Research Article

On L_p -Approximation by Iterative Combination of Bernstein-Durrmeyer Polynomials

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Received 11 November 2010; Accepted 9 December 2010

Academic Editors: O. Guès and R. Stenberg

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We improve the degree of approximation by Bernstein-Durrmeyer polynomials taking their iterates and obtain error estimate in higher-order approximation.

1. Introduction

The Bernstein-Durrmeyer polynomials

$$M_n(f; t) = (n+1) \sum_{v=0}^n p_{n,v}(t) \int_0^1 p_{n,v}(u) f(u) du, \quad (1.1)$$

where $p_{n,v}(t) = \binom{n}{v} t^v (1-t)^{n-v}$, $t \in [0, 1]$, were introduced by Durrmeyer [1] and extensively studied by Derriennic [2] and several other researchers. It turns out that the order of approximation by these operators is, at best, $\mathcal{O}(n^{-1})$ however smooth the function may be. In order to improve this rate of approximation, we consider an iterative combination $T_{n,k}(f; t)$ of the operators $M_n(f; t)$. This technique of improving the rate of convergence was given by Micchelli [3] who first used it to improve the order of approximation by Bernstein polynomials $B_n(f; t)$. Recently, this technique has been applied to obtain some direct and inverse theorems in ordinary and simultaneous approximation by several sequences of linear positive operators in uniform norm (c.f., e.g., [4–6]). The object of this paper is to study some direct theorems in L_p -approximation by the operators $T_{n,k}(f; t)$.

For $f \in L_p[0, 1]$, the operators $M_n(f; t)$ can be expressed as

$$M_n(f; t) = \int_0^1 W_n(u, t) f(u) du, \quad (1.2)$$

where

$$W_n(u, t) = (n+1) \sum_{v=0}^n p_{n,v}(t) p_{n,v}(u) \quad (1.3)$$

is the kernel of the operators.

For $m \in N^0$ (the set of nonnegative integers), the m th order moment for the operators M_n is defined as

$$\mu_{n,m}(t) = M_n((u-t)^m; t). \quad (1.4)$$

Let $L_B[a, b]$, $C^\infty[a, b]$, $AC[a, b]$, and $BV[a, b]$ denote the classes of bounded Lebesgue integrable, infinitely differentiable, absolutely continuous functions, and functions of bounded variations, respectively, on the interval $[a, b]$.

The Iterative combination $T_{n,k} : L_B[0, 1] \rightarrow C^\infty[0, 1]$ of the operators $M_n(f; t)$ is defined as

$$T_{n,k}(f; t) = (I - (I - M_n)^k)(f; t) = \sum_{r=1}^k (-1)^{r+1} \binom{k}{r} M_n^r(f; t), \quad k \in N, \quad (1.5)$$

where $M_n^0 = I$ and $M_n^r = M_n(M_n^{r-1})$ for $r \in N$.

In Section 2 of this paper, we give some definitions and auxiliary results which will be needed to prove the main results. In Section 3, we obtain an estimate of error in L_p -approximation ($1 \leq p < \infty$) by the iterative combination $T_{n,k}(\cdot; t)$ in terms of L_p -norm of derivatives of the function. From these estimates, we obtain a general error estimate in terms of $2k + 2$ th integral modulus of smoothness of the function.

In what follows, we suppose that $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < 1$ and $I_j = [a_j, b_j]$, $j = 1, 2, 3$. Further, C is a constant not always the same.

2. Preliminaries and Auxiliary Results

In the sequel, we will require the following results.

Lemma 2.1 (see [5]). *For the function $\mu_{n,m}(t)$, one has $\mu_{n,0}(t) = 1$, $\mu_{n,1}(t) = (1 - 2t)/(n + 2)$, and there holds the recurrence relation*

$$(n + m + 2)\mu_{n,m+1}(t) = t(1 - t)\{\mu'_{n,m}(t) + 2m\mu_{n,m-1}(t)\} + (m + 1)(1 - 2t)\mu_{n,m}(t), \quad (2.1)$$

for $m \geq 1$.

