Research Article

Some Results on Lorentzian Para-Sasakian Manifolds

Venkatesha, C. S. Bagewadi, and K. T. Pradeep Kumar

Department of Mathematics, Kuvempu University, Shankaraghatta, Karnataka, Shimoga 577 451, India

Correspondence should be addressed to Venkatesha, vensprem@gmail.com

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The object of the present paper is to study Lorentzian para-Sasakian (briefly LP-Sasakian) manifolds satisfying certain conditions on the W_2 -curvature tensor.

1. Introduction

In 1989, Matsumoto [1] introduced the notion of Lorentzian para-Sasakian manifold. Then, Mihai and Roşca [2] introduced the same notion independently and they obtained several results on this manifold. LP-Sasakian manifolds have also been studied by Matsumoto and Mihai [3], Mihai et al. [4], and Venkatesha and Bagewadi [5].

On the other hand, Pokhariyal and Mishra [6] have introduced new curvature tensor called W_2 -curvature tensor in a Riemannian manifold and studied their properties. Further, Pokhariyal [7] has studied some properties of this curvature tensor in a Sasakian manifold. Matsumoto et al. [8], and Yıldız and De [9] have studied W_2 -curvature tensor in P-Sasakian and Kenmotsu manifolds, respectively.

In the present paper, we study some curvature conditions on LP-Sasakian manifolds. Firstly, we study LP-Sasakian manifolds satisfying $W_2 = 0$ and W_2 -semisymmetric LP-Sasakian manifolds. Further, we study LP-Sasakian manifolds which satisfy $P \cdot W_2 = 0$, $\widetilde{M} \cdot W_2 = 0$ and $C \cdot W_2 = 0$, where P is the projective curvature tensor, $\widetilde{M}$ is the M -projective curvature tensor, and C is the conformal curvature tensor.

2. Preliminaries

An n -dimensional differentiable manifold M is called an LP-Sasakian manifold [1, 2] if it admits a $(1,1)$ tensor field ϕ , a contravariant vector field ξ , a 1-form η , and a Lorentzian

metric g which satisfy

$$\begin{aligned}
 \eta(\xi) &= -1, \\
 \phi^2 X &= X + \eta(X)\xi, \\
 g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \\
 g(X, \xi) &= \eta(X), \\
 \nabla_X \xi &= \phi X, \\
 (\nabla_X \phi)Y &= g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,
 \end{aligned} \tag{2.1}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g .

It can be easily seen that, in an LP-Sasakian manifold, the following relations hold:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \text{rank } \phi = n - 1. \tag{2.2}$$

If we put

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.3}$$

for any vector fields X and Y , then the tensor field $\Phi(X, Y)$ is a symmetric $(0, 2)$ tensor field [1]. Also, since the 1-form η is closed in an LP-Sasakian manifold, we have [1, 4]

$$(\nabla_X \eta)(Y) = \Phi(X, Y), \quad \Phi(X, \xi) = 0, \tag{2.4}$$

for any vector fields X and Y .

Also in an LP-Sasakian manifold, the following relations hold [3, 4]:

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.5}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.6}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$S(X, \xi) = (n - 1)\eta(X), \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \tag{2.9}$$

for any vector fields X , Y , and Z , where R is the Riemannian curvature tensor and S is the Ricci tensor of M .

An LP-Sasakian manifold M is said to be Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y), \quad (2.10)$$

for any vector fields X and Y , where a is a function on M .

In [6], Pokhariyal and Mishra have defined the curvature tensor W_2 , given by

$$W_2(X, Y, U, V) = R(X, Y, U, V) + \frac{1}{n-1} [g(X, U)S(Y, V) - g(Y, U)S(X, V)], \quad (2.11)$$

where S is a Ricci tensor of type $(0, 2)$.

Now, consider an LP-Sasakian manifold satisfying $W_2 = 0$; then, (2.11) becomes

$$R(X, Y, U, V) = \frac{1}{n-1} [g(Y, U)S(X, V) - g(X, U)S(Y, V)]. \quad (2.12)$$

Taking $X = U = \xi$ in (2.12) and using (2.7) and (2.8), we have

$$S(Y, V) = (n-1)g(Y, V). \quad (2.13)$$

Therefore, M is an Einstein manifold.

