Research Article

Criteria for Strongly Starlike Functions

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Let \( f(z) \) be analytic in the unit disk \( U = \{ z : |z| < 1 \} \) with \( f(0) = f'(0) - 1 = 0 \) and \( (f(z)/z)f'(z) \neq 0 \). By using the method of differential subordinations, we determine the largest number \( \alpha(\beta, \lambda, \mu, m) \) such that, for some \( \beta, \lambda, \mu, \) and \( m \), the differential subordination

\[
\lambda (zf'(z)/f(z))^1 + (1 + (zf''(z)/f'(z)) - zf'(z)/f(z)) + (zf'(z)/f(z))^m < (1 + z/1 - z)^{\beta} \quad (z \in U)
\]

implies \( zf'(z)/f(z) < (1 + z/1 - z)^{\beta} \). Some useful consequences of this result are also given.

1. Introduction

Let \( A \) be the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in \( U = \{ z : |z| < 1 \} \). A function \( f(z) \in A \) is said to be starlike of order \( \alpha \) if

\[
\text{Re} \frac{zf'(z)}{f(z)} > \alpha \quad (z \in U)
\]

for some \( \alpha \) \((0 \leq \alpha < 1)\). We denote this class by \( S^*(\alpha) \). A function \( f(z) \in A \) is said to be convex of order \( \alpha \) if

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U)
\]
for some \(0 < \alpha < 1\). We denote this class by \(C(\alpha)\). Further, a function \(f(z) \in A\) is said to be strongly starlike of order \(\beta\) if

\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \beta \quad (z \in U)
\]

for some \(0 < \beta \leq 1\). Also we denote this class by \(\tilde{S}^*(\beta)\) \((0 < \beta \leq 1)\). Clearly \(S^*(\alpha) \subset S^*(0) = S^*\) for \(0 \leq \alpha < 1\), \(\tilde{S}^*(1) = S^*\).

Let \(f(z)\) and \(g(z)\) be analytic in \(U\). Then the function \(f(z)\) is said to be subordinate to \(g(z)\), written as \(f(z) < g(z)\), if there exists an analytic function \(w(z)\) with \(w(0) = 0\) and \(|w(z)| < 1\) \((z \in U)\) such that \(f(z) = g(w(z))\) for \(z \in U\). If \(g(z)\) is univalent in \(U\), then \(f(z) < g(z)\) is equivalent to \(f(0) = g(0)\) and \(f(U) \subset g(U)\). It is easy to see that a function \(f(z) \in \tilde{S}^*(\beta)\) \((\alpha < \beta \leq 1)\) if and only if

\[
\frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^\beta \quad (z \in U).
\]

A number of results for strongly starlike functions in \(U\) have been obtained by several authors (see, e.g., [1–7]). In this paper, by using the method of differential subordinations, we determine the largest number \(\alpha(\beta, \lambda, \mu, m)\) such that, for some \(\beta, \lambda, \mu, \) and \(m\), the differential subordination

\[
\lambda \left( \frac{zf'(z)}{f(z)} \right)^{1-\mu} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) + \left( \frac{zf'(z)}{f(z)} \right)^m < \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta, \lambda, \mu, m)} \quad (z \in U)
\]

implies

\[
\frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^\beta.
\]

Our results improved or extended the above results.

To prove our results, we need the following lemma due to Miller and Mocanu [3].

**Lemma 1.1.** Let \(g(z)\) be analytic and univalent in \(U\), and let \(\theta(w)\) and \(\varphi(w)\) be analytic in a domain \(D\) containing \(g(U)\), with \(\varphi(w) \neq 0\) when \(w \in g(U)\). Set

\[
Q(z) = zg'(z)\varphi(g(z)), \quad h(z) = \theta(g(z)) + Q(z)
\]

and suppose that

(i) \(Q(z)\) is starlike univalent in \(U\),

(ii) \(\Re zh'(z)/Q(z) = \Re \{\theta'(g(z))/\varphi(g(z)) + zQ'(z)/Q(z)\} > 0 \quad (z \in U)\).
If \( p(z) \) is analytic in \( U \), with \( p(0) = g(0) \), \( p(U) \subset D \),
\[
\theta(p(z)) + zp'(z)\varphi(p(z)) < \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z),
\]
then \( p(z) \prec g(z) \) and \( g(z) \) is the best dominant of (1.9).

