Research Article

On Differential Properties R_ν -Generalized Solution of the Dirichlet Problem with Coordinated Degeneration of the Input Data

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We consider the first boundary value problem for second-order differential equation with strong singularity caused by coordinated degeneration of the input data. For this problem, we study the differential properties of the R_ν -generalized solution, that is, the fact that it belongs to the space $H_{2,\nu+\beta/2}^{k+2}(\Omega)$.

1. Introduction

In this paper, we study of differential properties of the Dirichlet problem for elliptic equations possessing strong singularity. A boundary value problem is said to possess strong singularity if its solution $u(x)$ does not belong to the Sobolev space $W_2^1(H^1)$ or, in other words, the Dirichlet integral of the solution $u(x)$ diverges.

Boundary value problems with strong singularity caused by the singularity in the initial data or by the internal properties of solution are found in the physics of plasma and gas discharge, electrodynamics, nuclear physics, nonlinear optics, and other branches of physics. In particular cases, numerical methods for problems of electrodynamics and quantum mechanics with string singularity were constructed based on separation of singular and components, mesh refinement near singular points, multiplicative extraction of singularities, and so forth (see, e.g., [1–4]).

The notion of an R_ν -generalized solution was introduced in [5] for boundary value problems with strong singularity in a solution, that is, for problems in which it is impossible to define a generalized (weak) solution or the generalized solution does not have the desired regularity. Such a new concept of solution led to distinction of two classes of boundary value

problems: problems with coordinated and uncoordinated degeneration of input data; it also made it possible to study the existence and uniqueness of solutions as well as its coercivity and differential properties in the weighted Sobolev spaces (see [6–8]).

In [9–11], h, p and $h-p$ versions of the finite element method were constructed and investigated for a Dirichlet problem with strong singularity of solution.

In the present paper, we consider the first boundary value problem for second-order differential equation with strong singularity caused by coordinated degeneration of the input data. For this problem, we study the differential properties of the R_ν -generalized solution, that is, the fact that it belongs to the space $H_{2,\nu+\beta/2}^{k+2}(\Omega)$.

2. The Basic Designations

Let R^2 denote the two-dimensional Euclidean space with $x = (x_1, x_2)$. Let Ω be a bounded convex domain with boundary $\partial\Omega$, and let $\bar{\Omega}$ be the closure of Ω . We denote by $\partial\Omega^o$ the set of the points of $\partial\Omega$: $\partial\Omega^{(i)} \in \partial\Omega$, $\partial\Omega^o = \bigcup_{i=1}^n \partial\Omega^{(i)}$. Let us assume that boundary $\partial\Omega$ is a piecewise smooth and $\partial\Omega \setminus \partial\Omega^o \in C^2$.

Let $\rho(x)$ be a weight function that is infinitely differentiable and positive everywhere, except at the points of $\partial\Omega^o$, with $\rho(x)$ coinciding in some neighborhood of each point $\partial\Omega^{(i)}$ ($i = \overline{1, n}$) with the distance to $\partial\Omega^{(i)}$. Moreover, the derivatives of $\rho(x)$ satisfy the inequality

$$\left| \frac{\partial \rho(x)}{\partial x_j} \right| \leq \delta, \quad \left| \frac{\partial^{|i|} \rho^\gamma(x)}{\partial x_1^{i_1} \partial x_2^{i_2}} \right| \leq \sigma \rho^{\gamma-i}(x), \quad (2.1)$$

where $j = 1, 2$; $i = (i_1, i_2)$, $|i| = i_1 + i_2$ are nonnegative integers; δ, σ, γ is real, $\delta > 1$.

We introduce the weighted Sobolev space $H_{2,\alpha}^k(\Omega)$ which at fixed integer $k \geq 0$ and real $\alpha > -1$ is updating of the set $C^\infty(\bar{\Omega})$ infinitely differentiated in $\bar{\Omega}$ function on norm

$$\|u(x)\|_{H_{2,\alpha}^k(\Omega)} = \left(\sum_{|i| \leq k} \int_{\Omega} \rho^{2(\alpha+|i|-k)}(x) |D^i u(x)|^2 dx \right)^{1/2}. \quad (2.2)$$

Seminorms in the this space look like

$$|u(x)|_{H_{2,\alpha}^s(\Omega)} = \left(\sum_{|i|=s} \int_{\Omega} \rho^{2(\alpha+|i|-k)}(x) |D^i u(x)|^2 dx \right)^{1/2}, \quad (2.3)$$

where s is some integer, $0 \leq s \leq k$. As usual, $H_{2,\alpha}^0(\Omega) = L_{2,\alpha}(\Omega)$.

