Majorization for A Subclass of $\beta$-Spiral Functions of Order $\alpha$ Involving a Generalized Linear Operator

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Motivated by Carlson-Shaffer linear operator, we define here a new generalized linear operator. Using this operator, we define a class of analytic functions in the unit disk $U$. For this class, a majorization problem of analytic functions is discussed.

1. Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}$$

which are analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $f$ and $g$ be analytic in $U$. Then, we say that function $f$ is subordinate to $g$ if there exists a Schwarz function $\omega(z)$, analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$, $z \in U$ (see [1]). We denote this subordination by

$$f \prec g \quad (z \in U).$$
Further, \( f \) is said to be quasi subordinate to \( g \) if there exists an analytic function \( \varphi(z) \) such that \( f(z)/\varphi(z) \) is analytic in \( U \),

\[
\frac{f(z)}{\varphi(z)} \preceq g(z), \quad (z \in U)
\]  

(1.3)

and \( |\varphi(z)| \leq 1 \). Note that the quasi subordination (1.3) is equivalent to

\[
f(z) = \varphi(z)g(\omega(z)),
\]  

(1.4)

where \( |\varphi(z)| \leq 1 \) and \( |\omega(z)| \leq |z| < 1 \) (see [2]). If \( \varphi(z) = 1 \), then (1.3) becomes (1.2).

Let functions \( f \) and \( g \) be analytic functions in \( U \). If \(|f(z)| \leq |g(z)|\), then there exists a function \( \varphi \) analytic in \( U \) such that \(|\varphi(z)| \leq 1 \) in \( U \), for which

\[
f(z) = \varphi(z)g(z) \quad (z \in U).
\]  

(1.5)

In this case, we say that \( f \) is majorized by \( g \) in \( U \) (see [3]), and we write

\[
f(z) \ll g(z) \quad (z \in U).
\]  

(1.6)

If we take \( \omega(z) = z \) in (1.4), then the quasi subordination (1.3) becomes the majorization (1.6).

Also, let \( S \) denote the subclass of \( A \) consisting of all functions which are univalent in \( U \).

In [4], Robertson introduced star-like functions of order \( \alpha \) on \( U \).

**Definition 1.1.** Let \( 0 \leq \alpha < 1 \) and \( f \in A \); then, \( f \) is a star-like function of order \( \alpha \) on \( U \) if and only if

\[
\Re \left \{ \frac{zf'(z)}{f(z)} \right \} > \alpha \quad (z \in U).
\]  

(1.7)

Let \( S^*(\alpha) \) denote the whole star-like functions of order \( \alpha \) in \( U \).

Spaček [5] extended the class of \( S^* \) and obtained the class of \( \beta \)-spiral-like functions. In the same article, the author gave an analytical characterization of spirallikeness of type \( \beta \) on \( U \).

**Definition 1.2.** Let \(-\pi/2 < \beta < \pi/2 \) and \( f \in A \); then, \( f \) is \( \beta \)-spiral-like function on \( U \) if and only if

\[
\Re \left \{ e^{i\beta} \frac{f'(z)}{f(z)} \right \} > 0 \quad (z \in U).
\]  

(1.8)

We denote the whole \( \beta \)-spiral-like functions in \( U \) by \( S^*_\beta \).
Finally, Libera [6] introduced and studied the class of $\beta$-spiral-like functions of order $\alpha$.  

**Definition 1.3.** Let $0 \leq \alpha < 1$, $-\pi/2 < \beta < \pi/2$ and $f \in A$; then, $f$ is $\beta$-spiral function of order $\alpha$ if and only if

$$
\mathfrak{R}\{e^{i\beta}z f'(z)/f(z)\} > \alpha \cos \beta \quad (z \in U).
$$

We denote the whole $\beta$-spiral-like functions of order $\alpha$ in $U$ by $S_\beta^\alpha(a)$.

In particular, we consider the convolution with function $\phi(a,c)$ defined by

$$
L(a,b) f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1},
$$

where $a \in \mathbb{C}$, $b \not= 0, -1, -2, \ldots$, and $(a)_n$ is the Pochhammer symbol defined by

$$
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 
1, & n = 0, \\
(a+1) \cdots (a+n-1), & n = \{1, 2, 3, \ldots\}.
\end{cases}
$$

Function $\phi(a,c)$ is an incomplete beta-function related to the Gauss hypergeometric function by

$$
\phi(a,c; z) = z_2 F_1(1, a; c; z).
$$

It has an analytic continuation to the $z$-plane cut along the positive real line from 1 to $\infty$. We note that $\phi(1.1; z) = z/(1-z)^a$ and $\phi(2, 1; z)$ are the Koebe functions.

