Research Article

Solutions of Higher-Order Homogeneous Linear Matrix Differential Equations for Consistent and Non-Consistent Initial Conditions: Regular Case

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We study a class of linear matrix differential equations (regular case) of higher order whose coefficients are square constant matrices. By using matrix pencil theory and the Weierstrass canonical form of the pencil we obtain formulas for the solutions and we show that the solution is unique for consistent initial conditions and infinite for nonconsistent initial conditions. Moreover we provide some numerical examples. These kinds of systems are inherent in many physical and engineering phenomena.

1. Introduction

Linear matrix differential equations (LMDEs) are inherent in many physical, engineering, mechanical, and financial/actuarial models. Having in mind such applications, for instance in finance, we provide the well-known input-output Leondief model and its several important extensions, see [1, 2]. In this paper, our long-term purpose is to study the solution of LMDEs of higher order (1.1) into the mainstream of matrix pencil theory. This effort is significant, since there are numerous applications. Thus, we consider

$$A_n X^{(n)}(t) + A_{n-1} X^{(n-1)}(t) + \cdots + A_1 X'(t) + A_0 X(t) = \mathbb{O}, \quad (1.1)$$

where $A_i, i = 0, 1, \dots, n \in \mathcal{M}(m \times m; \mathbb{F})$, (i.e., the algebra of square matrices with elements in

the field $\mathbb{F}$) with $X \in \mathcal{C}^\infty(\mathbb{F}, \mathcal{M}(m \times 1; \mathbb{F}))$. For the sake of simplicity we set $\mathcal{M}_m = \mathcal{M}(m \times m; \mathbb{F})$ and $\mathcal{M}_{nm} = \mathcal{M}(n \times m; \mathbb{F})$. In the sequel we adopt the following notations:

$$\begin{aligned}
 Y_1(t) &= X(t), \\
 Y_2(t) &= X'(t), \\
 &\vdots \\
 Y_{n-1}(t) &= X^{(n-1)}(t), \\
 Y_n(t) &= X^{(n)}(t), \\
 Y_1'(t) &= X'(t) = Y_2(t), \\
 Y_2'(t) &= X''(t) = Y_3(t), \\
 &\vdots \\
 Y_{n-1}(t) &= X^{(n-1)}(t) = Y_n(t), \\
 A_n Y_n'(t) &= A_n X^{(n)}(t) = -A_{n-1} Y_n(t) - \cdots - A_1 Y_2(t) - A_0 Y_1(t).
 \end{aligned} \tag{1.2}$$

Or in Matrix form

$$FY'(t) = GY(t), \tag{1.3}$$

where $Y(t) = [Y_1^T(t) Y_2^T(t) \cdots Y_n^T(t)]^T$ (where $()^T$ is the transpose tensor) and the coefficient matrices F, G are given by

$$\mathbf{F} = \begin{bmatrix} I_m & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & I_m & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & I_m & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & A_n \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbb{O} & I_m & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & I_m & \cdots & \mathbb{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & I_m \\ -A_0 & -A_1 & -A_2 & \cdots & -A_n \end{bmatrix} \tag{1.4}$$

with corresponding dimension of $\mathbf{F}, \mathbf{G}$, and $\mathbf{Y}(t)$, $mn \times mn$ and $mn \times 1$, respectively. Matrix pencil theory has been extensively used for the study of linear differential equations (LDEs) with time invariant coefficients, see for instance [1–5]. Systems of type (1.1) are more general, including the special case when $A_n = I_n$, where I_n is the identity matrix of $\mathcal{M}_n$, since the well-known class of higher-order linear matrix differential equations of Apostol-Kolodner type is derived straightforwardly, see [6] for $n = 2$, [7, 8].

The paper is organized as follows: in Section 2, some notations and the necessary preliminary concepts from matrix pencil theory are presented. Section 3 contains the case that system (1.1) has consistent initial conditions. In Section 4, the nonconsistent initial condition case is fully discussed. In this case, the arbitrarily chosen initial conditions which have

physical meaning for (regular) systems, in some sense, can be created or structurally changed at a fixed time $t = t_0$. Hence, it is derived that (1.1) should adopt a generalized solution, in the sense of Dirac δ -solutions.