Let us count function $\varphi(x)$ from space $H_{2,\alpha}^{k-1/2}(\partial\Omega)$ if there is such function $\Phi(x)$ from space $H_{2,\alpha}^k(\Omega)$ that

$$\Phi(x)|_{\partial\Omega} = \varphi(x), \quad \|\varphi(x)\|_{H_{2,\alpha}^{k-1/2}(\partial\Omega)} = \inf_{\Phi|_{\partial\Omega}=\varphi} \|\Phi(x)\|_{H_{2,\alpha}^k(\Omega)}. \quad (2.4)$$

Denote by $H_{\infty,-\alpha}^k(\Omega, C)$ the set of functions with the norm satisfying the inequality

$$\|u(x)\|_{H_{\infty,-\alpha}^k(\Omega, C)} = \max_{|\lambda| \leq k} \text{vrai max}_{x \in \Omega} \left| \rho^{-\alpha+|\lambda|}(x) D^\lambda u(x) \right| \leq C, \quad (2.5)$$

where C is a positive constant independent of $u(x)$.

As usual, $H_{\infty,-\alpha}^0(\Omega, C) = L_{\infty,-\alpha}(\Omega, C)$.

3. Statement of Problem: Definition of the R_ν -Generalized Solution

Suppose that the differential equation

$$Lu(x) \equiv - \sum_{l,s=1}^2 a_{ls}(x) \frac{\partial^2 u(x)}{\partial x_l \partial x_s} + \sum_{l=1}^2 a_l(x) \frac{\partial u(x)}{\partial x_l} + a(x)u(x) = f(x), \quad x \in \Omega, \quad (3.1)$$

is given in the domain Ω , with the boundary condition

$$u(x) = \varphi(x), \quad x \in \partial\Omega. \quad (3.2)$$

We suppose that $a_{12}(x) = a_{21}(x)$ and for some real β

$$a_{ls}(x) \in H_{\infty,-\beta}^1(\Omega, C_1) \quad (l, s = 1, 2), \quad (3.3)$$

$$a_l(x) \in L_{\infty,-(\beta-1)}(\Omega, C_2) \quad (l = 1, 2), \quad (3.4)$$

$$a(x) \in L_{\infty,-(\beta-2)}(\Omega, C_3), \quad (3.5)$$

$$\sum_{l,s=1}^2 a_{ls}(x) \xi_l \xi_s \geq C_4 \rho^\beta(x) \sum_{l=1}^2 \xi_l^2, \quad (3.6)$$

$$a(x) \geq C_5 \rho^{\beta-2}(x) \quad \text{a.e. in } \Omega, \quad (3.7)$$

$$f(x) \in L_{2,\mu}(\Omega), \quad (3.8)$$

$$\varphi(x) \in H_{2,\mu+\beta-1}^{1/2}(\partial\Omega), \quad (3.9)$$

where ξ_1, ξ_2 are arbitrary real parameters, C_i are positive constants independent of x ($i = \overline{1, 5}$), and μ is nonnegative.

The boundary value problem (3.1), (3.2) under conditions (3.3)–(3.9) will be called *the Dirichlet problem with coordinated degeneration of the input data*.

We introduce the bilinear and linear forms

$$\begin{aligned} a(u, v) = \sum_{l,s=1}^2 \int_{\Omega} \left[a_{ls}(x) \rho^{2\nu}(x) \frac{\partial u(x)}{\partial x_s} \frac{\partial v(x)}{\partial x_l} + a_{ls}(x) \frac{\partial \rho^{2\nu}(x)}{\partial x_l} \frac{\partial u(x)}{\partial x_s} v(x) \right. \\ \left. + \frac{\partial a_{ls}(x)}{\partial x_l} \rho^{2\nu}(x) \frac{\partial u(x)}{\partial x_s} v(x) \right] dx \end{aligned}$$

$$\begin{aligned}
& + \sum_{l=1}^2 \int_{\Omega} a_l(x) \rho^{2\nu}(x) \frac{\partial u(x)}{\partial x_l} v(x) dx + \int_{\Omega} a(x) \rho^{2\nu}(x) u(x) v(x) dx, \\
(f, v) & = \int_{\Omega} \rho^{2\nu}(x) f(x) v(x) dx.
\end{aligned} \tag{3.10}$$

