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Research Article

Criteria for Strongly Starlike Functions

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Let $f(z)$ be analytic in the unit disk $U = \{z : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$ and $(f(z)/z)f'(z) \neq 0$. By using the method of differential subordinations, we determine the largest number $\alpha(\beta, \lambda, \mu, m)$ such that, for some β, λ, μ, and m, the differential subordination $\lambda(zf'(z)/f(z))^{1-\mu}(1+(zf''(z)/f'(z))-zf'(z)/f(z))+(zf'(z)/f(z))^m \prec (1+z/1-z)^{\alpha(\beta,\lambda,\mu,m)}$ $(z \in U)$ implies $zf'(z)/f(z) \prec (1+z/1-z)^{\beta}$. Some useful consequences of this result are also given.

1. Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in $U = \{z : |z| < 1\}$. A function $f(z) \in A$ is said to be starlike of order α if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad (z \in U) \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$. We denote this class by $S^*(\alpha)$. A function $f(z) \in A$ is said to be convex of order α if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U) \tag{1.3}$$

for some α $(0 \le \alpha < 1)$. We denote this class by $C(\alpha)$. Further, a function $f(z) \in A$ is said to be strongly starlike of order β if

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\beta \quad (z \in U) \tag{1.4}$$

for some β $(0 < \beta \le 1)$. Also we denote this class by $\widetilde{S}^*(\beta)$ $(0 < \beta \le 1)$. Clearly $S^*(\alpha) \subset S^*(0) \equiv S^*$ for $0 \le \alpha < 1, \widetilde{S}^*(1) = S^*$.

Let $f(z)$ and $g(z)$ be analytic in U. Then the function $f(z)$ is said to be subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ for $z \in U$. If $g(z)$ is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$. It is easy to see that a function $f(z) \in \widetilde{S}^*(\beta)$ $(o < \beta \le 1)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} \quad (z \in U). \tag{1.5}$$

A number of results for strongly starlike functions in U have been obtained by several authors (see, e.g., [1–7]). In this paper, by using the method of differential subordinations, we determine the largest number $\alpha(\beta, \lambda, \mu, m)$ such that, for some β, λ, μ, and m, the differential subordination

$$\lambda\left(\frac{zf'(z)}{f(z)}\right)^{1-\mu}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right)^{m} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\lambda,\mu,m)} \quad (z \in U) \tag{1.6}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}. \tag{1.7}$$

Our results improved or extended the above results.

To prove our results, we need the following lemma due to Miller and Mocanu [3].

Lemma 1.1. *Let $g(z)$ be analytic and univalent in U, and let $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(U)$, with $\varphi(w) \neq 0$ when $w \in g(U)$. Set*

$$Q(z) = zg'(z)\varphi(g(z)), \qquad h(z) = \theta(g(z)) + Q(z) \tag{1.8}$$

and suppose that

(i) *$Q(z)$ is starlike univalent in U,*

(ii) $\operatorname{Re} zh'(z)/Q(z) = \operatorname{Re}\{\theta'(g(z))/\varphi(g(z)) + zQ'(z)/Q(z)\} > 0 \quad (z \in U)$.

If $p(z)$ is analytic in U, with $p(0) = g(0), p(U) \subset D$,

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z), \tag{1.9}$$

then $p(z) \prec g(z)$ and $g(z)$ is the best dominant of (1.9).

2. Main Results

Theorem 2.1. *Let $0 < \beta < 1, \mu \in \{0,1,2\}, m$ an integer, $|m + \mu - 1|\beta < 1, m\lambda \geq 0$, and $\lambda \neq 0$. If $f(z) \in A$ satisfies $(f(z)/z)f'(z) \neq 0$ $(z \in U)$ and*

$$\lambda\left(\frac{zf'(z)}{f(z)}\right)^{1-\mu}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right)^{m} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\lambda,\mu,m)} \quad (z \in U), \tag{2.1}$$

where

$$\begin{aligned}
&\alpha(\beta,\lambda,\mu,m) \\
&= \frac{2}{\pi}\arctan \\
&\quad \times \left(\tan\frac{(m+\mu-1)\beta\pi}{2}\right. \\
&\qquad \left. + \frac{\lambda\beta}{(1-(m+\mu-1)\beta)^{1-(m+\mu-1)\beta/2}(1+(m+\mu-1)\beta)^{1+(m+\mu-1)\beta/2}\cos(m+\mu-1)\beta\pi/2}\right) \\
&\quad + (1-\mu)\beta,
\end{aligned} \tag{2.2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} \tag{2.3}$$

and $\alpha(\beta,\lambda,\mu,m)$ given by (2.2) *is the largest number such that* (2.3) *holds.*

