Research Article

On the Transcendental Entire Solutions of a Certain Differential Equation and Fixed Points

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We will first show that the following differential equation $F^{(k)} - z = e^\alpha(F - z)$ has transcendental entire solutions, where $F = f^n$ and α is an entire function, which improve what we have known and answer an open question raised in the work of Zhang and Yang (2009). And then, the examples are discussed.

1. Introduction and Main Results

In this paper, a meromorphic (resp., entire) function means meromorphic (resp., analytic) in the whole complex plane. We will adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions, such as the characteristic function $T(r, f)$, the counting function of the poles $N(r, f)$, the proximity function $m(r, f)$, and the reduction counting function $\bar{N}(r, f)$ (see [1–3]). In addition, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \rightarrow \infty$, possibly outside a set of r of finite linear measure.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share the value a IM if $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, we say that they share the value a CM. Moreover, if $f - z$ and $g - z$ share 0 CM (resp., IM), we say that f and g share z CM (resp., IM), or we say that f and g have the same fixed-points CM (resp., IM).

In 2008, Lei et al. [4] proved the following.

Theorem A. Suppose that f is a nonconstant meromorphic function, n, k are positive integers, and $n \geq k + 5$, a is a nonzero constant. If f^n and $(f^n)^{(k)}$ share a CM, then $f = ce^{wz/n}$, where c, w are nonzero constants such that $w^k = 1$.

Recently, Zhang and Yang investigated the power of an entire function sharing one value with its derivative and obtained the following results.

Theorem B (see [5]). *Let f be a nonconstant entire function, and let $n \geq 7$ be an integer. If f^n and $(f^n)'$ share 1 CM, then $f^n = (f^n)'$, and $f = ce^{z/n}$, where c is a nonzero constant.*

Theorem C (see [6]). *Let f be a non-constant entire function, and let n be a positive integer. If f^n and $(f^n)^{(k)}$ share 1 CM, and $n \geq k + 1$, then $f^n = (f^n)^{(k)}$, and $f = ce^{wz/n}$, where c, w are nonzero constants such that $w^k = 1$.*

According to the above theorems, one may ask an interesting question: what can be said “if f^n and $(f^n)^{(k)}$ share z CM”? Our purpose of this paper is to solve this question by giving the transcendental entire solutions of the equation

$$F' - z = e^\alpha(F - z), \quad (1.1)$$

where $F = f^n$, and α is an entire function. As an application, we now use a completely different method from that in [4–6] and give the following results.

Theorem 1.1. *Let n ($n \geq 2$) be a positive integer, and α be an entire function, then (1.1) has transcendental entire solutions f and*

$$f = ce^{z/n}, \quad (1.2)$$

where c is a nonzero constant.

Obviously, from Theorem 1.1, we have the following result.

Corollary 1.2. *Let f be a transcendental entire function, and let n ($n \geq 2$) be a positive integer. If f^n and $(f^n)'$ share z CM, then the conclusion of Theorem 1.1 is valid.*

Obviously, Corollary 1.2 improves the results in [6] and answers the open question in [7].

In order to illustrate our condition $n \geq 2$ is sharp, we give examples as follows.

Example 1.3. Let $f = e^{e^z} \int_0^z e^{-e^t} (1 - e^t) t \, dt$, then f is a non-constant solution of

$$f' - z = e^z(f - z), \quad (1.3)$$

and f', f share z CM, while $f \neq f'$.

Example 1.4. Let $f = e^{3z} + 2z/3 + 2/9$, then f is a non-constant solution of

$$f' - z = 3(f - z), \quad (1.4)$$

and f, f' share z CM, but $f \neq f'$.

Theorem 1.5. *Let f be a transcendental entire function, n, k positive integers, and $n \geq k + 2$; if $f^n, (f^n)^{(k)}$ share z IM, then $f = ce^{wz/n}$, where c, w are nonzero constants and $w^k = 1$.*

2. Preliminaries

In order to prove the theorems above, we need some lemmas.

Lemma 2.1 (see [8, 9]). *Suppose that f is a meromorphic and transcendental in the whole complex plane and that*

$$f^n P(f) = Q(f), \quad (2.1)$$

where $P(f)$ and $Q(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients, and the degree of $Q(f)$ is at most n , then

$$m(r, P(f)) = S(r, f). \quad (2.2)$$

Lemma 2.2 (see [1, 10]). *Let f be a transcendental meromorphic function, then*

$$m\left(r, \frac{f^{(l)}}{f}\right) = S(r, f), \quad (2.3)$$

for every positive integer l .