Again using (2.13) in (2.12), we get

$$R(X, Y, U, V) = [g(Y, U)g(X, V) - g(X, U)g(Y, V)]. \quad (2.14)$$

Corollary 2.1. *An LP-Sasakian manifold satisfying $W_2 = 0$ is a space of constant curvature -1 , that is, it is locally isometric to the hyperbolic space.*

Definition 2.2. An LP-Sasakian manifold is called W_2 -semisymmetric if it satisfies

$$R(X, Y) \cdot W_2 = 0, \quad (2.15)$$

where $R(X, Y)$ is to be considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X and Y .

It can be easily shown that in an LP-Sasakian manifold the W_2 -curvature tensor satisfies the condition

$$\eta(W_2(X, Y)Z) = 0. \quad (2.16)$$

Theorem 2.3. *A W_2 -semisymmetric LP-Sasakian manifold M is an Einstein manifold.*

Proof. Since $R(X, Y) \cdot W_2 = 0$, we have

$$\begin{aligned} & R(X, Y)W_2(U, V)Z - W_2(R(X, Y)U, V)Z \\ & - W_2(U, R(X, Y)V)Z - W_2(U, V)R(X, Y)Z = 0. \end{aligned} \quad (2.17)$$

Putting $X = \xi$ in (2.17) and then taking the inner product with ξ , we obtain

$$\begin{aligned} & g(R(\xi, Y)W_2(U, V)Z, \xi) - g(W_2(R(\xi, Y)U, V)Z, \xi) \\ & - g(W_2(U, R(\xi, Y)V)Z, \xi) - g(W_2(U, V)R(\xi, Y)Z, \xi) = 0. \end{aligned} \quad (2.18)$$

Using (2.6) in (2.18), we obtain

$$\begin{aligned} & -g(Y, W_2(U, V)Z) - \eta(W_2(U, V)Z)\eta(Y) - g(Y, U)\eta(W_2(\xi, V)Z) \\ & - g(Y, V)\eta(W_2(U, \xi)Z) - g(Y, Z)\eta(W_2(U, V)\xi) + \eta(U)\eta(W_2(Y, V)Z) \\ & + \eta(V)\eta(W_2(U, Y)Z) + \eta(Z)\eta(W_2(U, V)Y) = 0. \end{aligned} \quad (2.19)$$

By using (2.16) in (2.19), we get

$$W_2(U, V, Z, Y) = 0. \quad (2.20)$$

In view of (2.11) and (2.20), it follows that

$$R(U, V, Z, Y) = \frac{1}{n-1} [g(V, Z)S(U, Y) - g(U, Z)S(V, Y)]. \quad (2.21)$$

Contracting (2.21), we have

$$S(V, Z) = (n-1)g(V, Z). \quad (2.22)$$

Therefore, M is an Einstein manifold. $\square$

Again using (2.22) in (2.12), we get

$$R(U, V, Z, Y) = [g(V, Z)g(U, Y) - g(U, Z)g(V, Y)]. \quad (2.23)$$

Corollary 2.4. *A W_2 -semisymmetric LP-Sasakian manifold is a space of constant curvature -1 , that is, it is locally isometric to the hyperbolic space.*

3. LP-Sasakian Manifolds Satisfying $P(X, Y) \cdot W_2 = 0$

The projective curvature tensor P is defined as [10]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]. \quad (3.1)$$

Using (2.6) and (2.8), (3.1) reduces to

$$P(\xi, Y)Z = g(Y, Z)\xi - \frac{1}{n-1}S(Y, Z)\xi. \quad (3.2)$$

Let us suppose that in an LP-Sasakian manifold

$$P(X, Y) \cdot W_2 = 0. \quad (3.3)$$

This condition implies that

$$\begin{aligned} &P(X, Y)W_2(U, V)Z - W_2(P(X, Y)U, V)Z \\ &- W_2(U, P(X, Y)V)Z - W_2(U, V)P(X, Y)Z = 0. \end{aligned} \quad (3.4)$$

Putting $X = \xi$ in (3.4) and then taking the inner product with ξ , we obtain

$$\begin{aligned} &g(P(\xi, Y)W_2(U, V)Z, \xi) - g(W_2(P(\xi, Y)U, V)Z, \xi) \\ &- g(W_2(U, P(\xi, Y)V)Z, \xi) - g(W_2(U, V)P(\xi, Y)Z, \xi) = 0. \end{aligned} \quad (3.5)$$