2. Mathematical Background and Notation

This brief section introduces some preliminary concepts and definitions from matrix pencil theory, which are being used throughout the paper. Linear systems of type (1.1) are closely related to matrix pencil theory, since the algebraic geometric and dynamic properties stem from the structure by the associated pencil $sF - G$.

Definition 2.1. Given $F, G \in \mathcal{M}_{nm}$ and an indeterminate $s \in \mathbb{F}$, the matrix pencil $sF - G$ is called regular when $m = n$ and $\det(sF - G) \neq 0$. In any other case, the pencil will be called singular.

Definition 2.2. The pencil $sF - G$ is said to be *strictly equivalent* to the pencil $s\tilde{F} - \tilde{G}$ if and only if there exist nonsingular $P \in \mathcal{M}_n$ and $Q \in \mathcal{M}_m$ such as

$$P(sF - G)Q = s\tilde{F} - \tilde{G}. \quad (2.1)$$

In this paper, we consider the case that pencil is *regular*. Thus, the strict equivalence relation can be defined rigorously on the set of regular pencils as follows. Here, we regard (2.2) as the set of pair of nonsingular elements of $\mathcal{M}_n$

$$g := \{(P, Q) : P, Q \in \mathcal{M}_n, P, Q \text{ nonsingular}\} \quad (2.2)$$

and a composition rule $*$ defined on g as follows:

$$* : g \times g \quad \text{such that } (P_1, Q_1) * (P_2, Q_2) := (P_1 \cdot P_2, Q_2 \cdot Q_1). \quad (2.3)$$

It can be easily verified that $(g, *)$ forms a *nonabelian group*. Furthermore, an action $\circ$ of the group $(g, *)$ on the set of *regular* matrix pencils $\mathcal{L}_n^{\text{reg}}$ is defined as $\circ : g \times \mathcal{L}_n^{\text{reg}} \rightarrow \mathcal{L}_n^{\text{reg}}$ such that

$$((P, Q), sF - G) \longrightarrow (P, Q) \circ (sF - G) := P(sF - G)Q. \quad (2.4)$$

This group has the following properties:

- (a) $(P_1, Q_1) \circ [(P_2, Q_2) \circ (sF - G)] = (P_1, Q_1) * (P_2, Q_2) \circ (sF - G)$ for every nonsingular $P_1, P_2 \in \mathcal{M}_n$ and $Q_1, Q_2 \in \mathcal{M}_n$.
- (b) $e_g \circ (sF - G) = sF - G$, $sF - G \in \mathcal{L}_n^{\text{reg}}$ where $e_g = (I_n, I_n)$ is the *identity element* of the group $(g, *)$ on the set of $\mathcal{L}_n^{\text{reg}}$ defines a transformation group denoted by $\mathcal{N}$, see [9].

For $sF - G \in \mathcal{L}_n^{\text{reg}}$, the subset

$$g \circ (sF - G) := \{(P, Q) \circ (sF - G) : (P, Q) \in g\} \subseteq \mathcal{L}_n^{\text{reg}} \quad (2.5)$$

will be called the orbit of $sF - G$ at g . Also $\mathcal{N}$ defines an equivalence relation on $\mathcal{L}_n^{\text{reg}}$ which is called a *strict-equivalence relation* and is denoted by $\mathcal{E}_{s-e}$.

So, $(sF - G)\mathcal{E}_{s-e}(s\tilde{F} - \tilde{G})$ if and only if $P(sF - G)Q = s\tilde{F} - \tilde{G}$, where $P, Q \in \mathcal{M}_n$ are nonsingular elements of algebra $\mathcal{M}_n$.

The class of $\mathcal{E}_{s-e}(sF - G)$ is characterized by a uniquely defined element, known as a complex Weierstrass canonical form, $sF_w - Q_w$, see [9], specified by the complete set of invariants of $\mathcal{E}_{s-e}(sF - G)$.