### 2. Main Results

**Theorem 2.1.** Let \( 0 < \beta < 1, \mu \in \{0, 1, 2\}, m \) an integer, \( |m + \mu - 1|\beta < 1, m\lambda \geq 0 \), and \( \lambda \neq 0 \). If \( f(z) \in A \) satisfies (\( f(z)/z \)) \( f'(z) \neq 0 \) (\( z \in U \)) and
\[
\lambda\left(\frac{zf'(z)}{f(z)}\right)^{1-\mu} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right)^m < \left(\frac{1+z}{1-z}\right)^{\alpha(\beta, \lambda, \mu, m)} \quad (z \in U),
\]
where
\[
\alpha(\beta, \lambda, \mu, m)
= \frac{2}{\pi} \arctan \left(\frac{\tan \frac{(m + \mu - 1)\beta \pi}{2}}{\lambda \beta \left(1 - (m + \mu - 1)\beta\right)^{1-\left(m + \mu - 1\right)\beta/2} \left(1 + (m + \mu - 1)\beta\right)^{1+\left(m + \mu - 1\right)\beta/2} \cos (m + \mu - 1)\beta \pi/2}\right) + (1 - \mu)\beta,
\]
then
\[
\frac{zf'(z)}{f(z)} < \left(\frac{1+z}{1-z}\right)^{\beta}
\]
and \( \alpha(\beta, \lambda, \mu, m) \) given by (2.2) is the largest number such that (2.3) holds.

**Proof.** Let \( f(z) \in A \) with \( (f(z)/z)f'(z) \neq 0 \) (\( z \in U \)), and define the function \( p(z) \) in \( U \) by
\[
p(z) = \frac{zf'(z)}{f(z)}.
\]

Then \( p(z) \) is analytic in \( U \) and
\[
\lambda\left(\frac{zf'(z)}{f(z)}\right)^{1-\mu} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right)^m = \lambda \frac{zp'(z)}{p^\mu(z)} + p''(z).
\]
Let $0 < \beta < 1, \mu \in \{0, 1, 2\}, m$ an integer, $|m + \mu - 1|\beta < 1$, $m\lambda \geq 0$, $\lambda \neq 0$, and

\[
D = \begin{cases}
C, & \mu = 0, m \geq 0, \\
C \setminus \{0\}, & \mu = 1, 2 \text{ or } m < 0,
\end{cases}
\tag{2.6}
\]

and choose

\[
g(z) = \left(\frac{1 + z}{1 - z}\right)^\beta, \quad \theta(w) = w^m, \quad \varphi(w) = \frac{\lambda}{\omega^\beta}.
\tag{2.7}
\]

Then $g(z)$ is analytic and univalent in $U$, $p(0) = g(0) = 1$, $p(U) \subset D$, and $\theta(w)$ and $\varphi(w)$ satisfy the conditions of the lemma. The function

\[
Q(z) = zg'(z)\varphi(g(z)) = \frac{2\lambda\beta z}{(1 + z)^{(m+\mu-1)\beta}(1 - z)^{(1 - (\mu - 1)\beta)}}
\tag{2.8}
\]

is univalent and starlike in $U$ because

\[
\Re \frac{zQ'(z)}{Q(z)} = 1 + (1 + (\mu - 1)\beta) \Re \left(-\frac{z}{1 + z}\right) + (1 - (\mu - 1)\beta) \Re \frac{z}{1 - z} > 0 \quad (z \in U)
\tag{2.9}
\]

for $|1 - (\mu - 1)\beta < 1$. Further, we have that

\[
\theta(g(z)) + Q(z) = \left(\frac{1 + z}{1 - z}\right)^{m\beta} \frac{2\lambda\beta z}{(1 + z)^{(1 + (\mu - 1)\beta)}(1 - z)^{(1 - (\mu - 1)\beta)}} = h(z) \quad \text{(say)},
\tag{2.10}
\]

\[
\frac{zh'(z)}{Q(z)} = \frac{m}{\lambda} \left(\frac{1 + z}{1 - z}\right)^{(m+\mu-1)\beta} + \frac{zQ'(z)}{Q(z)} \quad (z \in U).
\tag{2.11}
\]

Since $0 \leq |m + \mu - 1|\beta < 1$, we have that

\[
\left|\arg \left(\frac{1 + z}{1 - z}\right)^{(m+\mu-1)\beta}\right| < \frac{\pi}{2} \quad (z \in U)
\tag{2.12}
\]

and so

\[
\Re \frac{zh'(z)}{Q(z)} = \frac{m}{\lambda} \Re \left\{\left(\frac{1 + z}{1 - z}\right)^{(m+\mu-1)\beta}\right\} + \Re \frac{zQ'(z)}{Q(z)} > 0 \quad (z \in U).
\tag{2.13}
\]

