International Scholarly Research Network
ISRN Applied Mathematics
Volume 2011, Article ID 218458, 8 pages
doi:10.5402/2011/218458

Research Article

Synchronization of Time Delayed Systems by Common Delay Time Modulations

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Received 12 March 2011; Accepted 19 April 2011

Academic Editors: J. H. Park and J. Shen

We investigate the synchronization phenomenon between two identical time delayed systems with the common time delay, modulated by a chaotic or random signal. The phenomenon is verified by the conditional Lyapunov exponent. The relation between the present form of synchronization with generalized one is also discussed.

1. Introduction

In 1990, Pecora and Carroll [1] focused on synchronization between identical subsystems under a common forcing. Since then, it is of fundamental importance in a variety of complex physical, chemical, and biological models. Due to finite signal transmission times, switching speeds, and memory effects, systems with both single and multiple delays are ubiquitous in nature and technology. It is well known that dissipative systems with a nonlinear time-delayed feedback or memory can produce chaotic dynamics, and the dimension of their chaotic attractors can be made arbitrarily large by increasing their delay time sufficiently. High-dimensional chaotic systems are important in secure communications. If we consider delayed systems in which time delays are not constant but modulated with time [2–5], then the corresponding communication model will be more secure. Therefore the study of chaos synchronization in modulated time-delayed system is of high practical importance.

It is interesting to observe that all real objects showing synchronous behavior to a variable extent are subject to the influence of noise [6]. For a system of nonlinear oscillators, in the absence of coupling or in the weakly coupling regime where synchronization does not occur, noise applied identically to each oscillator can induced synchronization [7]. Common

noise is also of great relevance to biological systems. In ecology, similar environment shocks may be responsible for synchronization of different populations over a large geographical region [8]. In neural systems, different neurons connected to another group of neurons will receive a common input signal which often approaches a Gaussian distribution as a result of integration of many independent synaptic currents [9].

It is well known that noise-induced synchronization has been widely studied [7, 10, 11]. Theoretical results have inspired some experimental works since the noise-induce synchronization was observed in a biological system between two pairs of uncoupled sensory neurons. In this paper, we show the synchronization between two uncoupled time-delayed systems where their common delays are modulated by chaotic signal or noise. In all of the previously studied noise-induced synchronizations, the noise is present explicitly in the coupled equation. In our synchronization phenomenon chaotic or noising force is not explicitly present in equation but it modulates the common delay time. The synchronization is also verified by evaluating the largest conditional Lyapunov exponent. The relation with generalized synchronization is also discussed.

The rest of this paper is organized as follows. In Section 2, the general theory of time-delay modulation-induced synchronization [12–16] is discussed. Numerical simulations are shown in Section 3. Lorenz system and time-delay Ikeda system are considered as drive and response systems, respectively. In Section 4, the relation with generalized synchronization is discussed. Finally some conclusions are drawn in Section 5.

2. General Theory for Time-Delay-Induced Synchronization

We consider two time-delayed systems which are driven by a common chaotic time delay as

$$\begin{aligned} \dot{x} &= f_1(x, x(t-\tau(t))), \\ \dot{y} &= f_2\big(y, y(t-\tau(t))\big), \end{aligned} \tag{2.1}$$

where x and y are the dynamical variables of the two systems that are governed by vector fields f_1 and f_2, respectively, and $\tau(t)$ is common (identical) chaotic signal which is driven by a signal of another chaotic system. The chaos-driven dynamical systems are illustrated in Figure 1. The two dynamical systems 1 and 2 are supposed to be identical and have different initial conditions.

For complete synchronization, it is assumed that $f_1 \approx f_2$. In general, synchronization can be achieved only when there is an interaction between the dynamical systems (2.1). Since there is no direct coupling between $x(t)$ and $y(t)$, the interaction must be provided by the common time-delay modulation $\tau(t)$. The synchronization is said to occur if $\lim_{t\to\infty}\|y(t) - x(t)\| = 0$ for any initial conditions, where $\|\cdots\|$ represents Euclidean norm. When the common delay time is constant, the two systems (2.1) are independent and they never synchronizes. To emphasize, when the common delay times are modulated in time, we observe that the two systems (2.1) are synchronized. We consider the two cases (2.1) that are driven by common time delay which are chaotic and random signals.