Carlson and Shafer [7] defined a convolution operator on $A$ involving an incomplete beta-function as

$$
L(a,b) f(z) = \phi(a,c; z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}.
$$

**Definition 1.4.** Let function $F$ be given by

$$
F(m, \ell, \lambda) = \sum_{n=0}^{\infty} \left( \frac{1 + \ell + \lambda n}{1 + \ell} \right)^m z^{n+1},
$$

where $\ell, \lambda \geq 0$ and $m \in \mathbb{Z}$. The generalized linear operator $L(m, \ell, \lambda, a, c) : A \rightarrow A$ is given as

$$
L(m, \ell, \lambda, a, b) f(z) = z + \sum_{n=1}^{\infty} \left( \frac{1 + \ell + \lambda n}{1 + \ell} \right)^m \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}.
$$
We note here some special cases.

1. $L(0, \ell, \lambda, a, b)f(z) = L(a, b)f(z)$ is the Carlson and Shaffer operator [7].
2. $L(0, \ell, \lambda, \delta + 1, 1)f(z)$, $\delta \in \mathbb{N}_0$, is the Ruscheweyh derivative [8].
3. $L(m, 0, \lambda, 1, 1)f(z)$, $m \in \mathbb{N}_0$, is the Al-Oboudi operator [9].
4. $L(m, 0, \lambda, a, b)f(z)$ is the linear operator introduced by Al-Refai and Darus [10].
5. $F(m, \ell, \lambda)$, $m \in \mathbb{N}_0$, is the generalized multiplier transformation which was introduced and studied by Catas [11].
6. $F(m, \ell, 1), m \in \mathbb{N}_0$, is the multiplier transformation which was introduced and studied by Cho and Srivastava [12] and Cho and Kim [13].

**Remark 1.5.** It follows from the above definition that

$$z(L(m, \ell, \lambda, a, c)f(z))' = aL(m, \ell, \lambda, a + 1, c)f(z) - (a - 1)L(m, \ell, \lambda, a, c)f(z) \quad (z \in U).$$

(1.16)

We introduce the class $S'_\beta(m, \ell, \lambda, a, c, a)$ as follows.

**Definition 1.6.** Let $a \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$, $\ell, \lambda \geq 0$, $m \in \mathbb{Z}$, $0 \leq a < 1$, $-\pi/2 < \beta < \pi/2$, and $f \in A$; then, one has $S'_\beta(m, \ell, \lambda, a, c, \ell, \lambda, a)$ if and only if

$$\Re \left\{ e^{i\phi} \frac{z(L(a, c)f(z))'}{L(a, c)f(z)} \right\} > a \cos \beta.$$  

(1.17)

Obviously, when $a = c = 1$ and $m = 0$ we obtain $f \in S'_\beta(\alpha)$, when $a = c = 1$ and $m = \beta = 0$, we obtain that $f(z)$ is a star-like function of order $\alpha$ on $U$, and also when $a = c = 1$ and $m = \alpha = 0$, we obtain that $f(z)$ is spiral-like function of type $\beta$ on $U$.

Biermannki [14] in 1936 obtained the first results of majorization-subordination theory. He showed that, if $g(z) \in S$ and $f(z) < g(z)$ in $U$, then $f(z) \ll g(z)$ in $|z| \leq (1/4)$. Goluzin [15] improved the result and Shah [16] obtained the complete solution for $S$ by showing that $f(z) \ll g(z)$ in $|z| \leq (3 - \sqrt{5})/2$ and that the result is the best possible. A majorization problem for star-like functions has been given by MacGregor [3]. Also, majorization problem for star-like functions of complex order has recently been investigated by Altintas et al. [17].

The main object of this paper is to investigate the problem of majorization of the class $S'_\beta(\ell, \lambda, a, c, a)$ defined by a generalized linear operator.

In order to prove our main theorem we need the following lemma.

**Lemma 1.7** (see [18]). Let $\varphi(z)$ be analytic in $U$ satisfying $|\varphi(z)| \leq 1$ for $z \in U$. Then,

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}.$$  

(1.18)
2. Main Results

Theorem 2.1. Let function \( f \in A \) and suppose that \( g \in S^*_\rho(m, \ell, \lambda, a, c, \alpha) \). If \( L(m, \ell, \lambda, a, c) f \) is majorized by \( L(m, \ell, \lambda, a, c) g \) in \( U \), then

\[
|L(m, \ell, \lambda, a+1,c) f(z)| \leq |L(m, \ell, \lambda, a+1,c) g(z)| \quad (|z| \leq r_1),
\]