This is the set of *elementary divisors* (e.d.) obtained by factorizing the invariant polynomials $f_i(s, \hat{s})$ into powers of homogeneous polynomials irreducible over field $\mathbb{F}$. In the case where $sF - G$ is a regular, we have e.d. of the following type:

- (i) e.d. of the type s^p are called *zero finite elementary divisors* (z. f.e.d.),
- (ii) e.d. of the type $(s - a)^\pi$, $a \neq 0$ are called *nonzero finite elementary divisors* (nz. f.e.d.),
- (iii) e.d. of the type $\hat{s}^q$ are called *infinite elementary divisors* (i.e.d.).

Let $B_1, B_2, \dots, B_n$ be elements of $\mathcal{M}_n$. The direct sum of them denoted by $B_1 \oplus B_2 \oplus \dots \oplus B_n$ is the block diag $\{B_1, B_2, \dots, B_n\}$.

Then, the complex Weierstrass form $sF_w - Q_w$ of the regular pencil $sF - G$ is defined by $sF_w - Q_w := sI_p - J_p \oplus sH_q - I_q$, where the first normal Jordan type element is uniquely defined by the set of f.e.d.

$$(s - a_1)^{p_1}, \dots, (s - a_v)^{p_v}, \quad \sum_{j=1}^v p_j = p \quad (2.6)$$

of $sF - G$ and has the form

$$sI_p - J_p := sI_{p_1} - J_{p_1}(a_1) \oplus \dots \oplus sI_{p_v} - J_{p_v}(a_v). \quad (2.7)$$

And also the q blocks of the second uniquely defined block $sH_q - I_q$ correspond to the i.e.d.

$$\hat{s}^{q_1}, \dots, \hat{s}^{q_\sigma}, \quad \sum_{j=1}^{\sigma} q_j = q \quad (2.8)$$

of $sF - G$ and has the form

$$sH_q - I_q := sH_{q_1} - I_{q_1} \oplus \dots \oplus sH_{q_\sigma} - I_{q_\sigma}. \quad (2.9)$$

Thus, H_q is a nilpotent element of $\mathcal{M}_n$ with index $\tilde{q} = \max\{q_j : j = 1, 2, \dots, \sigma\}$, where

$$H_q^{\tilde{q}} = \mathbb{O}, \quad (2.10)$$

and $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined as

$$\begin{aligned}
 I_{p_j} &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_{p_j}, & J_{p_j}(a_j) &= \begin{bmatrix} a_j & 1 & 0 & \cdots & 0 \\ 0 & a_j & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & a_j & 1 \\ 0 & 0 & 0 & 0 & a_j \end{bmatrix} \in \mathcal{M}_{p_j}, \\
 H_{q_j} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{M}_{q_j}.
 \end{aligned} \tag{2.11}$$

In the last part of this section, some elements for the analytic computation of $e^{A(t-t_0)}$, $t \in [t_0, \infty)$ are provided. To perform this computation, many theoretical and numerical methods have been developed. Thus, the interesting readers might consult papers [7, 8, 10–12] and the references therein. In order to have computational formulas, see the following Sections 3 and 4, the following known results should firstly be mentioned.

Lemma 2.3 (see [10]). $e^{J_{p_j}(a_j)(t-t_0)} = (d_{k_1 k_2})_{p_j}$, where

$$d_{k_1 k_2} = \begin{cases} e^{a_j(t-t_0)} \frac{(t-t_0)^{k_2-k_1}}{(k_2-k_1)!}, & 1 \leq k_1 \leq k_2 \leq p_j, \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

Another expression for the exponential matrix of Jordan block, see (2.11), is provided by the following lemma.

Lemma 2.4 (see [12]).

$$e^{J_{p_j}(a_j)(t-t_0)} = \sum_{i=0}^{p_j-1} f_i(t-t_0) [J_{p_j}(a_j)]^i, \tag{2.13}$$

where the $f_i(t-t_0)$'s are given analytically by the following p_j equations:

$$f_{p_j-1-k}(t-t_0) = e^{a_j(t-t_0)} \sum_{i=0}^k b_{k,i} a_j^{k-i} \frac{(t-t_0)^{p_j-1-i}}{(p_j-1-i)!}, \quad k = 0, 1, 2, \dots, p_j-1, \tag{2.14}$$

where

$$b_{k,i} = \sum_{l=0}^{k-i} \binom{p_j}{l} \binom{k-l}{i} (-1)^l, \quad (2.15)$$