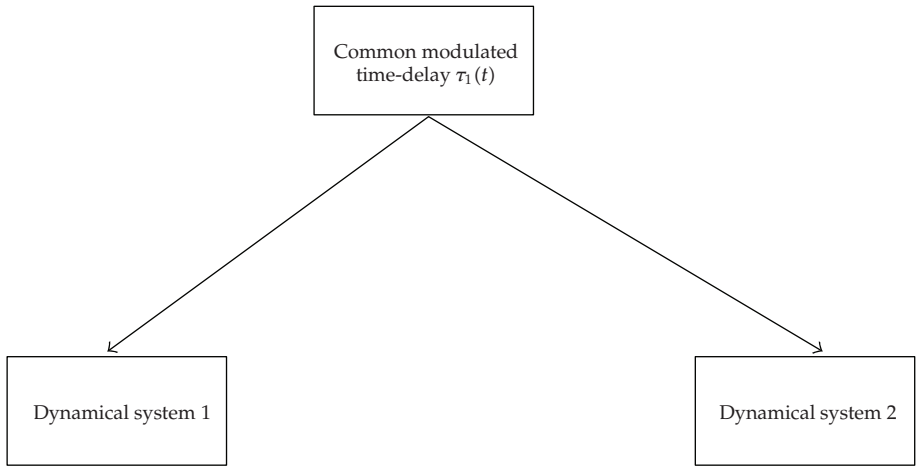

Figure 1: Configuration of chaos-driven dynamical systems.

We consider the system (2.1) in the form

$$\begin{aligned} \dot{x} &= -ax + m_1\Phi(x(t-\tau(t))), \\ \dot{y} &= -ay + m_1\Phi\big(y(t-\tau(t))\big). \end{aligned} \tag{2.2}$$

The class of system (2.2) covers many famous chaotic time-delayed systems, such as the Ikeda system [17], the Mackey-Glass system, Logistic system, and prototype system.

The linear stability of the complete synchronization state $x(t) = y(t)$ is characterized by a quantity called the Largest Conditional Lyapunov Exponent (LCLE). A precise and useful criterion for synchronization is the negativity of LCLE.

Let $\Delta(t) = y(t) - x(t)$ be the synchronization error. Then the linearized error equation is obtained as

$$\dot{\Delta}(t) = -a\Delta(t) + b\Phi'(x(t-\tau(t)))\Delta(t-\tau(t)). \tag{2.3}$$

We define the LCLE as follows [18]:

$$\lambda = \lim_{T\to\infty} \frac{1}{T} \ln \frac{\left\{\int_{T-1}^{T} \Delta^2(s)ds\right\}^{1/2}}{\left\{\int_{-1}^{0} \Delta^2(s)ds\right\}^{1/2}}. \tag{2.4}$$

3. Numerical Simulation

For numerical simulation we consider two uncoupled identical time-delayed systems where their common delay time is modulated by chaotic signal from another chaos-driven system. Take the Lorenz system [19] as driven:

$$\begin{aligned} \dot{u} &= \sigma(v-u), \\ \dot{v} &= ru - v - uw, \\ \dot{w} &= uv - bw. \end{aligned} \tag{3.1}$$

The system (3.1) is chaotic for the set of parameter values $\sigma = 10$, $r = 28$, $b = 8/3$ with initial condition $u(0) = 0.4$, $v(0) = 0.5$, $w(0) = 2.02$.

Consider the coupled Ikeda system as response system:

$$\begin{aligned} \dot{x} &= -ax + m_1 \sin(x(t-\tau_1(t))), \\ \dot{y} &= -ay + m_1 \sin\big(y(t-\tau_1(t))\big). \end{aligned} \tag{3.2}$$

Physically x is the phase lag of the electric field across the resonator, a is the relaxation coefficient for the dynamical variable, and m_1 is the laser intensity injected into the system. τ_1 is the round-trip time of the light in the resonator or feedback delay time in the coupled systems [20]. The Ikeda model was introduced to describe the dynamics of an optical bistable resonator and is well known for delay-induced chaotic behavior [20].