where

\[
r_1 = r(m, \ell, \lambda, a, c, \alpha, \beta) = \frac{2 + |a| + |2(1 - \alpha) \cos \beta - ae^{i\beta}|}{2|2(1 - \alpha) \cos \beta - ae^{i\beta}|} - \frac{\sqrt{\Theta(a, \alpha, \beta)}}{2|2(1 - \alpha) \cos \beta - ae^{i\beta}|},
\]

\[
\Theta(a, \alpha, \beta) = 4 + |a|^2 + |2(1 - \alpha) \cos \beta - ae^{i\beta}|^2 + 4|a| + 4|2(1 - \alpha) \cos \beta - ae^{i\beta}| - 2|a||2(1 - \alpha \cos \beta - ae^{i\beta}|,
\]

for \( a \in \mathbb{C}, \ c \neq 0, -1, -2, \ldots, \ell, \lambda \geq 0, m \in \mathbb{Z}, \ 0 \leq \alpha < 1, \ -\pi/2 < \beta < \pi/2, \) and \(|a| \geq |2(1 - \alpha) \cos \beta - ae^{i\beta}| \).

Proof. Since \( g \in S^*_\rho(m, \ell, \lambda, a, c, \alpha) \), we have

\[
e^{i\beta} z(L(m, \ell, \lambda, a, c) g(z))^' = \frac{1 + (1 - 2\alpha)\omega}{1 - \omega} \cos \beta + i \sin \beta,
\]

where \( \omega \) is analytic in \( U \), with \( \omega(0) = 0 \) and

\[
|\omega| \leq |z| < 1 \quad (z \in U).
\]

By using (1.16) in (2.4), we get

\[
e^{i\beta} \left[ \frac{aL(m, \ell, \lambda, a+1,c) g(z) - (a-1)L(m, \ell, \lambda, a,c) g(z)}{L(m, \ell, \lambda, a,c) g(z)} \right] = \frac{1 + (1 - 2\alpha)\omega}{1 - \omega} \cos \beta + i \sin \beta.
\]

Hence,

\[
\frac{L(m, \ell, \lambda, a+1,c) g(z)}{L(m, \ell, \lambda, a,c) g(z)} = \frac{ae^{i\beta} + (2(1 - \alpha) \cos \beta - ae^{i\beta})\omega}{ae^{i\beta}(1 - \omega)},
\]

which, in view of (2.5), immediately yields the inequality

\[
|L(m, \ell, \lambda, a,c) g(z)| \leq \frac{|e^{i\beta}| |a| (1 + |z|)}{|a| - |2(1 - \alpha \cos \beta - ae^{i\beta}| |z|} |L(m, \ell, \lambda, a+1,c) g(z)|.
\]
Next, since $L(m, \ell, \lambda, a, c)f$ is majorized by $L(m, \ell, \lambda, a, c)g$ in $U$, from (1.5) we have

$$z(L(m, \ell, \lambda, a, c)f(z))' = z\varphi'(z)L(m, \ell, \lambda, a, c)g(z) + z\varphi(z)(L(m, \ell, \lambda, a, c)g(z))'.$$

(2.9)

Also, by using (1.16) in (2.11), we get

$$aL(m, \ell, \lambda, a + 1, c)f(z) - (a - 1)L(m, \ell, \lambda, a, c)f(z) = z\varphi'(z)L(m, \ell, \lambda, a, c)g(z) + \varphi(z)[aL(m, \ell, \lambda, a + 1, c)g(z) - (a - 1)L(m, \ell, \lambda, a, c)g(z)];$$

(2.10)

then, we have

$$L(m, \ell, \lambda, a + 1, c)f(z) = \frac{1}{a} z\varphi'(z)L(m, \ell, \lambda, a, c)g(z) + \varphi(z)L(m, \ell, \lambda, a + 1, c)g(z).$$

(2.11)

Thus, by Lemma 1.7, since the Schwarz function $\phi$ satisfies the inequality in (1.18) and using (2.8) in (2.11), we get

$$|L(m, \ell, \lambda, a + 1, c)f(z)| \leq \frac{(1 - |\varphi(z)|^2)|z|}{(1 - |z|)(|a| - |2(1 - \alpha) \cos \beta - aei\beta||z|)} \times |L(m, \ell, \lambda, a + 1, c)g(z)| + |\varphi(z)| L(m, \ell, \lambda, a + 1, c)g(z)|.$$

(2.12)

Hence,

$$|L(m, \ell, \lambda, a + 1, c)f(z)| \leq \frac{(1 - |\varphi(z)|^2)|z| + (1 - |z|)(|a| - |2(1 - \alpha) \cos \beta - aei\beta||z|)|\varphi(z)|}{(1 - |z|)(|a| - |2(1 - \alpha) \cos \beta - aei\beta||z|)} \times |L(m, \ell, \lambda, a + 1, c)g(z)|;$$