Proof. Let $f(z) \in A$ with $(f(z)/z)f'(z) \neq 0$ $(z \in U)$, and define the function $p(z)$ in U by

$$p(z) = \frac{zf'(z)}{f(z)}. \tag{2.4}$$

Then $p(z)$ is analytic in U and

$$\lambda\left(\frac{zf'(z)}{f(z)}\right)^{1-\mu}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right)^{m} = \lambda\frac{zp'(z)}{p^{\mu}(z)} + p^{m}(z). \tag{2.5}$$

Let $0 < \beta < 1, \mu \in \{0,1,2\}, m$ an integer, $|m+\mu-1|\beta < 1,\ m\lambda \geq 0,\ \lambda \neq 0$, and

$$D = \begin{cases} C, & \mu = 0,\ m \geq 0, \\ C \setminus \{0\}, & \mu = 1,2 \text{ or } m < 0, \end{cases} \tag{2.6}$$

and choose

$$g(z) = \left(\frac{1+z}{1-z}\right)^{\beta}, \qquad \theta(w) = w^{m}, \qquad \varphi(w) = \frac{\lambda}{w^{\mu}}. \tag{2.7}$$

Then $g(z)$ is analytic and univalent in U, $p(0) = g(0) = 1$, $p(U) \subset D$, and $\theta(w)$ and $\varphi(w)$ satisfy the conditions of the lemma. The function

$$Q(z) = zg'(z)\varphi(g(z)) = \frac{2\lambda\beta z}{(1+z)^{1+(\mu-1)\beta}(1-z)^{1-(\mu-1)\beta}} \tag{2.8}$$

is univalent and starlike in U because

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = 1 + (1+(\mu-1)\beta)\operatorname{Re}\left(-\frac{z}{1+z}\right) + (1-(\mu-1)\beta)\operatorname{Re}\frac{z}{1-z} > 0 \quad (z \in U) \tag{2.9}$$

for $|(\mu-1)|\ \beta < 1$. Further, we have that

$$\theta(g(z)) + Q(z) = \left(\frac{1+z}{1-z}\right)^{m\beta} + \frac{2\lambda\beta z}{(1+z)^{1+(\mu-1)\beta}(1-z)^{1-(\mu-1)\beta}} = h(z) \quad (\text{say}), \tag{2.10}$$

$$\frac{zh'(z)}{Q(z)} = \frac{m}{\lambda}\left(\frac{1+z}{1-z}\right)^{(m+\mu-1)\beta} + \frac{zQ'(z)}{Q(z)} \quad (z \in U). \tag{2.11}$$

Since $0 \leq |m+\mu-1|\beta < 1$, we have that

$$\left|\arg\left(\frac{1+z}{1-z}\right)^{(m+\mu-1)\beta}\right| < \frac{\pi}{2} \quad (z \in U) \tag{2.12}$$

and so

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \frac{m}{\lambda}\operatorname{Re}\left\{\left(\frac{1+z}{1-z}\right)^{(m+\mu-1)\beta}\right\} + \operatorname{Re}\frac{zQ'(z)}{Q(z)} > 0 \quad (z \in U). \tag{2.13}$$