The systems are chaotic for the set of parameter values $a = 1$, $m_1 = 4$, and $\tau_1 = 2$. At this point, the response systems (3.2) are independent time-delayed systems. We take delay time in the form $\tau_1(t) = |w(t)|$. The modulated delay time $\tau_1(t)$ as a function of time is presented in Figure 2(a). In Figures 2(b) and 2(c), the temporal behavior of the response system (3.2) is, respectively shown, which is started from different initial conditions. From these figures it is shown that each of the response systems is in chaotic states. The corresponding synchronization error is shown in Figure 2(d). We emphasize that this phenomenon is purely originated from the common delay time modulation. If the modulation is turned off, the two response systems become two independent systems with a fixed time delay and so they cannot be synchronized. The largest conditional Lyapunov exponent is calculated and its negative (approximately -3.6784×10^{-2}).

Next we consider the Ikeda system with two time delays which are modulated by two chaotic signals. The response system is

$$\begin{aligned} \dot{x} &= -ax + m_1 \sin(x(t-\tau_1(t))) + m_2 \sin(x(t-\tau_2(t))), \\ \dot{y} &= -ay + m_1 \sin\big(y(t-\tau_1(t))\big) + m_2 \sin\big(y(t-\tau_2(t))\big), \end{aligned} \tag{3.3}$$

where $\tau_1(t) = |w(t)|$ and $\tau_2(t) = |u(t)|$. Take the parameter values of system (3.3) as $a = 1.0$, $m_1 = 4.0$, and $m_2 = 0.5$. The chaotic behavior of the response system (3.3) is shown in Figures 3(a) and 3(b). The two response systems are synchronized as shown in Figure 3(c), while each system is in chaotic state. At this position the largest conditional Lyapunov exponent is -8.4253×10^{-2}.