Definition 3.1. A function $u_\nu(x)$ in the space $H_{2,\nu+\beta/2}^1(\Omega)$ will be called R_ν -generalized solution of the Dirichlet problem with coordinated degeneration of the input data if $u_\nu(x) = \varphi(x)$ almost everywhere in $\partial\Omega$ and for all $v(x)$ in $H_{2,\nu+\beta/2}^1(\Omega)$, the identity $a(u_\nu, v) = (f, v)$ holds, where ν is arbitrary but fixed and satisfies the inequality

$$\nu \geq \mu + \frac{\beta}{2} - 1. \tag{3.11}$$

4. Existence, Uniqueness, and Coercive of the R_ν -Generalized Solution of the Dirichlet Problem with Coordinated Degeneration of the Input Data

Theorem 4.1 (see [6]). Assume that conditions (3.3)–(3.11) are satisfied and inequality

$$2(C_1(2\delta|\nu| + 1) + 0.5C_2)^2 < C_4C_5 \tag{4.1}$$

holds. Then there exists a unique R_ν -generalized solution $u_\nu(x)$ of the Dirichlet problem with coordinated degeneration of the input data in the space $H_{2,\nu+\beta/2}^1(\Omega)$ and the following estimate holds:

$$\|u_\nu(x)\|_{H_{2,\nu+\beta/2}^1(\Omega)} \leq C_6 \left(\|f(x)\|_{L_{2,\mu}(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta-1}^{1/2}(\partial\Omega)} \right), \tag{4.2}$$

where C_6 is a positive constant independent of $u_\nu(x)$, $f(x)$, and $\varphi(x)$.

Theorem 4.2. Assume that conditions (3.3)–(3.8), (4.1) are satisfied and also

$$\varphi(x) \in H_{2,\mu+\beta}^{3/2}(\partial\Omega), \tag{4.3}$$

$$\nu \geq \mu + \frac{\beta}{2}, \tag{4.4}$$

$$\nu + \frac{\beta}{2} > 2 \tag{4.5}$$

holds. Then R_ν -generalized solution $u_\nu(x)$ of the Dirichlet problem with coordinated degeneration of the input data belongs to space $H_{2,\nu+\beta/2}^2(\Omega)$ and coercivity inequality is valid

$$\|u_v(x)\|_{H_{2,\nu+\beta/2}^2(\Omega)} \leq C_7 \left(\|f(x)\|_{L_{2,\mu}(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{3/2}(\partial\Omega)} \right), \quad (4.6)$$

where C_7 is a positive constant independent of $u_v(x)$, $f(x)$, and $\varphi(x)$.

For case $\varphi(x) = 0$, the proof of the Theorem 4.2 resulted in [6, Theorem 3].

5. Differential Properties of an R_ν -Generalized Solution of the Dirichlet Problem with Coordinated Degeneration of the Input Data

Theorem 5.1. Let $a_{12}(x) = a_{21}(x)$ and conditions

$$a_{ls}(x) \in H_{\infty,-\beta}^{k+1}(\Omega, C_1) \quad (l, s = 1, 2), \quad (5.1)$$

$$a_l(x) \in H_{\infty,-(\beta-1)}^k(\Omega, C_2) \quad (l = 1, 2), \quad (5.2)$$

$$a(x) \in H_{\infty,-(\beta-2)}^k(\Omega, C_3), \quad (5.3)$$

$$f(x) \in H_{2,\mu}^k(\Omega), \quad (5.4)$$

$$\varphi(x) \in H_{2,\mu+\beta}^{k+3/2}(\partial\Omega), \quad (5.5)$$

$$\nu + \frac{\beta}{2} > k + 2 \quad (5.6)$$

are satisfied at some fixed integer $k \geq 1$ and also (3.6), (3.7), (4.1), and (4.4) hold. Then, the R_ν -generalized solution $u_v(x)$ of the Dirichlet problem with coordinated degeneration of the input data belongs to space $H_{2,\nu+\beta/2}^{k+2}(\Omega)$ and following estimate holds

$$\|u_v(x)\|_{H_{2,\nu+\beta/2}^{k+2}(\Omega)} \leq C_8 \left(\|f(x)\|_{H_{2,\mu}^k(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{k+3/2}(\partial\Omega)} \right), \quad (5.7)$$

where C_8 is a positive constant independent of $u_v(x)$, $f(x)$, and $\varphi(x)$.