Inequality (2.13) shows that the function $h(z)$ is close-to-convex and univalent in U. Letting $0 < \theta < \pi$ and $x = \cot(\theta/2) > 0$, then

$$h\left(e^{i\theta}\right) = \left(x^{m\beta}e^{(m+\mu-1)\beta\pi i/2} + \frac{\lambda\beta i}{2}\left(x^{1-(\mu-1)\beta} + \frac{1}{x^{1+(\mu-1)\beta}}\right)\right)e^{(1-\mu)\beta\pi i/2} \tag{2.14}$$

and so

$$\arg h\left(e^{i\theta}\right) = \arctan w(x) + \frac{(1-\mu)\beta\pi}{2}, \tag{2.15}$$

where

$$w(x) = \tan\frac{(m+\mu-1)\beta\pi}{2} + \frac{\lambda\beta}{2\cos(m+\mu-1)\beta\pi/2}\left(x^{1-(m+\mu-1)\beta} + \frac{1}{x^{1+(m+\mu-1)\beta}}\right). \tag{2.16}$$

It is easy to know that $w(x)$ takes its minimum value at $x = \sqrt{1+(m+\mu-1)\beta/1-(m+\mu-1)\beta}$. Hence, in view of $h(e^{-i\theta}) = \overline{h(e^{i\theta})}$, we deduce that

$$\begin{aligned}\inf_{|z|(z\neq 1)}\left|\arg h(z)\right| &= \arctan\left(w\left(\sqrt{\frac{1+(m+\mu-1)\beta}{1-(m+\mu-1)\beta}}\right)\right) + \frac{(1-\mu)\beta\pi}{2} \\ &= \alpha(\beta,\lambda,\mu,m)\frac{\pi}{2},\end{aligned} \tag{2.17}$$

where $\alpha(\beta,\lambda,\mu,m)$ is given by (2.2).

Now, if $f(z)$ satisfies (2.1), it follows from (2.17) that the subordination

$$\lambda\left(\frac{zf'(z)}{f(z)}\right)^{1-\mu}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)+\left(\frac{zf'(z)}{f(z)}\right)^{m} \prec h(z) \quad (z\in U) \tag{2.18}$$

holds. Hence it follows from (2.5), (2.7), (2.10), and (2.18) that

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z) \tag{2.19}$$

holds. Therefore, by virtue of the lemma, we conclude that $p(z) \prec g(z)$, that is, (2.3) holds.

Next we consider the extremal function

$$f(z) = z\exp\left\{\int_0^z \frac{1}{t}\left(\left(\frac{1+t}{1-t}\right)^{\beta}-1\right)dt\right\}. \tag{2.20}$$

Then $f(z)$ satisfies

$$\begin{aligned}&\lambda\left(\frac{zf'(z)}{f(z)}\right)^{1-\mu}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)+\left(\frac{zf'(z)}{f(z)}\right)^{m} \\ &\quad = \left(\frac{1+z}{1-z}\right)^{m\beta} + \frac{2\lambda\beta z}{(1+z)^{1+(\mu-1)\beta}(1-z)^{1-(\mu-1)\beta}} = h(z),\end{aligned} \tag{2.21}$$

and it follows from (2.17) that the bound $\alpha(\beta,\lambda,\mu,m)$ in (2.1) is sharp. The proof of the theorem is complete. □

Making use of the theorem, we can obtain a number of interesting results.

Letting $\mu = 0$ and $m = 2$ in the theorem, we have the following corollary.

Corollary 2.2. *Let $0 < \beta < 1$, and $\lambda > 0$. If $f(z) \in A$ satisfies $f(z) \neq 0$ $(0 < |z| < 1)$ and*

$$\frac{zf'(z)}{f(z)}\left(\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\lambda)\frac{zf'(z)}{f(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\alpha_3(\beta,\lambda)} \quad (z \in U), \tag{2.22}$$

where

$$\alpha_3(\beta,\lambda) = \frac{2}{\pi}\arctan\left(\tan\frac{\beta\pi}{2}+\frac{\lambda\beta}{(1-\beta)^{(1-\beta)/2}(1+\beta)^{(1+\beta)/2}\cos\beta\pi/2}\right)+\beta, \tag{2.23}$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} \tag{2.24}$$

and $\alpha_3(\beta,\lambda)$ given by (2.23) *is the largest number such that* (2.24) *holds.*

Remark 2.3. Ramesha et al. [6] have proved that, if $f(z) \in A$ satisfies $f(z) \neq 0$ $(0 < |z| < 1)$ and

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0 \quad (z \in U), \tag{2.25}$$

then $f(z) \in \widetilde{S}^*(1/2)$.