Consequently, we have

- (i) $\mu_{n,m}(t)$ is a polynomial in t of degree m ,
- (ii) for every $t \in [0, 1]$, $\mu_{n,m}(t) = \mathcal{O}(n^{-(m+1)/2})$, where $[\beta]$ is the integer part of β .

The m th order moment for the operator M_n^p is defined as $\mu_{n,m}^{[p]}(t) = M_n^p((u-t)^m; t)$, $p \in N$ (the set of natural numbers). We denote $\mu_{n,m}^{[1]}(t)$ by $\mu_{n,m}(t)$.

Lemma 2.2 (see [1]). For the function $p_{n,v}(t)$, there holds the result

$$t^r (1-t)^r \frac{d^r p_{n,v}(t)}{dt^r} = \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i (v-nt)^j q_{i,j,r}(t) p_{n,v}(t), \quad (2.2)$$

where $q_{i,j,r}(t)$ are certain polynomials in t independent of n and v .

Lemma 2.3 (see [7]). There holds the recurrence relation

$$\mu_{n,m}^{[p+1]}(t) = \sum_{j=0}^m \sum_{i=0}^{m-j} \binom{m}{j} \frac{1}{i!} D^i (\mu_{n,m-j}^{[p]}(t)) \mu_{n,i+j}(t). \quad (2.3)$$

Lemma 2.4 (see [7]). For $k, l \in N$, there holds $T_{n,k}((u-t)^l; t) = \mathcal{O}(n^{-k})$.

Using Lemmas 2.1 and 2.3, we can prove the following.

Lemma 2.5. For $p \in N$, $m \in N^0$, and $t \in [0, 1]$, we have

$$\mu_{n,m}^{[p]}(t) = \mathcal{O}(n^{-(m+1)/2}). \quad (2.4)$$

Let $f \in L_p[a, b]$, $1 \leq p < \infty$, and $I_1 \subset (a, b)$. Then, for sufficiently small $\eta > 0$ the Steklov mean $f_{\eta,m}$ of m th order corresponding to f is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left(f(t) + (-1)^{m-1} \Delta_{\sum_{i=1}^m t_i}^m f(t) \right) \prod_{i=1}^m dt_i, \quad t \in I_1, \quad (2.5)$$

where Δ_h^m is the forward difference operator with step length h .

Lemma 2.6. Let $f \in L_p[a, b]$, $1 \leq p < \infty$, and $I_1 \subset (a, b)$. Then, for the function $f_{\eta,m}$, we have

- (a) $f_{\eta,m}$ has derivatives up to order m over I_1 ,
- (b) $\|f_{\eta,m}^{(r)}\|_{L_p(I_1)} \leq C \eta^{-r} \omega_r(f, \eta, [a, b])$, $r = 1, 2, \dots, m$,
- (c) $\|f - f_{\eta,m}\|_{L_p(I_1)} \leq C \omega_m(f, \eta, [a, b])$,
- (d) $\|f_{\eta,m}\|_{L_p(I_1)} \leq C \eta^{-m} \|f\|_{L_p[a,b]}$,
- (e) $\|f_{\eta,m}^{(r)}\|_{L_p(I_1)} \leq C \|f\|_{L_p[a,b]}$,

where C is a constant that depends on i but is independent of f and η .

Following [8, Theorem 18.17] or [9, pages 163–165], the proof of the above lemma easily follows hence the details are omitted.

Let $f \in L_p[0, a]$, $1 \leq p < \infty$. Then, the Hardy-Littlewood majorant $h_f(x)$ of the function f is defined as

$$h_f(x) = \sup_{\xi \neq x} \frac{1}{\xi - x} \int_x^\xi f(t) dt. \quad (2.6)$$

Lemma 2.7. *If $1 < p < \infty$, $f \in L_p[0, a]$, then $h_f \in L_p[0, a]$ and*

$$\|h_f\|_{L_p[0,a]} \leq 2^{1/p} \frac{p}{p-1} \|f\|_{L_p[0,a]}. \quad (2.7)$$

The lemma follows from [10, page 32].

