} and $\partial u_{\text{app}}/\partial n$ are determined thanks to Theorems 2.2 and 2.3.

Let

$$\mu(u, u_{\text{app}}) = \mu_2(v) - \mu_{\text{app}}(v) = \int_{\Gamma_2}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds + \int_{\Gamma_1}\left(u_{\text{app}}\frac{\partial v}{\partial n} - v\frac{\partial u_{\text{app}}}{\partial n}\right)ds. \tag{3.6}$$

Numerically, if the approximated solution u_{app} converges to the exact one u_{exa} and the approximated flux $\partial u_{\text{app}}/\partial n$ converges to the exact one $\partial u_{\text{exa}}/\partial n$, then (3.6) converges to zero.

On the other part, from (3.6), we deduce that the condition $RG(v) = 0$ yields

$$\int_{\Gamma_1}(u_{\text{app}} - u_{\text{exa}})\frac{\partial v}{\partial n} + \int_{\Gamma_1} v\left(\frac{\partial u_{\text{exa}}}{\partial n} - \frac{\partial u_{\text{app}}}{\partial n}\right) = 0 \tag{3.7}$$

which gives an error estimate of the recovering missing boundary data and then characterizes a nice stopping criteria. By considering the scalar product defined on the Hilbert space $L^2(\Omega)$, numerically, the stopping criterion is such that the quantity

$$\mu\left(F, G, u_{\text{app}}, \frac{\partial u_{\text{app}}}{\partial n}\right) = \left\langle (u_{\text{exa}} - u_{\text{app}}), \frac{\partial v}{\partial n}\right\rangle + \left\langle\left(\frac{\partial u_{\text{exa}}}{\partial n} - \frac{\partial u_{\text{app}}}{\partial n}\right), v\right\rangle \tag{3.8}$$

is small enough.

4. Application to Image Inpainting

As a potential application of present boundary data completion algorithm, one can think of the image processing area. We consider here the inpainting problem. The goal of inpainting is to restore a damaged or corrupted image where one or several pieces of information have been lost, see [20–25]. Inpainting may also be an interesting tool for those who need to remove some parts of an image artificially such as scratches. The third application we are dealing with in this work is exactly this kind of problem, which is an inverse problem: the reconstruction of the image processed from its damaged version. Our motivation is the following: image inpainting refers to image restoration techniques which can be based on cracks detection. So, as a crack can be detected as it was presented in [4], then it will be easy to reconstruct a particularly damaged part of an image using the same idea. More precisely, let us consider Ω the image processed, $\Gamma = \Gamma_1$ the whole external boundary, and ω the missing part of the image which represents in this work a given scratch. ω is determined by the intensity of its grey level and it splits the domain Ω into two subdomains Ω_1 and Ω_2. G will represent the value of the image on ω and F its corresponding flux. By considering the following Cauchy problem:

$$\begin{array}{lll} \Delta u & = 0, & \text{in } \Omega \setminus \omega, \\ u & = G, & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial n} & = F, & \text{on } \Gamma_1, \\ u & = 0 & \text{on } \omega, \end{array} \tag{4.1}$$

we give the inpainting algorithm as follows.

Reconstruction Algorithm

(i) Calculation of u_1 and u_2 solutions of (4.1).

(ii) detection of the crack ω by minimizing the following criterion:

$$\frac{1}{2}\|u_1 - u_2\|_2^2, \tag{4.2}$$

(iii) calculation of u solution of the Neumann problem with this crack ω

$$\begin{array}{ll} \Delta u = 0 & \text{in } \Omega \setminus \omega, \\ \dfrac{\partial u}{\partial n} = F & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \omega. \end{array} \tag{4.3}$$

Figure 2 illustrates this technique. We have considered the inpainting of a black line on the Lena image. This line divides the original domain Ω into two subdomains. The black line represents ω: the missing part of the image. The corresponding reconstruction is shown in Figure 2(d). Then we added a Gaussian noise such that $\sigma = 20$. The restored image with the missing part is shown in Figure 2(f). One can see that the image is still very well inpainted. However, it could be more interesting to generalize the inpainting problem for general missing parts and more general cracks in the image processed and not only parts determined by cracks in only one direction. This will be part of a forthcoming paper.