Proof. At performance of conditions (5.1)–(5.6) of the Theorem 5.1 conditions of the Theorem 4.2 are satisfied, therefore, function $u_v(x)$ belongs to space $H_{2,\nu+\beta/2}^2(\Omega)$.

The proof of the Theorem 5.1 we will lead a method of a mathematical induction to two stages: at the first stage, we will check up validity of the statement for $k = 1$; at the second stage, we will prove it for $k = m$ the assumption that it is true at $k = m - 1$. $\square$

Stage 1. Let conditions (3.6), (3.7), (4.1), and (4.4) be satisfied and also conditions

$$a_{ls}(x) \in H_{\infty,-\beta}^2(\Omega, C_1) \quad (l, s = 1, 2) \quad (5.1_1)$$

$$a_l(x) \in H_{\infty,-(\beta-1)}^1(\Omega, C_2) \quad (l = 1, 2), \quad (5.2_1)$$

$$a(x) \in H_{\infty, -(\beta-2)}^1(\Omega, C_3), \quad (5.3_1)$$

$$f(x) \in H_{2,\mu}^1(\Omega), \quad (5.4_1)$$

$$\varphi(x) \in H_{2,\mu+\beta}^{5/2}(\partial\Omega), \quad (5.5_1)$$

$$v + \frac{\beta}{2} > 3 \quad (5.6_1)$$

holds.

Let us designate $u_{vj}(x) = \partial u_v(x) / \partial x_j$ ($j = 1, 2$). We will prove that functions $u_{vj}(x)$ belong to space $H_{2,v+\beta/2}^1(\Omega)$. For this purpose we will fix j and we will establish the upper bound a square of norm

$$\|u_{vj}(x)\|_{H_{2,v+\beta/2}^1(\Omega)}^2 = \sum_{|\lambda| \leq 1} \int_{\Omega} \rho^{2(v+\beta/2+|\lambda|-1)}(x) \left| D^{\lambda} u_{vj}(x) \right|^2 dx. \quad (5.8)$$

We consider the sum

$$\begin{aligned} \sum_{|\lambda| \leq 1} \left| D^{\lambda} u_{v1}(x) \right|^2 &= |u_{v1}(x)|^2 + \left| \frac{\partial u_{v1}(x)}{\partial x_1} \right|^2 + \left| \frac{\partial u_{v1}(x)}{\partial x_2} \right|^2 \\ &= \left| \frac{\partial u_v(x)}{\partial x_1} \right|^2 + \left| \frac{\partial^2 u_v(x)}{\partial x_1^2} \right|^2 + \left| \frac{\partial^2 u_v(x)}{\partial x_1 \partial x_2} \right|^2. \end{aligned} \quad (5.9)$$

We add to the right part of this equality nonnegative composed $|u_v(x)|^2$, $|\partial u_v(x) / \partial x_2|^2$, $|\partial^2 u_v(x) / \partial x_2^2|^2$, $|\partial^2 u_v(x) / \partial x_2 \partial x_1|^2$. Then, we will receive an inequality

$$\sum_{|\lambda| \leq 1} \left| D^{\lambda} u_{v1}(x) \right|^2 \leq \sum_{|\alpha| \leq 2} |D^{\alpha} u_v(x)|^2. \quad (5.10)$$

Similarly, the inequality turns out

$$\sum_{|\lambda| \leq 1} \left| D^{\lambda} u_{v2}(x) \right|^2 \leq \sum_{|\alpha| \leq 2} |D^{\alpha} u_v(x)|^2. \quad (5.11)$$

Let's take advantage of designations $|\alpha| = |\lambda| + 1$ for $|\lambda| \leq 1$ and two previous inequalities. Then, from (5.8), we will receive an estimation

$$\|u_{vj}\|_{H_{2,v+\beta/2}^1(\Omega)}^2 \leq \sum_{|\alpha| \leq 2} \int_{\Omega} \rho^{2(v+\beta/2+(x)|\alpha|-2)} |D^{\alpha} u_v(x)|^2 dx = \|u_v\|_{H_{2,v+\beta/2}^2(\Omega)}^2. \quad (5.12)$$

From this estimation (as $u_v(x) \in H_{2,v+\beta/2}^2(\Omega)$) limitation of function $u_{vj}(x)$ in norm of space $H_{2,v+\beta/2}^1(\Omega)$ follows, that is, $u_{vj}(x) \in H_{2,v+\beta/2}^1(\Omega)$.