Lemma 2.3 (see [1, 3]). *Let f be a meromorphic function, and $\alpha_i(z)$ with $i = 1, 2, 3$ are three distinct small functions of $f(z)$, then*

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{f - \alpha_1}\right) + \overline{N}\left(r, \frac{1}{f - \alpha_2}\right) + \overline{N}\left(r, \frac{1}{f - \alpha_3}\right) + S(r, f). \quad (2.4)$$

3. Proof of Theorem 1.1

By differentiation to (1.1), we have

$$F'' - 1 = \alpha' e^\alpha (F - z) + e^\alpha (F' - 1). \quad (3.1)$$

Combining (1.1) and (3.1) yields

$$(F'' - 1)(F - z) = \alpha'(F' - z)(F - z) + (F' - z)(F' - 1), \quad (3.2)$$

that is,

$$F''F - \alpha'FF' - (F')^2 = (1 - \alpha'z)F + zF'' - (\alpha'z + z + 1)F' + \alpha'z^2. \quad (3.3)$$

Substituting $F' = n f^{n-1} f'$, $F = f^n$ into (3.3) results in

$$f^n f^{n-2} P = Q, \quad (3.4)$$

where

$$\begin{aligned} P &= n f f'' - n \alpha' f f' - n (f')^2, \\ Q &= (1 - \alpha' z) f^n + z \left\{ n(n-1) f^{n-2} (f')^2 + n f^{n-1} f'' \right\} - n(\alpha' z + z + 1) f^{n-1} f' + \alpha' z^2 \end{aligned} \quad (3.5)$$

are differential polynomials in f , and the degree of Q is n . Lemma 2.1 gives $m(r, f^{n-2}P) = S(r, f)$ and

$$T(r, f^{n-2}P) = S(r, f). \quad (3.6)$$

We now prove $P \equiv 0$. Suppose that $P \not\equiv 0$, and $n \geq 3$, then by (3.6) and Lemma 2.1, we obtain $T(r, f) = S(r, f)$, which is impossible. We may now assume that $n = 2$. So, we get

$$P = 2 f f'' - 2 \alpha' f f' - 2 (f')^2. \quad (3.7)$$

It follows from (3.4) and Lemma 2.2 that $T(r, P) = S(r, f)$, and $m(r, P/f^2) = S(r, f)$. These equalities above show that

$$m\left(r, \frac{1}{f}\right) = S(r, f). \quad (3.8)$$

In addition, we can see from the expression of P that the multiple zeros of f must be the zeros of P ; hence, $N(r, 1/f) = \overline{N}(r, 1/f) + S(r, f)$. This together with (3.8) and the first theorem (e.g., see [2, Theorem 1.2]) will result in

$$T(r, f) = \overline{N}\left(r, \frac{1}{f}\right) + S(r, f). \quad (3.9)$$

By (3.7), we find

$$P' = 2 f f''' - 2 \alpha'' f f' - 2 \alpha' f f'' - 2 \alpha' (f')^2 - 2 f' f''. \quad (3.10)$$

Let z_0 be a simple zero of f ; it follows from (3.7) and (3.10) that $P(z_0) + 2\{f'(z_0)\}^2 = 0$ and $P'(z_0) = -2\alpha'(z_0)\{f'(z_0)\}^2 - 2f'(z_0)f''(z_0)$, which implies that z_0 is a zero of $Pf'' + (\alpha'P - P')f'$. Let

$$g = \frac{P f'' + (\alpha' P - P') f'}{f}. \quad (3.11)$$

Thus,

$$T(r, g) = S(r, f). \quad (3.12)$$

By (3.11), we obtain

$$f'' = \alpha_1 f + \beta_1 f', \quad (3.13)$$

where $\alpha_1 = g/P$, $\beta_1 = (P' - \alpha'P)/P$,

$$T(r, \alpha_1) = S(r, f), \quad T(r, \beta_1) = S(r, f). \quad (3.14)$$

Substituting (3.13) into (3.7) will yield

$$P = 2\alpha_1 f^2 + 2(\beta_1 - \alpha') f f' - 2(f')^2. \quad (3.15)$$

Also from (3.13), we find

$$f''' = \alpha_2 f + \beta_2 f', \quad (3.16)$$

where $\alpha_2 = \alpha'_1 + \alpha_1 \beta_1$, $\beta_2 = \alpha_1 + \beta'_1 + \beta_1^2$,

$$T(r, \alpha_2) = S(r, f), \quad T(r, \beta_2) = S(r, f). \quad (3.17)$$

It follows from (3.16), (3.13), and (3.10) that

$$P' = 2(\alpha_2 - \alpha' \alpha_1) f^2 + 2(\beta_2 - \alpha' \beta_1 - \alpha'' - \alpha_1) f f' - 2(\alpha' + \beta_1) (f')^2. \quad (3.18)$$

From (3.15) and (3.18), we get

$$P' = 2(\alpha_2 - \alpha' \alpha_1) f^2 + 2(\beta_2 - \alpha' \beta_1 - \alpha'' - \alpha_1) f f' - 2(\alpha' + \beta_1) \left\{ \alpha_1 f^2 + (\beta_1 - \alpha') f f' - \frac{P}{2} \right\}. \quad (3.19)$$