For $\beta = 1/2$ and $\lambda > 0$, it follows from (2.23) that $\alpha_3(1/2,\lambda) > 1$. Hence, the image of $w = ((1+z)/(1-z))^{\alpha_3(1/2,\lambda)}(z \in U)$ is a region which properly contains the right half plane. Thus we conclude that Corollary 2.2 with $\beta = 1/2$ and $\lambda = 1$ is better than the result of Ramesha et al. [6].

Letting $\alpha_3(\beta,\lambda) = 1$ in Corollary 2.2, we have the following corollary.

Corollary 2.4. *Let $\lambda > 0$. If $f(z) \in A$ satisfies $f(z) \neq 0$ $(0 < |z| < 1)$ and*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\lambda)\frac{zf'(z)}{f(z)}\right)\right\} > 0 \quad (z \in U), \tag{2.26}$$

then $f(z) \in \widetilde{S}^*(\beta)$, where $\beta \in (0,1)$ is the root of the equation

$$\frac{2}{\pi}\arctan\left(\tan\frac{\beta\pi}{2}+\frac{\lambda}{(1-\beta)^{(1-\beta)/2}(1+\beta)^{(1+\beta)/2}\cos\beta\pi/2}\right)+\beta = 1. \tag{2.27}$$

Remark 2.5. For $\lambda = 1$, Corollary 2.4 reduces to a main result of Nunokawa et al. [5, Theorem 1] by a different method.

Remark 2.6. Note that $\lim_{\beta\to 1}\alpha_3(\beta,1)=2$. Hence it follows from Corollary 2.2 that

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\right|<\pi \quad (z\in U) \tag{2.28}$$

implies

$$f(z)\in S^*. \tag{2.29}$$

Letting $\mu=m=1$ in the theorem, we have the following corollary.

Corollary 2.7. *Let $0<\beta<1, \lambda>0$. If $f(z)\in A$ satisfies $(f(z)/z)f'(z)\neq 0$ $(z\in U)$ and*

$$\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\lambda)\frac{zf'(z)}{f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\alpha_1(\beta,\lambda)} \quad (z\in U), \tag{2.30}$$

where

$$\alpha_1(\beta,\lambda)=\frac{2}{\pi}\arctan\left(\tan\frac{\beta\pi}{2}+\frac{\lambda\beta}{(1-\beta)^{(1-\beta)/2}(1+\beta)^{(1+\beta)/2}\cos\beta\pi/2}\right), \tag{2.31}$$

then

$$\frac{zf'(z)}{f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\beta} \tag{2.32}$$

and $\alpha_1(\beta,\lambda)$ given by (2.31) *is the largest number such that* (2.32) *holds.*

Remark 2.8. Marjono and Thomas [2] proved the above result by a different method. For $\lambda=\delta/\gamma$ $(\delta>0,\gamma>0)$ Corollary 2.2 reduces to a result of Darus [1]. For $\lambda=1$, Corollary 2.2 reduces to a result of Nunokawa and Thomas [4].

Letting $\mu=2$ and $m=0$ in the theorem, we have the following corollary.

Corollary 2.9. *Let $0<\beta<1$ and $\lambda\neq 0$. If $f(z)\in A$ satisfies $f'(z)\neq 0$ $(0<|z|<1)$ and*

$$\frac{\lambda(1+zf''(z))/f'(z)+(1-\lambda)zf'(z)/f(z)}{zf'(z)/f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\alpha_2(\beta,\lambda)} \quad (z\in U), \tag{2.33}$$

where

$$\alpha_2(\beta,\lambda)=\frac{2}{\pi}\arctan\left(\tan\frac{\beta\pi}{2}+\frac{\lambda\beta}{(1-\beta)^{(1-\beta)/2}(1+\beta)^{(1+\beta)/2}\cos\beta\pi/2}\right)-\beta, \tag{2.34}$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} \tag{2.35}$$

and $\alpha_2(\beta, \lambda)$ *given by* (2.34) *is the largest number such that* (2.35) *holds.*

Remark 2.10. For $\lambda = 1$, Corollary 2.9 reduces to a result of Ravichandran and Darus [7].

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