Inequality (2.13) shows that the function $h(z)$ is close-to-convex and univalent in $U$. Letting $0 < \theta < \pi$ and $x = \cot(\theta/2) > 0$, then

\[
h(e^{i\theta}) = \left(x^{m\beta} e^{(m+\mu-1)\beta x/2} + \frac{\lambda\beta i}{2} \left(x^{1 - (\mu - 1)\beta} + \frac{1}{x^{1 - (\mu - 1)\beta}}\right)e^{(1-\mu)\beta x/2}\right).
\tag{2.14}
\]
and so
\[ \arg h(e^{i\theta}) = \arctan w(x) + \frac{(1 - \mu)\beta \pi}{2}, \]  
(2.15)

where
\[ w(x) = \tan \left( \frac{(m + \mu - 1)\beta \pi}{2} \right) + \frac{\lambda \beta}{2 \cos (m + \mu - 1)\beta \pi/2} \left( x^{1-(m+\mu-1)\beta} + \frac{1}{x^{1+(m+\mu-1)\beta}} \right). \]  
(2.16)

It is easy to know that \( w(x) \) takes its minimum value at \( x = \sqrt{1 + (m + \mu - 1)\beta}/1 - (m + \mu - 1)\beta \). Hence, in view of \( h(e^{-i\theta}) = \overline{h(e^{i\theta})} \), we deduce that
\[ \inf_{|z| \neq 1} |\arg h(z)| = \arctan \left( \frac{\sqrt{1 + (m + \mu - 1)\beta}}{1 - (m + \mu - 1)\beta} \right) + \frac{(1 - \mu)\beta \pi}{2} \]  
(2.17)

where \( a(\beta, \lambda, \mu, m) \) is given by (2.2).

Now, if \( f(z) \) satisfies (2.1), it follows from (2.17) that the subordination
\[ \lambda \left( \frac{zf'(z)}{f(z)} \right)^{1-\mu} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) + \left( \frac{zf'(z)}{f(z)} \right)^m < h(z) \quad (z \in U) \]  
(2.18)

holds. Hence it follows from (2.5), (2.7), (2.10), and (2.18) that
\[ \theta(p(z)) + zp'(z)\varphi(p(z)) < \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z) \]  
(2.19)

holds. Therefore, by virtue of the lemma, we conclude that \( p(z) < g(z) \), that is, (2.3) holds.

Next we consider the extremal function
\[ f(z) = z \exp \left\{ \int_0^x \frac{1}{t} \left( \left( \frac{1+t}{1-t} \right)^{\beta} - 1 \right) dt \right\}. \]  
(2.20)

Then \( f(z) \) satisfies
\[ \lambda \left( \frac{zf'(z)}{f(z)} \right)^{1-\mu} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) + \left( \frac{zf'(z)}{f(z)} \right)^m = \left( \frac{1+z}{1-z} \right)^{m\beta} + \frac{2\lambda \beta z}{(1+z)^{1+(\mu-1)\beta}(1-z)^{1-(\mu-1)\beta}} = h(z), \]  
(2.21)

and it follows from (2.17) that the bound \( a(\beta, \lambda, \mu, m) \) in (2.1) is sharp. The proof of the theorem is complete.
Making use of the theorem, we can obtain a number of interesting results. Letting $\mu = 0$ and $m = 2$ in the theorem, we have the following corollary.

**Corollary 2.2.** Let $0 < \beta < 1$, and $\lambda > 0$. If $f(z) \in A$ satisfies $f(z) \neq 0$ ($0 < |z| < 1$) and

$$\frac{zf'(z)}{f(z)} \left( \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \lambda) \frac{zf'(z)}{f(z)} \right) < \left( \frac{1 + z}{1 - z} \right)^{\alpha_3(\beta, \lambda)} \quad (z \in U),$$

(2.22)

where

$$\alpha_3(\beta, \lambda) = \frac{2}{\pi} \arctan \left( \frac{\tan \frac{\beta \pi}{2} + \frac{\lambda \beta}{(1 - \beta)(1 + \beta)(1 + \beta)^{1/2}/2 \cos \beta \pi/2}}{(1 - \beta)^{1-\beta/2}(1 + \beta)^{1+\beta/2}} \right) + \beta,$$

(2.23)

then

$$\frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta}$$

(2.24)

and $\alpha_3(\beta, \lambda)$ given by (2.23) is the largest number such that (2.24) holds.