(2.13)

which, upon setting

$$|z| = r, \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1)$$

(2.14)

yields

$$|L(m, \ell, \lambda, a + 1, c)f(z)| \leq \frac{\theta(\rho)}{(1 - r)(|a| - |2(1 - \alpha) \cos \beta - aei\beta| r)} |L(m, \ell, \lambda, a + 1, c)g(z)|.$$

(2.15)
where function $\theta(\rho)$ defined by

$$
\theta(\rho) = (1 - \rho^2) r + (1 - r) \left(|a| - |2(1 - \alpha) \cos \beta - ae^{i\beta}|r\right) \rho
$$

(2.16)

takes on its maximum value at $\rho = 1$ with

$$
r_1 = r(m, \ell, \lambda, a, c, \alpha, \beta) = \max\{r \in [0, 1] : \psi(r, \rho) \leq 1, \forall \rho \in [0, 1]\},
$$

(2.17)

where

$$
\psi(r, \rho) = \frac{\theta(\rho)}{(1 - r) (|a| - |2(1 - \alpha) \cos \beta - ae^{i\beta}|r)};
$$

(2.18)

then, we have

$$
\frac{\theta(\rho)}{(1 - r) (|a| - |2(1 - \alpha) \cos \beta - ae^{i\beta}|r)} \leq 1.
$$

(2.19)

A simple calculus in (2.19) is equivalent to

$$
-(1 + \rho)r + (1 - r) \left(|a| - |2(1 - \alpha) \cos \beta - ae^{i\beta}|r\right) \geq 0,
$$

(2.20)

while the inequality in (2.19) takes its minimum value at $\rho = 1$, that is,

$$
\left|2(1 - \alpha) \cos \beta - ae^{i\beta}\right|^2 - \left(2|a| + \left|2(1 - \alpha) \cos \beta - ae^{i\beta}\right|\right)r + |a| \geq 0,
$$

(2.21)

for all $r \in [0, r_1]$, where $r_1 = r(m, \ell, \lambda, a, c, \alpha, \beta)$ given in (2.2) holds true for $|z| \leq r(m, \ell, \lambda, a, c, \alpha, \beta)$, which proves the conclusion (2.1).

Putting $m = a = \beta = 0$ in Theorem 2.1, we obtain the following result.

**Corollary 2.2.** Let function $f \in A$ and suppose that $g \in S^*(a, c)$. If $L(a, c)f$ is majorized by $L(a, c)g$ in $U$, then

$$
|L(a + 1, c)f(z)| \leq |L(a + 1, c)g(z)| \quad (|z| \leq r_2 = r(a, c)),
$$

(2.22)

where

$$
r(a, c) = \frac{3 + |a| + |2 - a|}{2|2 - a|} - \frac{\sqrt{4 + |2 - a|^2 - 2|2a|2|2 - a| + 4|a| + |a|^2}}{2|2 - a|}.
$$

(2.23)

Further, putting $a = c = 1$ and $m = 0$ in Theorem 2.1, we obtain the result of Altintas et al. [17].
Corollary 2.3. Let function $f \in A$ and suppose that $g \in S^*((\alpha - 1)e^{i\beta}) = S^*_\beta(\alpha)$, where $0 \leq \alpha < 1$ and $-\pi/2 < \beta < \pi/2$. If $f$ is majorized by $g$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r_3 = r(\alpha, \beta)), \quad (2.24)$$

where

$$r(\alpha, \beta) = \frac{3 + |2(\alpha - 1)e^{i\beta} - 1|}{2|2(\alpha - 1)e^{i\beta} - 1|} - \frac{\sqrt{9 + |2(\alpha - 1)e^{i\beta} - 1|^2 + 2|2(\alpha - 1) - 1|}}{2|2(\alpha - 1)e^{i\beta} - 1|}. \quad (2.25)$$

Putting $\beta = 0$ in Corollary 2.3, we obtain the result as follows.

Corollary 2.4. Let function $f \in A$ and suppose that $g \in S^*(\alpha)$, where $0 \leq \alpha < 1$. If $f$ is majorized by $g$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r_4 = r(\alpha)), \quad (2.26)$$

where

$$r(\alpha) = \frac{3 + |1 - 2\alpha|}{2|1 - 2\alpha|} - \frac{\sqrt{9 + |1 - 2\alpha|^2 + 2|2(\alpha - 1) - 1|}}{2|1 - 2\alpha|}. \quad (2.27)$$

Also, putting $\alpha = \beta = 0$ in Corollary 2.3, we obtain the result of MacGregor [3].

Corollary 2.5. Let function $f \in A$ and suppose that $g \in S^*(0)$. If $f$ is majorized by $g$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq 2 - \sqrt{3}). \quad (2.28)$$

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References