$$\left[J_{p_j}(a_j) \right]^i = \left(c_{k_1 k_2}^{(i)} \right)_{p_j}, \quad \text{for } 1 \leq k_1, k_2 \leq p_j,$$

where

$$c_{k_1 k_2}^{(i)} = \binom{i}{k_2 - k_1} a_j^{i - (k_2 - k_1)}. \quad (2.16)$$

3. Solution Space Form of Consistent Initial Conditions

In this section, the main results for consistent initial conditions are analytically presented for the regular case. Moreover, it should be stressed out that these results offer the necessary mathematical framework for interesting applications; see also Introduction. Now, in order to obtain a unique solution, we deal with consistent initial value problem. More analytically, we consider the system

$$A_n X^{(n)}(t) + A_{n-1} X^{(n-1)}(t) + \cdots + A_1 X'(t) + A_0 X(t) = \mathbb{O} \quad (3.1)$$

with known initial conditions

$$X(t_0), X'(t_0), \dots, X^{(n-1)}(t_0). \quad (3.2)$$

Analytically, we consider the system

$$FY'(t) = GY(t), \quad Y(t_0). \quad (3.3)$$

From the regularity of $sF - G$, there exist nonsingular $\mathcal{M}(mn \times mn, \mathbb{F})$ matrices P and Q such that (see also Section 2), such as

$$PFQ = F_w = I_p \oplus H_q, \quad (3.4)$$

$$PGQ = G_w = J_p \oplus I_q,$$

where I_p , J_p , H_q , and I_q are given by (2.11) where

$$I_p = I_{p_1} \oplus \cdots \oplus I_{p_v},$$

$$J_p = J_{p_1}(a_1) \oplus \cdots \oplus J_{p_v}(a_v), \quad (3.5)$$

$$H_q = H_{q_1} \oplus \cdots \oplus H_{q_\sigma},$$

$$I_q = I_{q_1} \oplus \cdots \oplus I_{q_\sigma}.$$

Note that $\sum_{j=1}^v p_j = p$ and $\sum_{j=1}^\sigma q_j = q$, where $p + q = n$.

Lemma 3.1. *System (3.1) is divided into two subsystems: the so-called slow subsystem*

$$Z'_p(t) = J_p Z_p(t), \quad (3.6)$$

and the relative fast subsystem

$$H_q Z'_p(t) = Z_q(t). \quad (3.7)$$

Proof. Consider the transformation

$$Y(t) = QZ(t). \quad (3.8)$$

Substituting the previous expression into (3.1) we obtain

$$FQZ'(t) = GQZ(t). \quad (3.9)$$

Whereby, multiplying by P , we arrive at

$$F_w Z(t) = G_w Z(t). \quad (3.10)$$

Moreover, we can write $Z(t)$ as $Z(t) = \begin{bmatrix} Z_p(t) \\ Z_q(t) \end{bmatrix}$. Taking into account the above expressions, we arrive easily at (3.6) and (3.7). $\square$

Proposition 3.2. *The sub-system (3.6) has the unique solution*

$$Z_p(t) = e^{J_p(t-t_0)} Z_p(t_0), \quad t \geq t_0, \quad (3.11)$$

where $\sum_{j=1}^v p_j = p$.

Proof. See [5, 12]. $\square$

Proposition 3.3. *The fast subsystem (3.7) has only the zero solution.*

Proof. Let q_* be the index of the nilpotent matrix H_q , that is, $H_q^{q_*} = \mathbb{O}$, we obtain the following equations:

$$\begin{aligned} H_q Z'_q(t) &= Z_q(t), \\ H_q L Z'_q(t) &= L Z_q(t), \\ H_q [sY(s) - Z_q(t_0)] &= Y(s), \\ (sH_q - I_q)Y(s) &= H_q Z_q, \end{aligned} \quad (3.12)$$

where $Y(s) = LZ_q(t) = \int_{t_0}^{\infty} Z_q(t)e^{-s(t-t_0)}dt$ is by definition the Laplace transform of Z_q . It is easy to show that $\det(sH_q - I_q) \neq 0$ and that $(sH_q - I_q)^{-1} = -\sum_{n=0}^{q_{\max}-1} (sH_q)^n$, while $H_q^n = 0$ for $n \geq q_*$