We write down integrated identity $a(u_v, v) = (f, v)$ in the form of

$$\begin{aligned} \sum_{l,s=1}^2 \int_{\Omega} \frac{\partial(\rho^{2v}(x)a_{ls}(x)v(x))}{\partial x_l} \frac{\partial u_v(x)}{\partial x_s} dx + \sum_{l=1}^2 \int_{\Omega} \rho^{2v}(x)a_l(x) \frac{\partial u_v(x)}{\partial x_l} v(x) dx \\ + \int_{\Omega} \rho^{2v}(x)a(x)u_v(x)v(x)dx = \int_{\Omega} \rho^{2v}(x)f(x)v(x)dx. \end{aligned} \quad (5.13)$$

In identity (5.13), we will designate $v(x) = \partial v_1(x)/\partial x_j$ ($j = 1, 2$), where function $v_1(x)$ belongs to space $\dot{H}_{2,\nu+\beta/2+1}^2(\Omega)$.

Let us consider the first composed left part of identity (5.13). We will apply to it twice formula of integration in parts (we will consider that $v(x) = 0$ if $x \in \partial\Omega$) and also the formula of differentiation of product of functions. We will receive equality

$$\begin{aligned} \sum_{l,s=1}^2 \int_{\Omega} \frac{\partial(\rho^{2v}a_{ls}v)}{\partial x_l} \frac{\partial u_v}{\partial x_s} dx = \int_{\Omega} \frac{\partial}{\partial x_j} \left(- \sum_{l,s=1}^2 \rho^{2v}a_{ls} \frac{\partial^2 u_v}{\partial x_l \partial x_s} v_1 \right) dx \\ + \int_{\Omega} \frac{\partial \rho^{2v}}{\partial x_j} \sum_{l,s=1}^2 a_{ls} \frac{\partial^2 u_v}{\partial x_l \partial x_s} v_1 dx + \sum_{l,s=1}^2 \int_{\Omega} \rho^{2v} \frac{\partial a_{ls}}{\partial x_j} \frac{\partial^2 u_v}{\partial x_l \partial x_s} v_1 dx \\ - \sum_{l,s=1}^2 \int_{\Omega} \frac{\partial(\rho^{2v}a_{ls}v_1)}{\partial x_l} \frac{\partial u_{vj}}{\partial x_s} dx. \end{aligned} \quad (5.14)$$

To the second and the third composed left part also for the right part of identity (5.13), we will apply the formula of integration in parts and the formula of differentiation of product of functions. As a result, we will receive equalities

$$\begin{aligned} \sum_{l=1}^2 \int_{\Omega} \rho^{2v}a_l \frac{\partial u_v}{\partial x_l} v dx = \int_{\Omega} \frac{\partial}{\partial x_j} \left(\sum_{l=1}^2 \rho^{2v}a_l \frac{\partial u_v}{\partial x_l} v_1 \right) dx - \int_{\Omega} \frac{\partial \rho^{2v}}{\partial x_j} \sum_{l=1}^2 a_l \frac{\partial u_v}{\partial x_l} v_1 dx \\ - \sum_{l=1}^2 \int_{\Omega} \rho^{2v} \frac{\partial a_l}{\partial x_j} \frac{\partial u_v}{\partial x_l} v_1 dx - \sum_{l=1}^2 \int_{\Omega} \rho^{2v}a_l \frac{\partial u_{vj}}{\partial x_l} v_1 dx, \end{aligned} \quad (5.15)$$

$$\begin{aligned} \int_{\Omega} \rho^{2v}(x)a(x)u_v(x)v(x)dx = \int_{\Omega} \frac{\partial}{\partial x_j} (\rho^{2v}au_v v_1) dx - \int_{\Omega} \frac{\partial \rho^{2v}}{\partial x_j} au_v v_1 dx \\ - \int_{\Omega} \rho^{2v} \frac{\partial a}{\partial x_j} u_v v_1 dx - \int_{\Omega} \rho^{2v} au_{vj} v_1 dx, \end{aligned} \quad (5.16)$$

$$\int_{\Omega} \rho^{2v} f v dx = \int_{\Omega} \frac{\partial}{\partial x_j} (\rho^{2v} f v_1) dx - \int_{\Omega} \frac{\partial \rho^{2v}}{\partial x_j} f v_1 dx - \int_{\Omega} \rho^{2v} \frac{\partial f}{\partial x_j} v_1 dx. \quad (5.17)$$