The next lemma gives a bound for the intermediate derivatives of f in terms of the highest-order derivative and the function in L_p -norm.

Lemma 2.8 (see [11]). *Let $1 \leq p < \infty$, $f \in L_p[a, b]$. Suppose $f^{(k)} \in AC[a, b]$ and $f^{(k+1)} \in L_p[a, b]$. Then,*

$$\|f^{(j)}\|_{L_p[a,b]} \leq K_j \left(\|f^{(k+1)}\|_{L_p[a,b]} + \|f\|_{L_p[a,b]} \right), \quad j = 1, 2, \dots, k, \quad (2.8)$$

where K_j are certain constants independent of f .

The dual operator $\widehat{M}_n$ corresponding to the operator M_n is defined as

$$\widehat{M}_n(f; u) = \int_0^1 W_n(u, t) f(t) dt. \quad (2.9)$$

Then, the corresponding m th order moment is given by $\widehat{\mu}_{n,m}(u) = \widehat{M}_n((t - u)^m; u) = \int_0^1 W_n(u, t) (t - u)^m dt$.

Lemma 2.9. *For the function $\widehat{\mu}_{n,m}(u)$, there holds the recurrence relation*

$$(n - m - 2)\widehat{\mu}_{n,m+1}(u) = u(1 - u)\widehat{\mu}'_{n,m}(u) + (m + 1)(1 - 2u)\widehat{\mu}_{n,m}(u) + 2m(u - u^2)\widehat{\mu}_{n,m-1}(u). \quad (2.10)$$

Proof. In view of the relation $u(1-u)p'_{n,k}(u) = (k-nu)p_{n,k}(u)$, we get

$$\begin{aligned}
& u(1-u)\widehat{\mu}'_{n,m}(u) \\
&= (n+1) \sum_{k=0}^n \int_0^1 p_{n,k}(t) u(1-u) p'_{n,k}(u) (t-u)^m dt - mu(1-u)\widehat{\mu}_{n,m-1}(u) \\
&= (n+1) \sum_{k=0}^n (k-nu) p_{n,k}(u) \int_0^1 p_{n,k}(t) (t-u)^m dt - mu(1-u)\widehat{\mu}_{n,m-1}(u) \\
&= (n+1) \sum_{k=0}^n p_{n,k}(u) \int_0^1 ((k-nt) + n(t-u)) p_{n,k}(t) (t-u)^m dt - mu(1-u)\widehat{\mu}_{n,m-1}(u) \\
&= (n+1) \sum_{k=0}^n p_{n,k}(u) \int_0^1 (t-t^2) p'_{n,k}(t) (t-u)^m dt + n\widehat{\mu}_{n,m+1}(u) - mu(1-u)\widehat{\mu}_{n,m-1}(u).
\end{aligned} \tag{2.11}$$

Expanding $(t-t^2)$ as a polynomial in $(t-u)$ and integrating by parts, we get

$$\begin{aligned}
& u(1-u)\widehat{\mu}'_{n,m}(u) \\
&= -(n+1) \sum_{k=0}^n p_{n,k}(u) \int_0^1 p_{n,k}(t) \left[(m+1)(1-2u)(t-u) + m(u-u^2) - (m+2)(t-u)^2 \right] \\
&\quad \times (t-u)^{m-1} dt + n\widehat{\mu}_{n,m+1}(u) - mu(1-u)\widehat{\mu}_{n,m-1}(u) \\
&= -(m+1)(1-2u)\widehat{\mu}_{n,m}(u) - 2m(u-u^2)\widehat{\mu}_{n,m-1}(u) \\
&\quad - (m+2)\widehat{\mu}_{n,m+1}(u) + n\widehat{\mu}_{n,m+1}(u).
\end{aligned} \tag{2.12}$$

Rearrangement of the terms gives (2.10). $\square$

Remark 2.10. From (2.10), it follows that $\widehat{\mu}_{n,m}(u) = \mathcal{O}(n^{-[(m+1)/2]})$, where $[\beta]$ is the integer part of β .