$$\begin{aligned}
Y(s) &= (sH_q - I_q)^{-1} H_q Z_q, \\
Y(s) &= -\sum_{n=0}^{q_*-1} (sH_q)^n H_q Z_q = -\sum_{n=0}^{q_*-1} s^n H_q^{n+1} Z_q, \\
Y(s) &= -\sum_{n=1}^{q_*-2} (s)^{n-1} H_q^n Z_q, \\
L^{-1}Y(s) &= -\sum_{n=1}^{q_*-2} L^{-1}s^{n-1} H_q^n Z_q(t_0), \\
Z_q &= -\sum_{n=1}^{q_*-2} \delta^n(t-t_0) H_q^n Z_q(t_0), \\
Z_q &= 0,
\end{aligned} \tag{3.13}$$

where $\delta(t-t_0)$ is by definition the Dirac function

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(t-t_0)dt &= 1, \quad t = t_0, \\
\delta(t-t_0) &= 0, \quad t \neq t_0.
\end{aligned} \tag{3.14}$$

The conclusion, that is, $Y_q(t) = \mathbb{O}$, is obtained by repetitive substitution of each equation in the next one, and using the fact that $H_q^{q_*} = \mathbb{O}$. $\square$

Theorem 3.4. *Consider the system (3.1)-(3.2). Then the solution is unique if and only if the initial conditions are consistent. Moreover the analytic solution of (3.1)-(3.2) is given by*

$$X(t) = Q_p^1 e^{J_p(t-t_0)} Z_p(t_0). \tag{3.15}$$

Proof. Let $Q = [Q_p Q_q]$ where $Q_p \in \mathcal{M}_{(mn)p}$ and $Q_q \in \mathcal{M}_{(mn)q}$; combining (3.8) and (3.17), we obtain

$$Y(t) = QZ(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} Z_p(t) \\ \mathbb{O} \end{bmatrix} = Q_p e^{J_p(t-t_0)} Z_p(t_0), \tag{3.16}$$

solution that exists if and only if $Y(t_0) = Q_p Z_p(t_0)$ or $Y(t_0) \in \text{colspan} Q_p$. The columns of Q_p are the p eigenvectors of the finite elementary divisors (eigenvalues) of the pencil $sA - B$. In that case the system has the unique solution

$$X(t) = Q_p^1 e^{J_p(t-t_0)} Z_p(t_0), \tag{3.17}$$

where Q_p^1 is defined as

$$Q_p = \begin{bmatrix} Q_p^1 \\ Q_p^2 \end{bmatrix} \quad (3.18)$$

and $Q_p^1 \in \mathcal{M}_{pp}$. □

4. Form on Nonconsistent Initial Condition

In this short section, we describe the impulse behavior of the original system (1.1), at time t_0 . In that case, we reformulate Theorem 3.4, so the impulse solution is finally obtained.

Proposition 4.1. *Consider the system (3.3). Then for nonconsistent initial conditions ($Y(t_0) \notin \text{colspan} Q_p$) the system has infinite solutions.*

Proof. Let Q_p, Q_q be the matrices defined in Theorem 3.4. If the initial conditions are non-consistent then $Y(t_0) \notin \text{colspan} Q_p$ and $Z_q(t_0) \neq 0$. Moreover $Y(t_0) = Q_p Z_p(t_0) + Q_q Z_q(t_0)$.

This means (3.3) is defined for $t \neq t_0$ because if $t = t_0$ then $FY'(t_0) = GY(t_0)$ and $Z_q(t_0) = \mathbb{O}$ which is a contradiction. Let $H(t - t_0)$ be the Heaviside function and

$$f(t) = H(t - t_0) - H(t_0 - t) = \begin{cases} 1, & t > t_0, \\ 0, & t = t_0, \end{cases} \quad (4.1)$$

$$g(t) = H(t_0 - t) = \begin{cases} 1, & t = t_0, \\ 0, & t \neq t_0. \end{cases}$$