In view of equalities (5.14)–(5.15) and formulas of differentiation of the sum of function after a grouping and removal of identical multipliers for brackets identity (5.13), we will lead to the form

$$\begin{aligned}
& - \sum_{l,s=1}^2 \int_{\Omega} \frac{\partial(\rho^{2v} a_{ls} v_1)}{\partial x_l} \frac{\partial u_{vj}}{\partial x_s} dx - \sum_{l=1}^2 \int_{\Omega} \rho^{2v} a_l \frac{\partial u_{vj}}{\partial x_l} v_1 dx - \int_{\Omega} \rho^{2v} a u_{vj} v_1 dx \\
& + \int_{\Omega} \frac{\partial}{\partial x_j} (\rho^{2v} (Lu_v - f) v_1) dx - \int_{\Omega} \frac{\partial \rho^{2v}}{\partial x_j} (Lu_v - f) v_1 dx \\
& = - \int_{\Omega} \rho^{2v} \left(\frac{\partial f}{\partial x_j} + \sum_{l,s=1}^2 \frac{\partial a_{ls}}{\partial x_j} \frac{\partial^2 u_v}{\partial x_l \partial x_s} - \sum_{l=1}^2 \frac{\partial a_l}{\partial x_j} \frac{\partial u_v}{\partial x_l} - \frac{\partial a}{\partial x_j} u_v \right) v_1 dx.
\end{aligned} \tag{5.18}$$

Difference $Lu_v(x) - f(x)$ is equal to zero almost everywhere on Ω . The integrated identity will take the form

$$\begin{aligned}
& \sum_{l,s=1}^2 \int_{\Omega} \frac{\partial(\rho^{2v} a_{ls} v_1)}{\partial x_l} \frac{\partial u_{vj}}{\partial x_s} dx + \sum_{l=1}^2 \int_{\Omega} \rho^{2v} a_l \frac{\partial u_{vj}}{\partial x_l} v_1 dx + \int_{\Omega} \rho^{2v} a u_{vj} v_1 dx \\
& = \int_{\Omega} \rho^{2v} \left(\frac{\partial f}{\partial x_j} + \sum_{l,s=1}^2 \frac{\partial a_{ls}}{\partial x_j} \frac{\partial^2 u_v}{\partial x_l \partial x_s} - \sum_{l=1}^2 \frac{\partial a_l}{\partial x_j} \frac{\partial u_v}{\partial x_l} - \frac{\partial a}{\partial x_j} u_v \right) v_1 dx,
\end{aligned} \tag{5.19}$$

or $a(u_{vj}, v_1) = (F_j, v_1)$, where

$$F_j(x) = \frac{\partial f(x)}{\partial x_j} + \sum_{l,s=1}^2 \frac{\partial a_{ls}(x)}{\partial x_j} \frac{\partial^2 u_v(x)}{\partial x_l \partial x_s} - \sum_{l=1}^2 \frac{\partial a_l(x)}{\partial x_j} \frac{\partial u_v(x)}{\partial x_l} - \frac{\partial a(x)}{\partial x_j} u_v(x). \tag{5.20}$$

Suppose that the differential equation

$$Lu(x) = F_j(x), \quad x \in \Omega \tag{5.21}$$

is given in the domain Ω , with the boundary condition

$$u(x) = \varphi_j(x), \quad x \in \partial\Omega, \tag{5.22}$$

where $\varphi_j(x) = \partial u_v(x) / \partial x_j|_{\partial\Omega}$.

Let function $u^*(x)$ is the R_v -generalized solution of a boundary value problem (5.21), (5.22). Then, for any function $v^*(x) \in \dot{H}_{2,v+\beta/2}^1(\Omega)$ the integrated identity

$$\sum_{l,s=1}^2 \int_{\Omega} \frac{\partial(\rho^{2v} a_{ls} v^*)}{\partial x_l} \frac{\partial u^*}{\partial x_s} dx + \sum_{l=1}^2 \int_{\Omega} \rho^{2v} a_l \frac{\partial u^*}{\partial x_l} v^* dx + \int_{\Omega} \rho^{2v} a u^* v^* dx = \int_{\Omega} \rho^{2v} F_j v^* dx \tag{30*}$$

holds.

