Research Article

Multiple Periodic Solutions to a Suspension Bridge Wave Equation with Damping

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This paper is concerned with the existence of multiple periodic solutions for a suspension bridge wave equation with damping. By using Leray-Schauder degree theory, the authors prove that the damped wave equation has multiple periodic solutions.

1. Introduction

In [1], also see [2–6], the author considered a horizontal cross-section of the center span of a suspension bridge and proposed a partial differential equation model for the torsional motion of the cross-section and treated the center span of the bridge as a beam of length $L$ and width $2l$ suspended by cables. Consider the horizontal cross-section of mass $m$ located at position $x$ along the length of the span. She treated this cross-section as a rod of length $2l$ and mass $m$ suspended by cables. Let $y(x,t)$ denote the downward distance of the center of gravity of the rod from the unloaded state and let $\theta(x,t)$ denote the angle of the rod from horizontal at time $t$. Assume that the cables do not resist compression, but resist elongation according to Hooke’s law with spring constant $K$. Then the torsional and vertical motion of the span satisfy

$$
\theta_{tt} - \varepsilon_1 \theta_{xx} = \frac{3K}{ml} \cos[(y - l \sin \theta)^+ - \delta \theta] + h_1(x,t),
$$

$$
y_{tt} + \varepsilon_2 y_{xxxx} = -\frac{K}{m} [(y - l \sin \theta)^+ + (y + l \sin \theta)^+] - \delta y_t + g + h_2(x,t),
$$

$$
\theta(0,t) = \theta(L,t) = y(0,t) = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0,
$$

(1.1)
where \( u^* = \max \{u, 0\}, \varepsilon_1, \varepsilon_2 \) are physical constants related to the flexibility of the beam, \( \delta \) is the damping constant, \( h_1 \) and \( h_2 \) are external forcing terms, and \( g \) is the acceleration due to gravity. The spatial derivatives describe the restoring force that the beam exerts, and the time derivatives \( \theta_t \) and \( y_t \) represent the force due to friction. The boundary conditions reflect the fact that the ends of the span are hinged.

Throughout the paper [1] the author assumes that the cables never lose tension; that is, it is assumed that \((y \pm \sin \theta) \geq 0\). In this case, we see that (1.1) becomes uncoupled, and the torsional and vertical motions satisfy, respectively,

\[
\theta_t - \varepsilon_1 \theta_{xx} = \frac{6K}{m} \cos \theta \sin \theta - \delta \theta_t + h_1(x, t),
\]

\[
\theta(0, t) = \theta(L, t) = 0,
\]

\[
y_t + \varepsilon_2 y_{xxxx} = -\frac{2K}{m} y - \delta y_t + g + h_2(x, t),
\]

\[
y(0, t) = y(L, t) = y_{xx}(0, t) = y_{xx}(L, t) = 0.
\]

In paper [1], removing the damping term; that is, let \( \delta = 0 \), changing variables, and imposing boundary and periodicity conditions, the author rewrites (1.2) as

\[
u_{tt} - u_{xx} + b \sin u = \varepsilon h(x, t),
\]

\[
u(0, t) = \nu(\pi, t) = 0,
\]

\[
u(x, 0) = \nu(x, \pi), \quad \nu_t(x, 0) = \nu_t(x, \pi),
\]

\[
u(x, t) = \nu(\pi - x, t), \quad \nu(x, t) = \nu(x, \pi - t).
\]

And it proves that (1.4) has at least two solutions in the subspace \( H \) of \( L^2 \). Where \( H \) is defined as

\[
H = \{ \nu \in L^2(\Omega) u(x, t) = u(\pi - x, t), u(x, t) = u(x, \pi - t), \; \nu \text{ is } \pi - \text{periodic in } t \}. \tag{1.5}
\]

Notice that (1.4) is particular in no damping and the selection of \( H \). Hence, in [1] the author left a problem which is relevant to this case.