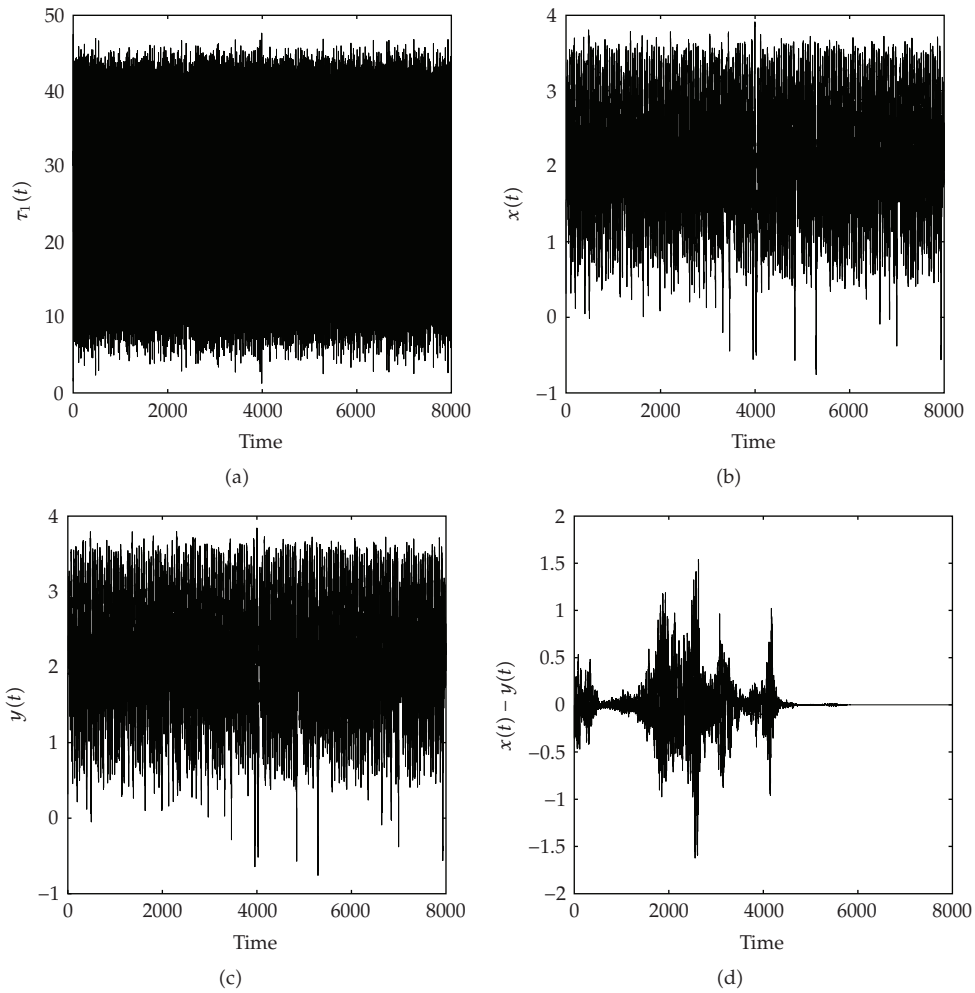

Figure 2: (a) Variation of modulated time-delay $\tau_1(t)$; (b) temporal behavior of $x(t)$ and (c) $y(t)$; (d) corresponding synchronization error.

Next we consider the synchronization where the time delay is modulated by Gaussian noise. We take the system (3.2) where $\tau(t) = |\xi(t)|$, $\xi(t)$ is the Gaussian noise where $\langle \xi(t) \rangle = 0$ and $\langle \xi(t), \xi(t') \rangle = 2N\delta(t - t')$ in which N is the noise intensity and $\langle \cdot \rangle$ denotes the time average. We have integrated numerically the above-mentioned equation using the standard Euler method [21], and specifically the evolution algorithm reads

$$\begin{aligned} x(t + \Delta t) &= x(t) + \Delta t[-ax(t) + m_1 \sin(x(t - \tau_1(t)))], \\ y(t + \Delta t) &= y(t) + \Delta t\big[-ay(t) + m_1 \sin\big(y(t - \tau_1(t))\big)\big], \end{aligned} \tag{3.4}$$

The time step used is $\Delta t = 0.001$, and simulation range typically for a total time is of the order of $t = 10^5$. Figure 4(a) shows the variation of $\tau_1(t)$ by noising force, and the corresponding synchronization error is depicted in Figure 4(b).