Let us prove that function $u^*(x)$ belongs to space $H_{2,\nu+\beta/2}^2(\Omega)$. For this purpose, we will estimate the right parts of (5.21) and a boundary condition (5.22) in norms of corresponding spaces.

First we will prove that the right part of the differential equation (5.21) belongs to space $L_{2,\mu}(\Omega)$. For this purpose, we will estimate composed in expression $F_j(x)$. The first composed we will estimate in norm of space $L_{2,\mu}(\Omega)$

$$\left\| \frac{\partial f(x)}{\partial x_j} \right\|_{L_{2,\mu}(\Omega)} \leq \|f(x)\|_{H_{2,\mu}^1(\Omega)}. \quad (5.23)$$

The second, the third, and the fourth composed at performance conditions (5.1₁)–(5.3₁) we will estimate in norm of space $L_{2,\nu-\beta/2}(\Omega)$

$$\begin{aligned} \left\| \sum_{l,s=1}^2 \frac{\partial a_{ls}(x)}{\partial x_j} \frac{\partial^2 u_\nu(x)}{\partial x_l \partial x_s} \right\|_{L_{2,\nu-\beta/2}(\Omega)} &\leq 2C_1 \|u_\nu(x)\|_{H_{2,\nu+\beta/2-1}^2(\Omega)}, \\ \left\| \sum_{s=1}^2 \frac{\partial a_s(x)}{\partial x_j} \frac{\partial u_\nu(x)}{\partial x_s} \right\|_{L_{2,\nu-\beta/2}(\Omega)} &\leq \sqrt{2}C_2 \|u_\nu(x)\|_{H_{2,\nu+\beta/2-2}^1(\Omega)}, \\ \left\| \frac{\partial a(x)}{\partial x_j} u_\nu(x) \right\|_{L_{2,\nu-\beta/2}(\Omega)} &\leq C_3 \|u_\nu(x)\|_{L_{2,\nu+\beta/2-3}(\Omega)}. \end{aligned} \quad (5.24)$$

The received estimations are fair for all values of parameter ν which satisfy condition (4.4). Therefore, they will be fair and for $\nu = \mu + \beta/2$, whence $\nu - \beta/2 = \mu$. From this fact and four previous estimations, we will receive inequalities

$$\begin{aligned} \|F_j(x)\|_{L_{2,\mu}(\Omega)} &\leq \left\| \frac{\partial f(x)}{\partial x_j} \right\|_{L_{2,\mu}(\Omega)} + \left\| \sum_{l,s=1}^2 \frac{\partial a_{ls}(x)}{\partial x_j} \frac{\partial^2 u_\nu(x)}{\partial x_l \partial x_s} \right\|_{L_{2,\mu}(\Omega)} \\ &\quad + \left\| \sum_{s=1}^2 \frac{\partial a_s(x)}{\partial x_j} \frac{\partial u_\nu(x)}{\partial x_s} \right\|_{L_{2,\mu}(\Omega)} + \left\| \frac{\partial a(x)}{\partial x_j} u_\nu(x) \right\|_{L_{2,\mu}(\Omega)} \\ &\leq \|f(x)\|_{H_{2,\mu}^1(\Omega)} + 2C_1 \|u_\nu(x)\|_{H_{2,\nu+\beta/2-1}^2(\Omega)} \\ &\quad + \sqrt{2}C_2 \|u_\nu(x)\|_{H_{2,\nu+\beta/2-2}^1(\Omega)} + C_3 \|u_\nu(x)\|_{L_{2,\nu+\beta/2-3}(\Omega)}. \end{aligned} \quad (5.25)$$

Let us strengthen this inequality, and we will receive an estimation

$$\|F_j(x)\|_{L_{2,\mu}(\Omega)} \leq \|f(x)\|_{H_{2,\mu}^1(\Omega)} + C_9 \|u_\nu(x)\|_{H_{2,\nu+\beta/2-1}^2(\Omega)}, \quad (5.26)$$

where $C_9 = \max\{2C_1; \sqrt{2}C_2; C_3\}$.