3. Main Result

In this section, we obtain an error estimate in terms of L_p norm. The proof of the case $p > 1$ makes use of Lemma 2.7 regarding Hardy-Littlewood majorant and Lemma 2.8, while for $p = 1$, we require only Lemma 2.8.

Theorem 3.1. *If $p > 1$, $f \in L_p[0, 1]$, f has derivatives of order $2k$ on I_1 with $f^{(2k-1)} \in AC(I_1)$, and $f^{(2k)} \in L_p(I_1)$, then for sufficiently large n*

$$\|T_{n,k}(f; \cdot) - f(\cdot)\|_{L_p(I_2)} \leq Cn^{-k} \left\{ \|f^{(2k)}\|_{L_p(I_1)} + \|f\|_{L_p[0,1]} \right\}. \quad (3.1)$$

Moreover, if $f \in L_1[0, 1]$, f has derivatives up to the order $(2k-1)$ on I_1 with $f^{(2k-2)} \in AC(I_1)$, and $f^{(2k-1)} \in BV(I_1)$, then for sufficiently large n there holds

$$\|T_{n,k}(f; \cdot) - f(\cdot)\|_{L_1(I_2)} \leq Cn^{-k} \left\{ \|f^{(2k-1)}\|_{BV(I_1)} + \|f^{(2k-1)}\|_{L_1(I_2)} + \|f\|_{L_1(I_2)} \right\}, \quad (3.2)$$

where C is a certain constant independent of f and n .

Proof. Let $p > 1$, then for all $u \in I_1$ and $t \in I_2$, we can write

$$f(u) - f(t) = \sum_{j=1}^{2k-1} \frac{f^{(j)}(t)}{j!} (u-t)^j + \frac{1}{(2k-1)!} \int_t^u \varphi(u)(u-v)^{2k-1} f^{(2k)}(v) dv + F(u, t)(1 - \varphi(u)), \quad (3.3)$$

where $\varphi(u)$ is the characteristic function of the interval I_1 and

$$F(u, t) = f(u) - \sum_{j=0}^{2k-1} \frac{f^{(j)}(t)}{j!} (u-t)^j. \quad (3.4)$$

Therefore, operating by $T_{n,k}$ on both sides of (3.3), we obtain three terms, say E_1 , E_2 , and E_3 corresponding to the three terms in the right-hand side of (3.3).

In view of Lemmas 2.4 and 2.8, we get

$$\|E_1\|_{L_p(I_2)} \leq Cn^{-k} \left(\|f^{(2k)}\|_{L_p(I_1)} + \|f\|_{L_p(I_2)} \right). \quad (3.5)$$

Let $h_{f^{(2k)}}$ be the Hardy-Littleood majorant of $f^{(2k)}$ on I_1 . Then, in order to estimate E_2 , it is sufficient to consider the estimate for J_1

$$J_1 = M_n \left(\varphi(u)(u-t)^{2k} \left| h_{f^{(2k)}}(u) \right|; t \right). \quad (3.6)$$

Applying Hölder's inequality, Lemma 2.1, and then Fubini's theorem, we get

$$\begin{aligned} \|J_1\|_{L_p(I_2)}^p &\leq \left(M_n \left(\varphi(u)(u-t)^{2kq}; t \right) \right)^{p/q} \times M_n \left(\varphi(u)(u-t)^p \left| h_{f^{(2k)}}(u) \right|^p; t \right) \\ &\leq Cn^{-kp} \int_{a_1}^{b_1} \int_{a_2}^{b_2} W_n(t, u) \left| h_{f^{(2k)}}(u) \right|^p dt du. \end{aligned} \quad (3.7)$$

Now, in view of Lemmas 2.1 and 2.7, we have

$$\|J_1\|_{L_p(I_2)}^p \leq Cn^{-kp} \int_{a_1}^{b_1} |h_{f^{(2k)}}(u)|^p du \leq Cn^{-kp} \|f^{(2k)}\|_{L_p(I_1)}^p. \quad (3.8)$$