**Problem 1.** “Under appropriate hypotheses on the forcing term, does a similar result hold for the damped equation?”

Motivated by this problem, in this paper, we suppose that the damping is present, that is, \( \delta \neq 0 \), and study the following problem:

\[
u_{tt} - u_{xx} + \delta u_t + b \sin u = \varepsilon h(x, t),
\]

\[
u(0, t) = \nu(\pi, t) = 0,
\]

\[
u(x, 0) = \nu(x, \pi), \quad \nu_t(x, t) = \nu_t(x, \pi),
\]

\[
u(x, t) = \nu(\pi - x, t).
\]

\[
\theta_t - \varepsilon_1 \theta_{xx} = \frac{6K}{m} \cos \theta \sin \theta - \delta \theta_t + h_1(x, t),
\]
2. Preliminaries

Let \( N = \{0, 1, \ldots\} \) and \( Z \) be the set of integers, \( \Lambda = N \times N \). Let \( \Omega = (0, \pi) \times (0, \pi) \) and \( L^2(\Omega) \) be the usual space of square integrable functions with usual inner product \( \langle \cdot, \cdot \rangle \) and corresponding norm \( \| \cdot \| \). For the Sobolev space \( H^1(\Omega) \), we denote the standard inner product by \( \langle u, v \rangle_1 = (u, v) + (u_x, v_x) + (u_t, v_t) \) and norm by \( \| u \|_1 \).

Define the operator \( L_\delta u = u_{tt} - u_{xx} + \delta u_t : H \to H \) by

\[
D(L_\delta) = \left\{ u \in H \mid u(x, t) = \sum_{\Lambda} u_{mn} \Phi_{mn} \sum_{\Lambda} \left( \left( (2n + 1)^2 - 4m^2 \right)^2 + 4m^2 \delta^2 \right) |u_{mn}|^2 < \infty \right\},
\]

\[
L_\delta u = \sum_{\Lambda} \left( (2n + 1)^2 - 4m^2 + 2m\delta \right) u_{mn} \Phi_{mn}, \quad \text{for all} \ u \in H.
\]

(2.1)

We know that the eigenvalues and corresponding eigenfunctions of \( L_\delta \) are

\[
\lambda_{mn} = (2n + 1)^2 - 4m^2 + 2m\delta, \quad (m, n) \in \Lambda,
\]

\[
\Phi_{mn}(x, t) = e^{2mti} \sin(2n + 1)x, \quad (m, n) \in \Lambda.
\]

(2.2)

In order to seek the solutions of (1.6), we first investigate the properties of operator \( L_\delta \). We have the following Lemma.

**Lemma 2.1.** \( L_\delta^{-1} \) exists, \( L_\delta^{-1} : H \to H \) is compact, and \( \| L_\delta^{-1} \| = 1 \).

**Proof.** Because we are restricted to the subspace \( H \) of \( L^2 \), and \( \lambda_{mn} \neq 0 \), we easily know \( L_\delta^{-1} \) exists.

We prove \( L_\delta^{-1} : H \to H \) is compact below. We find that

\[
L_\delta^{-1} u = \sum_{\Lambda} \frac{1}{(2n + 1)^2 - 4m^2 + 2m\delta} u_{mn} \Phi_{mn},
\]

(2.3)

for all \( u = \sum_{\Lambda} u_{mn} \Phi_{mn} \in H \). For any \( (m, n) \in \Lambda \), we have

\[
\left| (2n + 1)^2 - 4m^2 + 2m\delta \right|^2 \geq 1,
\]

(2.4)

then

\[
\left\| L_\delta^{-1} u \right\|^2 = \sum_{\Lambda} \left| \frac{1}{(2n + 1)^2 - 4m^2 + 2m\delta} u_{mn} \right|^2 \leq \sum_{\Lambda} |u_{mn}|^2 = \| u \|^2.
\]

(2.5)
Hence,
\[ \|L^{-1}_\delta u\| \leq \|u\|. \] (2.6)

On the other hand,
\[ \left\| L^{-1}_\delta u \right\|_1^2 \leq \sum_\Lambda \left| \frac{1 + (2n + 1)^2 + 4m^2}{(2n + 1)^2 - 4m^2 + 4m^2 \delta^2} u_{mn} \right|^2, \] (2.7)

while
\[
\frac{1 + (2n + 1)^2 + 4m^2}{(2n + 1)^2 - 4m^2 + 4m^2 \delta^2} = \frac{1 + (2n + 1)^2 + 4|m|^2}{(2n + 2|m| + 1)^2 + 4|m|^2 \delta^2} \\
\leq \frac{(2n + 2|m| + 1)^2 + 1}{(2n + 2|m| + 1)^2} \\
\leq 2.
\]