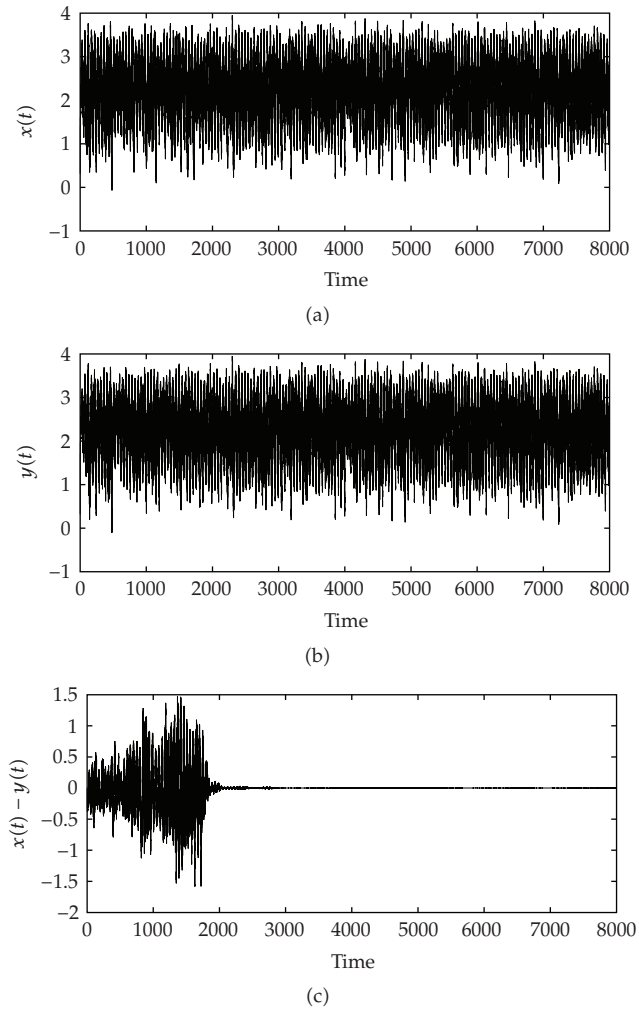

Figure 3: Variation of (a) $x(t)$ and (b) $y(t)$; (c) synchronization error.

4. Relationship with Generalized Synchronization

It is very important to discuss the relationship between the above synchronization and the generalized synchronization. Suppose that one wishes to determine whether there is a generalized synchronization between two unidirectionally coupled systems X and Y. One can imagine an auxiliary response system Z, which is identical to Y and subject to the same driving signal. Regarding whether there is a complete synchronization between Y and Z Abarbanel et al. [22] showed that an affirmative answer would imply a generalized synchronization between X and Y. But in the above synchronization phenomenon, common time delay is modulated by a chaotic forcing which is generated from another driving system or noising forcing. The functional dependency between the driving and response systems is not explicitly presented. That is to say, the effective force acting in the two response systems is quite different form that of the generalized synchronization. The feedback signal is not a common feeding signal as in the generalized synchronization but is proportional to the value

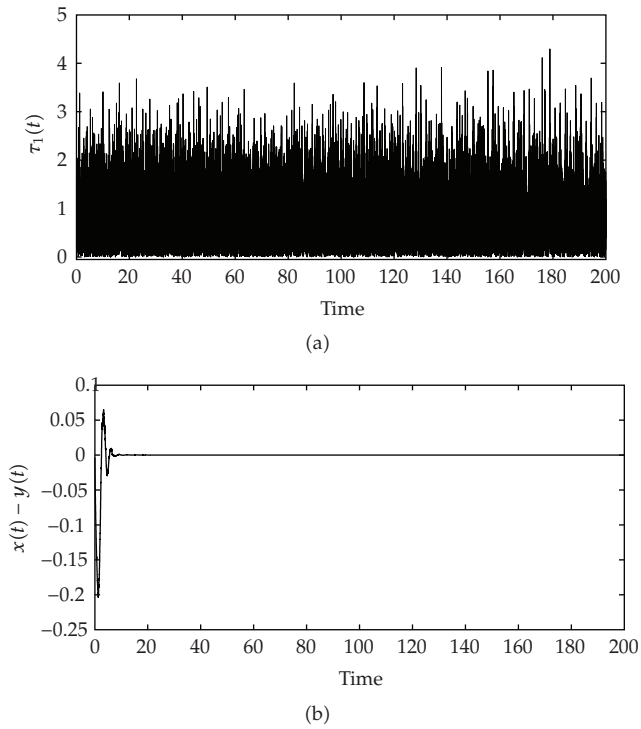

Figure 4: (a) Time delay $\tau_1(t)$ is modulated by noise with noise intensity $N = 1.0$; (b) corresponding synchronization error.

of its own state vector. However, the force is proportional to the driving signal as well as the response one in this case, while the feedback force is proportional to the response signal only in our case. In this respect the above synchronization can be classified as an extension of the generalized synchronization.

5. Conclusions

We have investigated the synchronization between two identical time-delay systems where their common delay time is modulated by chaotic or random signals. The condition for synchronization through the analysis of the conditional largest Lyapunov exponent is discussed. The difference from the previous study is that in our analysis the driving common signal is fed into the delay time implicitly while in the previous study the driving signal is explicitly introduced. We also discussed the relation to generalized synchronization. This type of chaos synchronization is highly applicable in population dynamics, neural network, secure communication, and so forth.

Acknowledgment

S. Banerjee is thankful to Professor T. Kapitaniak for helpful discussion.

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