Consequently,

$$\|E_2\|_{L_p(I_2)} \leq Cn^{-k} \|f^{(2k)}\|_{L_p(I_1)}. \quad (3.9)$$

For $u \in [0, 1] \setminus I_1$, $t \in I_2$, we can find a $\delta > 0$ such that $|u - t| \geq \delta$. Thus,

$$\begin{aligned} & |M_n(F(u, t)(1 - \varphi(u)); t)| \\ &= \left| M_n \left(\left(f(u) - \sum_{j=0}^{2k-1} \frac{f^{(j)}(t)}{j!} (u - t)^j \right) (1 - \varphi(u)); t \right) \right| \\ &\leq \delta^{-2k} \left\{ M_n(|f(u)|(u - t)^{2k}; t) + \sum_{j=0}^{2k-1} \frac{|f^{(j)}(t)|}{j!} M_n(|u - t|^{2k+j}; t) \right\} = J_2 + J_3, \quad \text{say.} \end{aligned} \quad (3.10)$$

On an application of Hölder's inequality, Lemma 2.1, and Fubini's theorem, we get

$$\|J_2\|_{L_p(I_2)} \leq Cn^{-k} \left(\int_{a_2}^{b_2} \int_0^1 |f(u)|^p W_n(t, u) dt du \right)^{1/p} \leq Cn^{-k} \|f\|_{L_p[0,1]}. \quad (3.11)$$

Now in view of Lemmas 2.1 and 2.8, we have the inequality

$$\|J_3\|_{L_p(I_2)} \leq Cn^{-k} \left(\|f\|_{L_p(I_2)} + \|f^{(2k)}\|_{L_p(I_2)} \right). \quad (3.12)$$

Combining the estimates (3.5)–(3.12), (3.1) follows.

Now, let $p = 1$. Then, we can expand $f(u)$ for almost all $t \in I_2$ and for all $u \in I_1$, as

$$f(u) - f(t) = \sum_{j=1}^{2k-1} \frac{f^{(j)}(t)}{j!} (u - t)^j + \frac{1}{(2k-1)!} \int_t^u \varphi(u)(u - v)^{2k-1} df^{(2k-1)}(v) + F(u, t)(1 - \varphi(u)), \quad (3.13)$$

where $\varphi(u)$ and $F(u, t)$ are defined as above. Therefore, operating by $T_{n,k}$ on both sides of (3.13), we obtain three terms E_4 , E_5 , and E_6 , say corresponding to the three terms in the right-hand side of (3.13).

Now proceeding as in the case of the estimate of E_1 , we have

$$\|E_4\|_{L_1(I_2)} \leq Cn^{-k} \left(\|f\|_{L_1(I_2)} + \|f^{(2k-1)}\|_{L_1(I_2)} \right). \quad (3.14)$$

It can easily be shown that

$$\|M_n(f, t)\|_{L_1(I)} \leq \|f(t)\|_{L_1(I)}. \quad (3.15)$$

Consequently, by induction, we get

$$\|M_n^r(f, t)\|_{L_1(I)} = \|M_n(M_n^{r-1}(f, u), t)\|_{L_1(I)} \leq \|f(t)\|_{L_1(I)}. \quad (3.16)$$

Therefore, in order to get an estimate for E_5 , it is sufficient to consider the estimate for K

$$\begin{aligned} K &= \left\| M_n \left(\int_t^u \varphi(u) |u - v|^{2k-1} df^{(2k-1)}(v); t \right) \right\|_{L_1(I_2)} \\ &\leq \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} W_n(t, u) |u - t|^{2k-1} \left| \int_t^u df^{(2k-1)}(v) \right| du \right) dt. \end{aligned} \quad (3.17)$$