Using (3.2) in (3.5), we obtain

$$\begin{aligned} &-g(Y, W_2(U, V)Z) - g(Y, U)\eta(W_2(\xi, V)Z) - g(Y, V)\eta(W_2(U, \xi)Z) - g(Y, Z)\eta(W_2(U, V)\xi) \\ &+ \frac{1}{n-1}[S(Y, W_2(U, V)Z) + S(Y, U)\eta(W_2(\xi, V)Z) \\ &+ S(Y, V)\eta(W_2(U, \xi)Z) + S(Y, Z)\eta(W_2(U, V)\xi)] = 0. \end{aligned} \quad (3.6)$$

By using (2.16) in (3.6), we get

$$g(Y, W_2(U, V)Z) - \frac{1}{n-1}S(Y, W_2(U, V)Z) = 0. \quad (3.7)$$

Taking $U = Z = \xi$ in (3.7) and using (2.11) and (2.6), we have

$$S(QY, V) = 2(n-1)S(Y, V) - (n-1)^2g(Y, V). \quad (3.8)$$

This implies that

$$QY = (n-1)Y. \quad (3.9)$$

From this, we get

$$S(Y, V) = (n-1)g(Y, V). \quad (3.10)$$

Thus, we can state the following.

Theorem 3.1. *An LP-Sasakian manifold M satisfying the condition $P(X, Y) \cdot W_2 = 0$ is an Einstein manifold.*

4. LP-Sasakian Manifold Satisfying $\widetilde{M}(X, Y) \cdot W_2 = 0$

The M -projective curvature tensor $\widetilde{M}$ is defined as [11]

$$\begin{aligned} \widetilde{M}(X, Y)Z = R(X, Y)Z - \frac{1}{2(n-1)}[S(Y, Z)X - S(X, Z)Y \\ + g(Y, Z)QX - g(X, Z)QY]. \end{aligned} \quad (4.1)$$

Using (2.6) and (2.8), (4.1) reduces to

$$\widetilde{M}(\xi, Y)Z = \frac{1}{2}[g(Y, Z)\xi - \eta(Z)Y] - \frac{1}{2(n-1)}[S(Y, Z)\xi - \eta(Z)QY]. \quad (4.2)$$

Suppose that in an LP-Sasakian manifold

$$\widetilde{M}(X, Y) \cdot W_2 = 0. \quad (4.3)$$

This condition implies that

$$\begin{aligned} \widetilde{M}(X, Y)W_2(U, V)Z - W_2(\widetilde{M}(X, Y)U, V)Z \\ - W_2(U, \widetilde{M}(X, Y)V)Z - W_2(U, V)\widetilde{M}(X, Y)Z = 0. \end{aligned} \quad (4.4)$$

Putting $X = \xi$ in (4.4) and then taking the inner product with ξ , we obtain

$$\begin{aligned} g(\widetilde{M}(\xi, Y)W_2(U, V)Z, \xi) - g(W_2(\widetilde{M}(\xi, Y)U, V)Z, \xi) \\ - g(W_2(U, \widetilde{M}(\xi, Y)V)Z, \xi) - g(W_2(U, V)\widetilde{M}(\xi, Y)Z, \xi) = 0. \end{aligned} \quad (4.5)$$

Using (4.2) in (4.5), we obtain

$$\begin{aligned}
& \frac{1}{2} [-g(Y, W_2(U, V)Z) - g(Y, U)\eta(W_2(\xi, V)Z) - g(Y, V)\eta(W_2(U, \xi)Z) \\
& \quad - g(Y, Z)\eta(W_2(U, V)\xi) + \eta(U)\eta(W_2(Y, V)Z) + \eta(V)\eta(W_2(U, Y)Z) \\
& \quad + \eta(Z)\eta(W_2(U, V)Y)] \\
& + \frac{1}{2(n-1)} [S(Y, W_2(U, V)Z) + S(Y, U)\eta(W_2(\xi, V)Z) \\
& \quad + S(Y, V)\eta(W_2(U, \xi)Z) + S(Y, Z)\eta(W_2(U, V)\xi) - \eta(U)\eta(W_2(QY, V)Z) \\
& \quad - \eta(V)\eta(W_2(U, QY)Z) - \eta(Z)\eta(W_2(U, V)QY)] = 0.
\end{aligned} \tag{4.6}$$

By using (2.16) in (4.6), we get

$$\frac{1}{2}g(Y, W_2(U, V)Z) - \frac{1}{2(n-1)}S(Y, W_2(U, V)Z) = 0. \tag{4.7}$$

Taking $U = Z = \xi$ in (4.7) and using (2.11) and (2.6), we have

$$S(QY, V) = 2(n-1)S(Y, V) - (n-1)^2g(Y, V). \tag{4.8}$$

This implies that

$$QY = (n-1)Y, \tag{4.9}$$

which gives

$$S(Y, V) = (n-1)g(Y, V). \tag{4.10}$$

Thus, we can state the following.