As noted from $P' = (\alpha' + \beta_1)P$ and the definitions of α_2 and β_2 , we can deduce from (3.19) that

$$(\alpha'_1 - 2\alpha_1 \alpha') f + \left\{ \beta'_1 - \alpha'' - \alpha' \beta_{1+} (\alpha')^2 \right\} f' = 0. \quad (3.20)$$

Suppose that $\alpha'_1 - 2\alpha_1 \alpha' \equiv 0$, then $\alpha_1 = c e^{2\alpha}$, where c is a nonzero constant. If $e^\alpha \equiv 1$, then (1.1) gives $f = c e^{z/n}$, which is the desired result. If $e^\alpha \not\equiv 1$, this together with (1.1) will lead to $N(r, 1/f) = S(r, f)$, which contradicts with (3.9). We now assume that $\alpha'_1 - 2\alpha_1 \alpha' \not\equiv 0$, in the same way, (3.20) shows that $N(r, 1/f) = S(r, f)$, which also contradicts with (3.9). Hence, $P \equiv 0$. Then, it follows from (3.4) that $Q \equiv 0$ and from (3.3) that

$$F'' F - \alpha' F' F - (F')^2 = 0. \quad (3.21)$$

Obviously, we obtain from (3.21) that

$$\frac{F''}{F'} = \alpha' + \frac{F'}{F}. \quad (3.22)$$

By integration, we have $F' = dFe^\alpha$, where d is a nonzero constant. Substituting this into (1.1) will yield

$$(d-1)f^n = \frac{z(1-e^\alpha)}{e^\alpha}. \quad (3.23)$$

If $d = 1$, by this and (1.1), we have $F' = F$, that is, $f = ce^{z/n}$, where c is a nonzero constant.

If $d \neq 1$, suppose that f has infinitely many zeros, and e^α is not a constant, then the zeros of f are the zeros of $1 - e^\alpha$, with multiplicities at least n . It follows from the equation above and the second fundamental theorem (or Lemma 2.3) that

$$T(r, e^\alpha) \leq \overline{N}(r, e^\alpha) + \overline{N}\left(r, \frac{1}{e^\alpha}\right) + \overline{N}\left(r, \frac{1}{e^\alpha - 1}\right) + S(r, e^\alpha), \quad (3.24)$$

which with $n \geq 2$ may lead to a contradiction. Now, we suppose that f has infinitely many zeros, and e^α is a constant. We suppose that $e^\alpha = B$, then we can get a contradiction.

If $d \neq 1$, suppose that f has finitely many zeros, and let

$$f = m(z)e^\beta, \quad (3.25)$$

where $m(z)$ is a polynomial and β is an entire function. In this case, (3.23) can be written as

$$\frac{(1-d)m^n}{z} e^{n\beta} + e^{-\alpha} = 1. \quad (3.26)$$

Applying the second fundamental theorem (or Lemma 2.3) to (3.26), we have

$$n(T(r, f) = T(r, e^{n\beta}) + S(r, f) = S(r, f), \quad (3.27)$$

which is impossible. This also proves Theorem 1.1.

Remark 3.1. By examining the proof of Theorem 1.1 more carefully, here as an extension of Theorem 1.1, we would like to pose the following.

Conjecture 3.2. Let f be a transcendental entire function, and let n be a positive integer. If f^n and $(f^n)^{(k)}$ share z CM, and $n \geq k + 1$, then $f^n = (f^n)^{(k)}$,

$$f = ce^{wz/n}, \quad (3.28)$$

where c, w are nonzero constants such that $w^k = 1$.

4. Proof of Theorem 1.5

Let $F = f^n$, if $F \neq F^{(k)}$, Lemma 2.3 implies

$$\begin{aligned}
 T(r, F) &\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-z}\right) + S(r, F) \\
 &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{F^{(k)}/F-1}\right) + S(r, f) \\
 &\leq \overline{N}\left(r, \frac{1}{f}\right) + T\left(r, \frac{F^{(k)}}{F}\right) + S(r, f) \\
 &\leq \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{F^{(k)}}{F}\right) + S(r, f) \\
 &\leq \overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}\left(r, \frac{1}{f}\right) + S(r, f) \\
 &\leq (k+1)T(r, f) + S(r, f),
 \end{aligned} \tag{4.1}$$

which contradicts with $n \geq k+2$, and hence $F \equiv F^{(k)}$, and our desired result follows.

Finally, we conclude the paper with the following.

Question 1. Let R be a rational function, and let α be an entire function, then for two integers n, k such that $n \geq k+1$, what can be said about the transcendental entire solution of $(f^n)^{(k)} - z = \text{Re}^\alpha(f^n - z)$, can f be expressed as $f = ce^{wz/n}$ for some constants $c (\neq 0), w^k = 1$?

Question 2. What can be said if the condition in Theorem 1.5 “ $n \geq k+2$ ” is replaced by “ $n \geq k+1$ ”?

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