Then the system can be written as

$$f(t)FY'(t) = GY(t) - g(t)GY(t_0), \quad t \geq t_0. \quad (4.2)$$

This is a linear matrix differential equation of first order with $Y_h(t) = f(t)Q_p e^{J_p(t-t_0)}C$ being the solution of the homogeneous and $Y_p(t) = g(t)Y(t_0)$ a partial solution. And we obtain the general solution

$$Y(t) = f(t)Q_p e^{J_p(t-t_0)}C + g(t)Y(t_0), \quad t \geq t_0, \quad (4.3)$$

where $C = [C_1 C_2 \cdots C_p]^T$ is constant vector and the dimension of the solution vector space is p . □

Theorem 4.2. *Consider the system (3.1)-(3.2) with nonconsistent initial conditions. Then the system has infinite solutions.*

Proof. We rewrite the system (3.1) in the following form:

$$\begin{aligned}
 A_n X^{(n)}(t) + A_{n-1} X^{(n-1)}(t) + \cdots + A_1 X'(t) + A_0 X(t) &= \mathbb{O}, \quad t > t_0, \\
 \sum_{i=0}^n A_i X^{(i)}(t) &= \mathbb{O}, \quad t > t_0, \\
 A_0 X(t) + f(t) \sum_{i=1}^n A_i X^{(i)}(t) &= g(t) A_0 \sum_{i=0}^{n-1} \frac{(t-t_0)^i}{i!} X^{(i)}(t_0), \quad t \geq t_0,
 \end{aligned} \tag{4.4}$$

where $f(t)$ and $g(t)$ are the functions defined in Theorem 4.2. Combining the results of Theorem 4.2 and the above discussion the solution of the system is

$$X(t) = f(t) Q_p e^{J_p(t-t_0)} C + g(t) \sum_{i=0}^{n-1} \frac{(t-t_0)^i}{i!} X^{(i)}(t_0), \quad t \geq t_0. \tag{4.5}$$

The dimension of the solution vector space is p . □

Remark 4.3. For $t > t_0$, it is obvious that (4.5) is satisfied. Thus, we should stress out that the system (3.1)-(3.2) has the above impulse behaviour at time instant where a non-consistent initial value is assumed, while it returns to smooth behaviour at any subsequent time instant.

5. Numerical Example

Let

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} X'''(t) + \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} X''(t) + \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix} X'(t) + \begin{bmatrix} 4 & -2 \\ -1 & -1 \end{bmatrix} X(t) = \mathbb{O}, \tag{5.1}$$

where $X(t) = [X_1(t)^T X_2(t)^T]^T$. We adopt the following notations:

$$\begin{aligned}
 Y_1(t) &= X(t), \\
 Y_2(t) &= X'(t), \\
 Y_3(t) &= X''(t), \\
 Y_1'(t) &= X'(t) = Y_2(t), \\
 Y_2'(t) &= X''(t) = Y_3(t), \\
 A_3 Y_3'(t) &= A_3 X'''(t) = -A_2 Y_3(t) - A_1 Y_2(t) - A_0 Y_1(t).
 \end{aligned} \tag{5.2}$$

Or in Matrix form

$$FY'(t) = GY(t), \tag{5.3}$$

where $Y(t) = [Y_1^T(t)Y_2^T(t)Y_3^T(t)]^T$ (where $()^T$ is the transpose tensor) and the coefficient matrices F, G are given by

$$\mathbf{F} = \begin{bmatrix} I_2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & I_2 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & A_3 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbb{O} & I_2 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & I_2 \\ -A_0 & -A_1 & -A_2 \end{bmatrix},$$

$$sF - G = s \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & 2 & 2 & -3 & -2 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix} \quad (5.4)$$

$s-1, s-2, s-3$ finite elementary divisors and $\bar{s}^3$ the infinite elementary divisor of degree 3 of the pencil $sF - G$. There exist matrices nonsingular P, Q such that $PAQ = F_w$ and $PGQ = G_w$, where

$$\mathbf{F}_w = \begin{bmatrix} I_3 & \mathbb{O} \\ \mathbb{O} & H_3 \end{bmatrix}, \quad \mathbf{G}_w = \begin{bmatrix} J_3 & \mathbb{O} \\ \mathbb{O} & I_3 \end{bmatrix}. \quad (5.5)$$