5. Conclusion

We have presented in this work an application of the moment method used for crack detection to a particular problem in image processing related to image inpainting. A stopping criterion based on the reciprocity gap functional has been also introduced in this work. Our data recovering has a major advantage: it concerns the cost of the present method. In fact, since the initial Cauchy problem for the Laplace equation has been converted to a moment

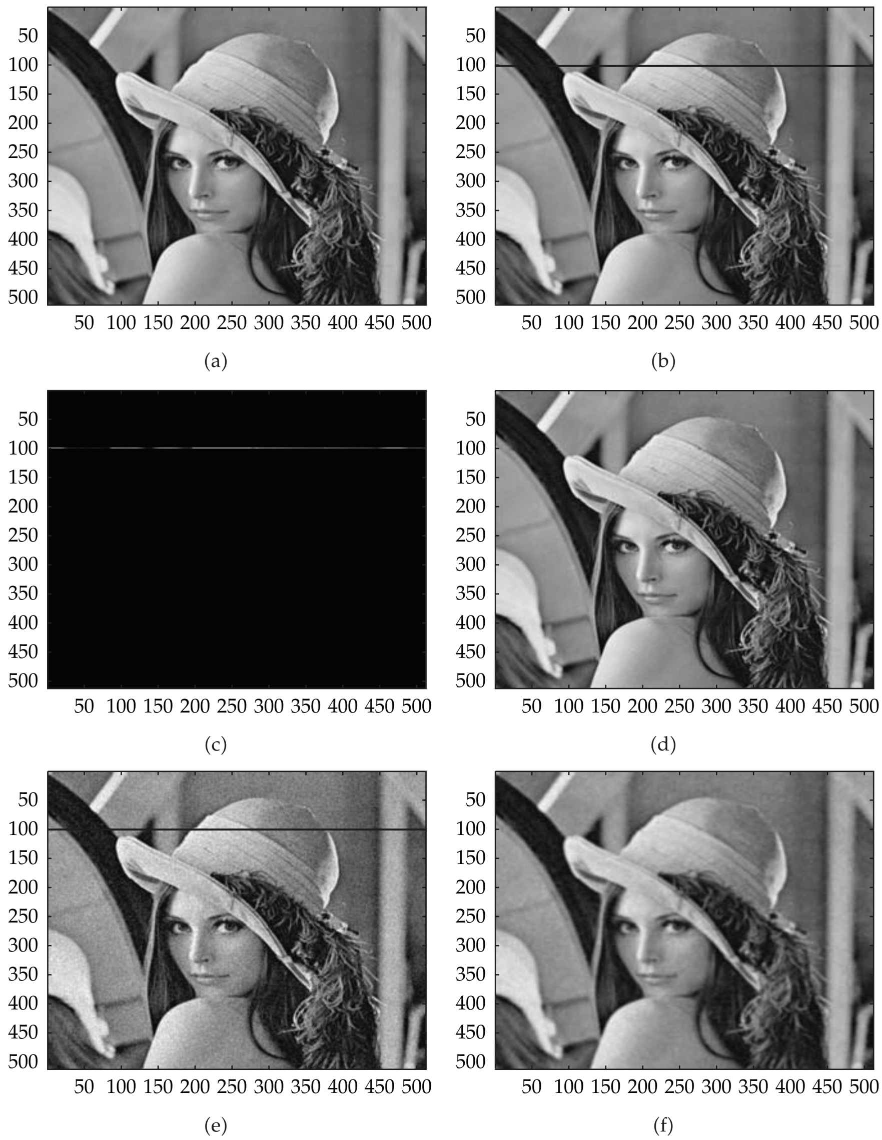


Figure 2: Inpainting of a black line on Lena image: original image (a), occluded image (b), detection of the missing part (c), inpainted image (d), occluded noisy image with $\sigma = 20$ (e), and inpainted noisy image (f).

one, then the recovering process turns out to be quasi-explicit; it does not require any resolution of the forward problem. However, one can consider the very particular character of our method concerning the application to the inpainting problem, which represents here a major drawback, in the sense that this recovering is strictly related to a flat boundary. This drawback will be considered in a forthcoming paper by considering the reciprocity gap functional related to geometrical moments. Finally, some attention should be devoted to noisy data. We propose to study the robustness of our model. In fact, we propose to define a stopping criterion based on the reciprocity gap functional in case of Cauchy data noise, for a better comparison to real-life applications, and particularly for image processing applications as restoration.

Acknowledgment

This paper has been supported by the Deanship of Scientific Research at the University of Dammam under the Reference 201036.

References

[1] V. Isakov, *Inverse Problems for Partial Differential Equations*, vol. 127, Springer, New York, NY, USA, 1998.

[2] A. M. Denisov, *Elements of the Theory of Inverse Problems*, VSP, Utrecht, the Netherlands, 1999.

[3] V. Y. Arsenin and A. N. Tikhonov, *Solutions of Ill-Posed Problems*, Winston and Sons, Washington, DC, USA, 1997.