Hence,
\[ \left\| L^{-1}_\delta u \right\|_1^2 \leq 2 \sum_\Lambda |u|^2 = 2\|u\|. \] (2.9)

By (2.6) and (2.9), we can find that the operator \(L^{-1}_\delta : H \to H\) is compact since the embedding \(H^1 \to L^2\) is compact.

Finally, we prove \(\|L^{-1}_\delta\| = 1\). By (2.2) and
\[ |\lambda_{mn}|^2 = \left| (2n + 1)^2 - 4m^2 + 2m6i \right|^2 \geq 1. \] (2.10)

Set \(u = \Phi_{00}\), such that \(\|L^{-1}_\delta u\|/\|u\| = 1\). Therefore,
\[ \left\| L^{-1}_\delta \right\| = 1. \] (2.11)

Hence, we complete the proof of this lemma. \(\square\)
Definition 2.2. One says that \( u \in H \) is a solution to (1.6) if
\[
    u = L_0^{-1}(\varepsilon h - b \sin u). \tag{2.12}
\]

To establish the existence of multiple periodic solutions to (1.6), we use Leray-Schauder degree theory to prove the existence of multiple zeros of a related operator \( T_1 \). To compute the degree of \( T_1 \), we continuously deform it to a linear operator \( T_0 \), the Gâteaux derivative of \( T_1 \), and compute its degree via a direct calculation.

It is not difficult to show that the homotopy property of Leray-Schauder degree ensures that the degree of an operator \( T_1 \) is preserved as \( T_1 \) is continuously deformed to its Fréchet derivative under appropriate hypotheses. However, the nonlinear term in (1.6), \( f(u) = \sin u \), is not Fréchet differentiable in \( L^2 \) at \( u = 0 \).

There is a theorem in paper [1], in which, the author shows that, under certain conditions on the nonlinear term \( f \) and the differential operator \( L \), Leray-Schauder degree is indeed preserved under homotopy from the operator \( T_1 \) to its Gâteaux derivative \( T_0 \). This result can be used to establish multiplicity of solutions to equations of the form (1.6). The result follows.

Lemma 2.3. Let \( I_1, I_2 \) be open, bounded intervals in \( \mathbb{R} \), and define \( Q := I_1 \times I_2 \). Let \( B \) be a subspace of \( L^p(Q) \), \( p \geq 1 \), and define \( \|u\| := \|u\|_{L^p} \). Consider the problem
\[
    Lu + f(u) = \varepsilon h(x,t), \tag{2.13}
\]
where \( L, f, \) and \( h \) satisfy the following:

(H1) \( L^{-1} \) is compact;
(H2) \( \|L^{-1}\| \leq 1 \);
(H3) \( f(0) = 0 \);
(H4) \( f \) is Lipschitz with Lipschitz constant \( M \);
(H5) \( h \in B \) and \( h \leq 1 \);
(H6) the Gâteaux derivative \( df(0, u) \) exists and satisfies \( df(0, u) = \rho u \), where \( \rho > 0 \) and \(-\rho\) is not an eigenvalue of \( L \).

Define \( T_0 : B \to B \) by
\[
    T_0(u) = u + \rho L^{-1}(u), \tag{2.14}
\]
and \( T_1 : B \to B \) by
\[
    T_1(u) = u - L^{-1}(\varepsilon h - f(u)). \tag{2.15}
\]

Then for \( \varepsilon \) sufficiently small, there exists \( \gamma > 0 \) such that
\[
    \deg(T_1, B_{\gamma}(0), 0) = \deg(T, B_{\gamma}(0), 0). \tag{2.16}
\]
3. Result and Proof

The main result of this paper is as follows.