**Remark 2.3.** Ramesha et al. [6] have proved that, if $f(z) \in A$ satisfies $f(z) \neq 0$ ($0 < |z| < 1$) and

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \quad (z \in U),$$

(2.25)

then $f(z) \in \bar{S}(1/2)$.

For $\beta = 1/2$ and $\lambda > 0$, it follows from (2.23) that $\alpha_3(1/2, \lambda) > 1$. Hence, the image of $w = \left( (1 + z)/(1 - z) \right)^{\alpha_3(1/2, \lambda)} (z \in U)$ is a region which properly contains the right half plane. Thus we conclude that Corollary 2.2 with $\beta = 1/2$ and $\lambda = 1$ is better than the result of Ramesha et al. [6].

Letting $\alpha_3(\beta, \lambda) = 1$ in Corollary 2.2, we have the following corollary.

**Corollary 2.4.** Let $\lambda > 0$. If $f(z) \in A$ satisfies $f(z) \neq 0$ ($0 < |z| < 1$) and

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \lambda) \frac{zf'(z)}{f(z)} \right) \right\} > 0 \quad (z \in U),$$

(2.26)

then $f(z) \in \bar{S}(\beta)$, where $\beta \in (0, 1)$ is the root of the equation

$$\frac{2}{\pi} \arctan \left( \frac{\tan \frac{\beta \pi}{2} + \frac{\lambda}{(1 - \beta)(1 + \beta)(1 + \beta)^{1/2}/2 \cos \beta \pi/2}}{(1 - \beta)^{1-\beta/2}(1 + \beta)^{1+\beta/2}} \right) + \beta = 1.$$

(2.27)

**Remark 2.5.** For $\lambda = 1$, Corollary 2.4 reduces to a main result of Nunokawa et al. [5, Theorem 1] by a different method.
Remark 2.6. Note that \( \lim_{\beta \to 1} \alpha_3(\beta, 1) = 2 \). Hence it follows from Corollary 2.2 that

\[
\left| \arg \left\{ \frac{zf''(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \right| < \pi \quad (z \in U)
\] (2.28)

implies

\[
f(z) \in S^*.
\] (2.29)

Letting \( \mu = m = 1 \) in the theorem, we have the following corollary.

**Corollary 2.7.** Let \( 0 < \beta < 1, \lambda \beta > 0 \). If \( f(z) \in A \) satisfies \( (f(z)/z)f'(z) \neq 0 \) \((z \in U)\) and

\[
\lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \lambda) \frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\alpha_1(\beta, \lambda)} \quad (z \in U),
\] (2.30)

where

\[
\alpha_1(\beta, \lambda) = \frac{2}{\pi} \arctan \left( \tan \frac{\beta \pi}{2} + \frac{\lambda \beta}{(1 - \beta)^{1/2} (1 + \beta)^{1/2} \cos \beta \pi / 2} \right),
\] (2.31)

then

\[
\frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta}
\] (2.32)

and \( \alpha_1(\beta, \lambda) \) given by (2.31) is the largest number such that (2.32) holds.

**Remark 2.8.** Marjono and Thomas [2] proved the above result by a different method. For \( \lambda = \delta / \gamma \) \((\delta > 0, \gamma > 0)\) Corollary 2.2 reduces to a result of Darus [1]. For \( \lambda = 1 \), Corollary 2.2 reduces to a result of Nunokawa and Thomas [4].

Letting \( \mu = 2 \) and \( m = 0 \) in the theorem, we have the following corollary.

**Corollary 2.9.** Let \( 0 < \beta < 1 \) and \( \lambda \neq 0 \). If \( f(z) \in A \) satisfies \( f'(z) \neq 0 \) \((0 < |z| < 1)\) and

\[
\frac{\lambda (1 + zf''(z)) / f'(z) + (1 - \lambda) zf'(z) / f(z)}{zf'(z) / f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\alpha_2(\beta, \lambda)} \quad (z \in U),
\] (2.33)

where

\[
\alpha_2(\beta, \lambda) = \frac{2}{\pi} \arctan \left( \tan \frac{\beta \pi}{2} + \frac{\lambda \beta}{(1 - \beta)^{1/2} (1 + \beta)^{1/2} \cos \beta \pi / 2} \right) - \beta,
\] (2.34)
then

$$\frac{zf'(z)}{f(z)} < \left(\frac{1 + z}{1 - z}\right)^\beta$$

(2.35)

and $\alpha_2(\beta, \lambda)$ given by (2.34) is the largest number such that (2.35) holds.

Remark 2.10. For $\lambda = 1$, Corollary 2.9 reduces to a result of Ravichandran and Darus [7].

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References

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