Theorem 4.1. *An LP-Sasakian manifold M satisfying the condition $\widetilde{M}(X, Y) \cdot W_2 = 0$ is an Einstein manifold.*

5. LP-Sasakian Manifolds Satisfying $C(X, Y) \cdot W_2 = 0$

The conformal curvature tensor C is defined as [12]

$$\begin{aligned}
C(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\
&+ \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y].
\end{aligned} \tag{5.1}$$

Using (2.6) and (2.8), (5.1) reduces to

$$C(\xi, Y)Z = \frac{1-n+r}{(n-1)(n-2)} [g(Y, Z)\xi - \eta(Z)Y] - \frac{1}{n-2} [S(Y, Z)\xi - \eta(Z)QY]. \quad (5.2)$$

Now consider an LP-Sasakian manifold satisfying

$$C(X, Y) \cdot W_2 = 0. \quad (5.3)$$

This condition implies that

$$\begin{aligned} C(X, Y)W_2(U, V)Z - W_2(C(X, Y)U, V)Z \\ - W_2(U, C(X, Y)V)Z - W_2(U, V)C(X, Y)Z = 0. \end{aligned} \quad (5.4)$$

Putting $X = \xi$ in (5.4) and then taking the inner product with ξ , we obtain

$$\begin{aligned} g(C(\xi, Y)W_2(U, V)Z, \xi) - g(W_2(C(\xi, Y)U, V)Z, \xi) \\ - g(W_2(U, C(\xi, Y)V)Z, \xi) - g(W_2(U, V)C(\xi, Y)Z, \xi) = 0. \end{aligned} \quad (5.5)$$

Using (5.2) in (5.5), we obtain

$$\begin{aligned} \frac{1-n+r}{(n-1)(n-2)} [-g(Y, W_2(U, V)Z) - g(Y, U)\eta(W_2(\xi, V)Z) - g(Y, V)\eta(W_2(U, \xi)Z) \\ - g(Y, Z)\eta(W_2(U, V)\xi) - \eta(Y)\eta(W_2(U, V)Z) + \eta(U)\eta(W_2(Y, V)Z) \\ + \eta(V)\eta(W_2(U, Y)Z) + \eta(Z)\eta(W_2(U, V)Y)] \\ + \frac{1}{n-2} [S(Y, W_2(U, V)Z) + S(Y, U)\eta(W_2(\xi, V)Z) + S(Y, V)\eta(W_2(U, \xi)Z) \\ + S(Y, Z)\eta(W_2(U, V)\xi) + (n-1)\eta(Y)\eta(W_2(U, V)Z) \\ - \eta(U)\eta(W_2(QY, V)Z) \\ - \eta(V)\eta(W_2(U, QY)Z) - \eta(Z)\eta(W_2(U, V)QY)] = 0. \end{aligned} \quad (5.6)$$

By using (2.16) in (5.6), we get

$$\left\{ \frac{1-n+r}{(n-1)(n-2)} \right\} g(Y, W_2(U, V)Z) - \frac{1}{n-2} S(Y, W_2(U, V)Z) = 0. \quad (5.7)$$

Taking $U = Z = \xi$ in (5.7) and then using (2.11) and (2.6), we have

$$S(QY, V) = \left\{ \frac{2+n(n-3)+r}{n-1} \right\} S(Y, V) - (n-1+r)g(Y, V). \quad (5.8)$$

This implies that

$$QY = (n - 1)Y, \quad (5.9)$$

and it follows that

$$S(Y, V) = (n - 1)g(Y, V). \quad (5.10)$$

Thus, we can state the following.

Theorem 5.1. *An LP-Sasakian manifold M satisfying the condition $C(X, Y) \cdot W_2 = 0$ is an Einstein manifold.*

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