[4] A. B. Abda, L. J. Belaid, and A. Sakat, "Data recovering problem using moments theory and applications to some inverse problems," *International Journal of Tomography and Statistics*, vol. 17, no. S11, pp. 1–16, 2011.

[5] J. Cheng, Y. C. Hon, T. Wei, and M. Yamamoto, "Numerical computation of a Cauchy problem for Laplace's equation," *Zeitschrift fur Angewandte Mathematik und Mechanik*, vol. 81, no. 10, pp. 665–674, 2001.

[6] Y. C. Hon and T. Wei, "An orthonormal basis functions method for moment problems," *Engineering Analysis with Boundary Elements*, vol. 26, no. 10, pp. 855–860, 2002.

[7] Y. C. Hon and T. Wei, "Backus-Gilbert algorithm for the Cauchy problem of the Laplace equation," *Inverse Problems*, vol. 17, no. 2, pp. 261–271, 2001.

[8] D. N. Hào and D. Lesnic, "The Cauchy problem for Laplace's equation via the conjugate gradient method," *IMA Journal of Applied Mathematics*, vol. 65, no. 2, pp. 199–217, 2000.

[9] Z. Qian, C.-L. Fu, and X.-T. Xiong, "Fourth-order modified method for the Cauchy problem for the Laplace equation," *Journal of Computational and Applied Mathematics*, vol. 192, no. 2, pp. 205–218, 2006.

[10] C.-L. Fu, H.-F. Li, Z. Qian, and X.-T. Xiong, "Fourier regularization method for solving a Cauchy problem for the Laplace equation," *Inverse Problems in Science and Engineering*, vol. 16, no. 2, pp. 159–169, 2008.

[11] C.-Y. Qiu and C.-L. Fu, "Wavelets and regularization of the Cauchy problem for the Laplace equation," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1440–1447, 2008.

[12] G. Talenti, "Recovering a function from a finite number of moments," *Inverse Problems*, vol. 3, no. 3, pp. 501–517, 1987.

[13] S. Andrieux and A. Ben Abda, "Identification de fissures planes par une donnée de bord unique: un procédé direct de localisation et d'identification," *Comptes Rendus de l'Académie des Sciences. Série I*, vol. 315, no. 12, pp. 1323–1328, 1992.

[14] S. Andrieux and A. Ben Abda, "Identification of planar cracks by complete overdetermined data: inversion formulae," *Inverse Problems*, vol. 12, no. 5, pp. 553–563, 1996.

[15] G. Alessandrini, L. Del Piero, and L. Rondi, "Stable determination of corrosion by a single electrostatic boundary measurement," *Inverse Problems*, vol. 19, no. 4, pp. 973–984, 2003.

[16] G. Alessandrini and E. Sincich, "Solving elliptic Cauchy problems and the identification of nonlinear corrosion," *Journal of Computational and Applied Mathematics*, vol. 198, no. 2, pp. 307–320, 2007.

[17] S. Chaabane and M. Jaoua, "Identification of Robin coefficients by the means of boundary measurements," *Inverse Problems*, vol. 15, no. 6, pp. 1425–1438, 1999.

[18] D. Colton and H. Haddar, "An application of the reciprocity gap functional to inverse scattering theory," *Inverse Problems*, vol. 21, no. 1, pp. 383–398, 2005.

[19] N. T. Hariga, A. Ben Abda, R. Bouhlila, and J.-R. de Dreuzy, "Definition and interests of reciprocity and reciprocity gap principles for groundwater flow problems," *Advances in Water Resources*, vol. 33, no. 8, pp. 899–904, 2010.

[20] G. Aubert and P. Kornprobst, "Explicit reconstruction for image inpainting," Rapport de recherche 5905, 2006.

[21] D. Auroux and M. Masmoudi, "A one-shot inpainting algorithm based on the topological asymptotic analysis," *Computational & Applied Mathematics*, vol. 25, no. 2-3, pp. 251–267, 2006.

[22] M. Bertalmio, G. Sapiro, V. Caselles, and C. Ballestu, "Image inpainting," in *Proceedings of ACM Conference on Computer Graphics and Interactive Techniques*, pp. 417–424, 2000.

[23] M. Bertalmio, L. Vese, G. Sapiro, and S. Osher, "Simultaneous structure and texture image inpainting," in *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, 2003.

[24] T. Chan and J. Shen, "Mathematical models for local deterministic inpaintings," Tech. Rep. 00 11, UCLA CAM, 2000.

[25] A. Criminisi, P. Pérez, and K. Toyama, "Region filling and object removal by exemplar-based image inpainting," *IEEE Transactions on Image Processing*, vol. 13, no. 9, pp. 1200–1212, 2004.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