Let $Y(t) = QZ(t)$ then

$$\begin{aligned} FY'(t) &= GY(t), \\ PFQZ'(t) &= PGQZ(t), \\ F_w Z'(t) &= G_w Z(t). \end{aligned} \quad (5.6)$$

For consistent initial conditions the solution is

$$Y(t) = Q_3^1 e^{J_3(t-t_0)} Z_3(t_0) X(t) = Q_3^1 e^{J_3(t-t_0)} Z_3(t_0) \quad (5.7)$$

and for nonconsistent initial conditions the solution is

$$X(t) = f(t)Q_3 e^{J_3(t-t_0)} C + g(t) \sum_{i=0}^2 \frac{(t-t_0)^i}{i!} X^{(i)}(t_0), \quad (5.8)$$

where

$$J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

$$e^{J_3 t} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}.$$
(5.9)

The columns of Q_p are the eigenvectors of the eigenvalues 1,2,3

$$Q_p^T = \begin{bmatrix} 3 & -5 & 3 & -5 & 3 & -5 \\ 1 & -1 & 2 & -2 & 4 & -4 \\ 1 & -1 & 3 & -3 & 9 & -9 \end{bmatrix}^T.$$
(5.10)

Let the initial values of the system be

$$X(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad X'(0) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad X''(0) = \begin{bmatrix} -10 \\ 8 \end{bmatrix},$$

$$Y(0)^T = [1 \quad -3 \quad -2 \quad 0 \quad -10 \quad 8]^T.$$
(5.11)

Then $Y(0) \in \text{colspan} Q_p$ (consistent initial conditions) and the solution of the system is

$$Y(t) = Q_3 e^{J_3 t} Z_3(0)$$
(5.12)

and by calculating $Z_p(0)$ we get

$$Y(0) = Q_3 Z_3(0),$$

$$Z_3(0)^T = [1 \quad -1 \quad -1]^T$$
(5.13)

and the solution of the system is

$$\begin{aligned}
 Y(t) &= Q_3 e^{J_3 t} Z_3(0), \\
 Y(t) &= \begin{bmatrix} 3e^t - e^{2t} - e^{3t} \\ -5e^t + e^{2t} + e^{3t} \\ 3e^t - 2e^{2t} - 3e^{3t} \\ -5e^t + 2e^{2t} + 3e^{3t} \\ 3e^t - 4e^{2t} - 9e^{3t} \\ -5e^t + 4e^{2t} + 9e^{3t} \end{bmatrix}, \\
 X(t) &= \begin{bmatrix} 3e^t - e^{2t} - e^{3t} \\ -5e^t + e^{2t} + e^{3t} \end{bmatrix}.
 \end{aligned} \tag{5.14}$$

Next assume the initial conditions

$$\begin{aligned}
 X(0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad X'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad X''(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
 Y(0)^T &= [0 \ 0 \ 0 \ 0 \ 1 \ 1]^T.
 \end{aligned} \tag{5.15}$$

Then $Y(0) \notin \text{colspan} Q_p$ nonconsistent initial conditions and the solution is

$$\begin{aligned}
 X(t) &= f(t) Q_3^1 e^{J_3 t} C + g(t) \sum_{i=0}^2 \frac{t_i}{i!} X^{(i)}(0), \\
 X(t) &= f(t) \begin{bmatrix} 3e^t c_1 + e^{2t} c_2 + e^{3t} c_3 \\ -5e^t c_1 - e^{2t} c_2 - e^{3t} c_3 \end{bmatrix} + g(t) \frac{t^2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \geq 0.
 \end{aligned} \tag{5.16}$$

The dimension of the domain that describes the solutions of the system is 3.

6. Conclusions

In this paper we investigate systems of the form suggested in [1–3, 5, 7], but from another point of view. By taking into consideration that the relevant pencil is regular, we use the Weierstrass canonical form in order to decompose the differential system into two sub-systems. Afterwards, we give necessary and sufficient conditions for existence and uniqueness of solutions for that general class of linear matrix differential equations of higher order and we provide analytical formulas when we have consistent and non-consistent initial conditions. Moreover, as a further extension of the present paper, we can discuss the case where the pencil is singular. Thus, the Kronecker canonical form is required. The non-homogeneous case has also a special interest, since it appears often in applications. For all these, there is some research in progress.

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