**Theorem 3.1.** Let \( h \in H \) with \( \| h \| \leq 1 \), and let \( b \in (-\sqrt{25 + 4\delta^2}, -\sqrt{9 + 4\delta^2}) \), \( 0 < \delta^2 < 14 \). Then there exists \( \epsilon_0 > 0 \) such that if \( |\epsilon| < \epsilon_0 \), (1.6) has at least two solutions in \( H \).

Proof. Let \( L = L_\delta \) and \( f(u) = b \sin u \), it is easy to know that \( L \) and \( f \) satisfy the conditions (H1–H5) in Lemma 2.3.

Reply Lemma 2.3, we define \( T_0: H \to H \) by

\[
T_0(u) = u + bL_\delta^{-1}(u),
\]

and \( T_1: H \to H \) by

\[
T_1(u) = u - L_\delta^{-1}(\epsilon h - b \sin(u)).
\]

And note that zeros of \( T_1 \) correspond to solutions of (1.6). To prove the theorem, we will show the following:

- (C1) there exists \( R_0 > 0 \) such that for \( R > R_0 \), \( \deg(T_1, B_R(0), 0) = 1 \);
- (C2) there exists \( \gamma \in (0, R_0) \) such that \( \deg(T_1, B_\gamma(0), 0) = -1 \).

Then, since \( \deg(T_1, B_\gamma(0), 0) \neq 0 \), there exists a zero of \( T_1 \) (i.e., a solution of (1.6)) in \( B_\gamma(0) \). Moreover, by the additivity property of degree, \( \deg(T_1, B_R(0) \setminus B_\gamma(0), 0) \neq 0 \) and hence (1.6) has a second solution in the annulus \( B_R(0) \setminus B_\gamma(0) \).

To establish (C1), define

\[
T_\beta u = u - \beta L_\delta^{-1}(\epsilon h - b \sin(u)),
\]

or \( \beta \in [0, 1] \), and note that this definition of \( T_1 \) is consistent with our previous definition. Note also that \( T_0 \) is simply the identity map; hence, for any \( R > 0 \) we have \( \deg(T_0, B_R(0), 0) = 1 \).

The homotopy property of degree ensures that \( \deg(T_\beta, B_R(0), 0) \) is constant provided that

\[
0 \in T_\beta(\partial B_R(0)) \quad \text{for all} \quad \beta \in [0, 1].
\]

Fix \( \beta \in [0, 1] \) and suppose \( u \in H \) solves \( T_\beta u = 0 \). We will show that \( u \) is bounded above by some \( R_0 > 0 \) and that this bound is independent of \( \beta \).

Since \( T_\beta u = 0 \), we have

\[
\|u\| = \beta \left\| L_\delta^{-1}(\epsilon h - b \sin u) \right\| \leq \beta(\epsilon_0 + b\|\sin u\|)
\]

\[
\leq \left[ \epsilon_0 + b \sin(\Omega) \right] < \left[ \epsilon_0 + b \sqrt{2\pi} \right] < R_0,
\]

if we choose \( R_0 > \epsilon_0 + b \sqrt{2\pi} \).

Thus, for \( R > R_0 \), we have

\[
\deg(T_1, B_R(0), 0) = \deg(T_0, B_R(0), 0) = 1,
\]

and (C1) above holds.
To establish (C2), let \( \varepsilon < \varepsilon_0 \); we will determine the value of \( \varepsilon_0 \) later. For \( \mu \in [0, 1] \) define

\[
T_\mu u = u + (1 - \mu)L^{-1}_0(\varepsilon h - b \sin u),
\]

and note again that this definition of \( T_1 \) is consistent with our previous definitions. We will again apply the homotopy property of degree (via Lemma 2.3) and a standard degree calculation to show that for some \( \gamma > 0 \)

\[
\text{deg}(T_1, B_\gamma(0), 0) = \text{deg}(T_0, B_\gamma(0), 0) = -1.
\]

Observe that for \( L = L_\delta \) and \( f(u) := b \sin u \), hypotheses (H1)–(H5) of Lemma 2.3 are satisfied. To verify hypothesis (H6), we need to show that

\[
df(0, u) = bu.
\]

By definition of the \( Gâteaux \) derivative,

\[
df(0, u) = \frac{d}{dt}f(0 + tu)|_{t=0} = \lim_{h \to 0} \frac{f((t + h)u) - f(tu)}{h} |_{t=0} = \lim_{h \to 0} \frac{b \sin(hu)}{h}.
\]

We will show that the limit above (in \( H \)) is \( bu \).