For each n , there exists the integer $r = r(n)$ s.t. $r/\sqrt{n} \leq \max(b_1 - a_2, b_2 - a_1) \leq (r+1)/\sqrt{n}$

$$\begin{aligned} K &\leq \sum_{l=0}^r \int_{a_2}^{b_2} \left\{ \int_{t+l/\sqrt{n}}^{t+(l+1)/\sqrt{n}} W_n(t, u) |u - t|^{2k-1} \int_t^{t+(l+1)/\sqrt{n}} \varphi(v) \left| df^{(2k-1)}(v) \right| du \right. \\ &\quad \left. + \int_{t-(l+1)/\sqrt{n}}^{t-l/\sqrt{n}} W_n(t, u) |u - t|^{2k-1} \int_{t-(l+1)/\sqrt{n}}^t \varphi(v) \left| df^{(2k-1)}(v) \right| du \right\} dt \\ &\leq \sum_{l=1}^r \int_{a_2}^{b_2} \left\{ \int_{t+l/\sqrt{n}}^{t+(l+1)/\sqrt{n}} W_n(t, u) \frac{n^2}{l^4} |u - t|^{2k+3} \int_t^{t+(l+1)/\sqrt{n}} \varphi(v) \left| df^{(2k-1)}(v) \right| du \right. \\ &\quad \left. + \int_{t-(l+1)/\sqrt{n}}^{t-l/\sqrt{n}} W_n(t, u) \frac{n^2}{l^4} |u - t|^{2k+3} \int_{t-(l+1)/\sqrt{n}}^t \varphi(v) \left| df^{(2k-1)}(v) \right| du \right\} dt \\ &\quad + \int_{a_2}^{b_2} \int_{a_2-1/\sqrt{n}}^{b_2+1/\sqrt{n}} W_n(t, u) |u - t|^{2k-1} \int_{t-n^{-1/2}}^{t+n^{-1/2}} \varphi(v) \left| df^{(2k-1)}(v) \right| du dt \\ &\leq 2 \sum_{l=1}^r \frac{n^2}{l^4} \frac{l+1}{\sqrt{n}} \int_{a_1}^{b_1} W_n(t, u) |u - t|^{2k+3} du \|f^{(2k-1)}\|_{BV(I_1)} \\ &\quad + \frac{2}{\sqrt{n}} \int_{a_1}^{b_1} W_n(t, u) |u - t|^{2k-1} du \|f^{(2k-1)}\|_{BV(I_1)} \\ &\leq Cn^{-k} \|f^{(2k-1)}\|_{BV(I_1)}. \quad (\text{applying Fubini's theorem}). \end{aligned} \quad (3.18)$$

To estimate E_6 , for all $u \in [0, 1] \setminus I_1$, $t \in I_2$, we can choose a $\delta > 0$ such that $|u - t| \geq \delta$. Therefore, we get the inequality

$$\begin{aligned} \|E_4\|_{L_1(I_2)} &\leq C \int_{a_2}^{b_2} \int_0^1 W_n(t, u) |f(u)(1 - \varphi(u))| du dt \\ &\quad + \sum_{i=0}^{2k-1} \frac{1}{i!} \int_{a_2}^{b_2} \int_0^1 W_n(t, u) |f^{(i)}(t)(u - t)^i(1 - \varphi(u))| du dt \\ &= S_1 + S_2, \quad \text{say.} \end{aligned} \quad (3.19)$$

Since, $W_n(t, u)$ is symmetric in t and u , there follows

$$S_1 \leq Cn^{-k} \|f\|_{L_1(I_2)}. \quad (3.20)$$

In view of Lemma 2.8, we obtain

$$S_2 \leq Cn^{-k} \left(\|f\|_{L_1(I_2)} + \|f^{(2k-1)}\|_{L_1(I_2)} \right). \quad (3.21)$$

From the estimates (3.14)–(3.21) and the definition of $T_{n,k}$, we get (3.2). $\square$

Theorem 3.2. *If $p \geq 1$, $f \in L_p[0, 1]$. Then, for all n sufficiently large there holds*

$$\|T_{n,k}(f; \cdot) - f\|_{L_p(I_2)} \leq C \left(\omega_{2k} \left(f, \frac{1}{\sqrt{n}}, p, I_1, \cdot \right) + n^{-k} \|f\|_{L_p[0,1]} \right), \quad (3.22)$$

where C is a constant independent of f and n .