Note first that in \( R \) we have

\[
\lim_{h \to 0} \frac{\sin(hu)}{h} = \lim_{h \to 0} \frac{\sin(hu)}{h} \cdot \frac{u}{u} = u,
\]

and hence

\[
\left| \frac{\sin(hu)}{h} - u \right|^2 \to 0,
\]

as \( h \to 0 \). Invoking the convexity of \( u^2 \), we have

\[
\left| \frac{\sin(hu)}{h} - u \right|^2 \leq 4 \left[ \frac{1}{2} \left| \frac{\sin(hu)}{h} \right|^2 + \frac{1}{2} |u|^2 \right] \leq 4u^2.
\]
Since \( u \in L^2 \), \(|(\sin(hu)/h) - u|^2\) is dominated in \( L^1 \); thus by the dominated convergence theorem,
\[
\left\| \frac{b \sin(hu)}{h} - bu \right\| \to 0,
\]
(3.13)
as \( h \to 0 \); therefore (3.8) holds. Moreover, by the form of eigenvalue of \( L_\delta \) and our choice of \( b \), \(-b\) is not an eigenvalue of \( L_\delta \); therefore hypothesis (H6) of Lemma 2.3 holds. Thus, by Lemma 2.3, for sufficiently small \( \gamma, \varepsilon > 0 \), we have
\[
\deg(T_0, B_1(0), 0) = \deg(T_1, B_1(0), 0).
\]
(3.14)
Finally, we will show that
\[
\deg(T_0, B_1(0), 0) = \deg(I + bL^{-1}_\delta, B_1(0), 0) = -1.
\]
(3.15)
Consider the finite dimensional subspace \( MN = \text{span}\{\Phi_{mn}\}_{n=0}^N \) of \( H \) and recall that, by compactness, \( bL^{-1}_\delta \) can be approximated in operator norm by the operators \( B_N : M_N \to M_N \) given by
\[
B_N(u) = b \sum_{N} \sum_{N} \frac{c_{mn}}{\lambda_{mn}} \Phi_{mn}.
\]
(3.16)
By definition of Leray-Schauder degree, for \( N \) sufficiently large,
\[
\deg(T_0, B_1(0), 0) = \deg\left(I + B_N, B_1(0) \cap M_N, 0\right) = \sum_{u \in (I + B_N)^{-1}(0)} \text{sgn} J_{I + B_N}(u),
\]
(3.17)
where \( J_{\Phi}(u) \) is the Jacobian determinant of \( \Phi \) at \( u \).
Since \( I + B_N \) can be identified with an \((2N + 1)^2 \times (2N + 1)^2\) diagonal matrix whose entries are \( 1 + b/\lambda_{mn} \), we have
\[
\deg\left(I + B_N, B_1(0) \cap M_N, 0\right) = \text{sgn} \prod_{m=-N}^{N} \prod_{n=-N}^{N} \left(1 + \frac{b}{\lambda_{mn}}\right).
\]
(3.18)
Now we consider the following two cases.

(D1) If \( \lambda_{mn} \) contains imaginary part, suppose a pair of conjugate complex numbers are \( a \pm ci \) (\( c \neq 0 \)), then,
\[
\left(1 + \frac{b}{a + ci}\right)\left(1 + \frac{b}{a - ci}\right) = \frac{(a + b)^2 + c^2}{a^2 + c^2} > 0.
\]
(3.19)
(D2) If $\lambda_{mn}$ is real, then $m = 0$, here $\lambda_{nn} = (2n + 1)^2$.

Since $b \in (-\sqrt{25 + 4\delta^2}, -\sqrt{9 + 4\delta^2})$, and $0 < \delta^2 < 14$, the only negative value of $1 + b/(2n + 1)^2$ occurs at $\lambda_{30}$. Therefore,

$$\deg\left(I + B_N, B_T(0) \cap M_N, 0\right) = -1. \quad (3.20)$$

The proof of the theorem is complete.

References


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