Proof. In order to prove the theorem, it is sufficient to prove it for the function fg , where $g \in C_0^\infty$ be such that $\text{supp } g \subset I_1$ and $g = 1$ in I_2 . Let for convenience $\hat{f} = fg$. Let $\hat{f}_{\eta,2k}$ be the Steklov mean of order $2k$ corresponding to the function $\hat{f}$, where $\eta > 0$ is sufficiently small. Then, we have

$$\begin{aligned} \|T_{n,k}(\hat{f}; \cdot) - \hat{f}\|_{L_p(I_2)} &\leq \|T_{n,k}(\hat{f} - \hat{f}_{\eta,2k}; \cdot)\|_{L_p(I_2)} + \|T_{n,k}(\hat{f}_{\eta,2k}; \cdot) - \hat{f}_{\eta,2k}\|_{L_p(I_2)} \\ &\quad + \|\hat{f}_{\eta,2k} - \hat{f}\|_{L_p(I_2)} = J_1 + J_2 + J_3, \quad \text{say.} \end{aligned} \quad (3.23)$$

Let φ be the characteristic function of I_3 . Then,

$$\begin{aligned} M_n(\hat{f} - \hat{f}_{\eta,2k}; t) &= M_n(\varphi(u)(\hat{f} - \hat{f}_{\eta,2k}); t) + M_n((1 - \varphi(u))(\hat{f} - \hat{f}_{\eta,2k}); t) \\ &= J_4 + J_5 \quad \text{say.} \end{aligned} \quad (3.24)$$

For $p > 1$ using Hölder's inequality and then applying Fubini's theorem, we get

$$\begin{aligned} \|J_4\|_{L_p(I_2)}^p &\leq \int_{a_2}^{b_2} \int_{a_3}^{b_3} W_n(t, u) \left| \hat{f} - \hat{f}_{\eta, 2k} \right|^p du dt \\ &\leq C \left\| \hat{f} - \hat{f}_{\eta, 2k} \right\|_{L_p(I_3)}^p. \end{aligned} \quad (3.25)$$

This in view of property (c) of Steklov's mean implies

$$\|J_4\|_{L_p(I_2)} \leq C \omega_{2k}(\hat{f}, \eta, p, I_2). \quad (3.26)$$

In the case $p = 1$, (3.22) is obtained by boundedness of the operator M_n . Now, as above in Theorem 3.1 for $u \in I_2 \setminus I_3$, we can choose a $\delta > 0$ such that $|u - t| \geq \delta$. Then, from Fubini's theorem and moment estimates 1 of dual operator, we get

$$J_5 \leq \delta^{-2k} \int_{I_2 \setminus I_3} W_n(t, u) (\hat{f} - \hat{f}_{\eta, 2k})(u - t)^{2k} du. \quad (3.27)$$

Therefore,

$$\|J_5\|_{L_p(I_2)} \leq C n^{-k} \left\| \hat{f} - \hat{f}_{\eta, 2k} \right\|_{L_p(I_2)}. \quad (3.28)$$

Consequently, we have the estimate

$$J_1 \leq C \left(\omega_{2k}(\hat{f}, \eta, p, I_2) + n^{-k} \left\| \hat{f} - \hat{f}_{\eta, 2k} \right\|_{L_p(I_2)} \right). \quad (3.29)$$

Now using Theorem 3.1 and Lemma 2.8,

$$J_2 \leq C n^{-k} \left(\left\| \hat{f}_{\eta, 2k} \right\|_{L_p[0,1]} + \left\| \hat{f}_{\eta, 2k}^{(2k)} \right\|_{L_p(I_1)} \right). \quad (3.30)$$

In view of property (c) of Steklov's means, we get the inequality

$$J_3 \leq C \omega_{2k}(\hat{f}, \eta, p, I_1). \quad (3.31)$$

Choosing $\eta = 1/\sqrt{n}$, the result follows from the estimates of J_1 – J_3 . $\square$

Acknowledgment

A. R. Gairola is thankful to the *Council of Scientific and Industrial Research*, New Delhi, India for financial support to carry out the above work.

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