The first composed the right part of received estimation is limited under condition of (5.4₁). Let us prove that second composed too is limited. For this purpose, we will notice that,

by analogy with [6, Theorem 2], the R_ν -generalized solution of a boundary value problem (3.1), (3.2) is unique (same) for various values of parameter ν . Therefore, function $u_\nu(x)$ will satisfy integrated identity (5.13) at ν and at $\nu - 1$. From here and from the Theorem 4.2, the belonging of function $u_\nu(x)$ follows space $H_{2,\nu+\beta/2-1}^2(\Omega)$ and also validity of an estimation

$$\|u_\nu(x)\|_{H_{2,\nu+\beta/2-1}^2(\Omega)} \leq C_7 \left(\|f(x)\|_{L_{2,\mu}(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{3/2}(\partial\Omega)} \right). \quad (5.27)$$

From this estimation at performance of inequalities

$$\begin{aligned} \|f(x)\|_{L_{2,\mu}(\Omega)} &\leq \max_{x \in \bar{\Omega}} \rho(x) \|f(x)\|_{H_{2,\mu}^1(\Omega)} \\ \|\varphi(x)\|_{H_{2,\mu+\beta}^{3/2}(\partial\Omega)} &\leq \max_{x \in \bar{\Omega}} \rho(x) \|\varphi(x)\|_{H_{2,\mu+\beta}^{5/2}(\partial\Omega)}, \end{aligned} \quad (5.28)$$

the estimation

$$\|u_\nu(x)\|_{H_{2,\nu+\beta/2-1}^2(\Omega)} \leq C_7 \max_{x \in \bar{\Omega}} \rho(x) \left(\|f(x)\|_{H_{2,\mu}^1(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{5/2}(\partial\Omega)} \right) \quad (5.29)$$

follows. Therefore, function $F_j(x)$ is limited in norm of space $L_{2,\mu}(\Omega)$, that is, $F_j(x) \in L_{2,\mu}(\Omega)$. Except for the right part of the differential equation (5.21) on the basis of inequalities (5.26) and (5.29) the estimation is valid

$$\|F_j(x)\|_{L_{2,\mu}(\Omega)} \leq C_{10} \left(\|f(x)\|_{H_{2,\mu}^1(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{5/2}(\partial\Omega)} \right), \quad (5.30)$$

where C_{10} is a positive constant independent of $F_j(x)$, $f(x)$ and $\varphi(x)$.

The right part $\varphi_j(x)$ of a boundary condition (5.22) belongs to space $H_{2,\mu+\beta}^{3/2}(\partial\Omega)$. Validity of this fact under condition (5.51) follows from inequality

$$\|\varphi_j(x)\|_{H_{2,\mu+\beta}^{3/2}(\partial\Omega)} \leq \|\varphi(x)\|_{H_{2,\mu+\beta}^{5/2}(\partial\Omega)}. \quad (5.31)$$

This inequality is obvious if to use definition of norm in space $H_{2,\alpha}^{k+1/2}(\partial\Omega)$.

Therefore, at performance of conditions (5.11)–(5.31) and (5.61) conditions (3.3)–(3.5) and (4.5) are carried out automatically.

Thus, for a boundary value problem (5.21), (5.22) conditions of the Theorem 4.2 according to which function $u^*(x)$ belongs to space $H_{2,\nu+\beta/2}^2(\Omega)$ are satisfied all and the estimation

$$\|u^*(x)\|_{H_{2,\nu+\beta/2}^2(\Omega)} \leq C_7 \left(\|F_j(x)\|_{L_{2,\mu}(\Omega)} + \|\varphi_j(x)\|_{H_{2,\mu+\beta}^{3/2}(\partial\Omega)} \right) \quad (5.32)$$

is fair. Besides, under the Theorem 4.1 by virtue of belonging $\varphi_j(x) \in H_{2,\mu+\beta-1}^{1/2}(\partial\Omega)$ it is received that function $u^*(x)$ (the R_ν -generalized solution of a boundary value problem (5.21), (5.22)) exists and unique in space $H_{2,\nu+\beta/2}^1(\Omega)$.

Let us specify that the integrated identity (30*) is fair for all functions $v^*(x)$ from $H_{2,v+\beta/2}^1(\Omega)$, therefore, it will be fair and for functions $v_1(x)$. On the basic of this remark, identities (5.13) and (30*), and also uniqueness of the R_v -generalized solution of a boundary value problem (5.21), (5.22) it is received, that $u_{vj}(x) \equiv u^*(x)$, that is, function $u_{vj}(x)$ belongs to space $H_{2,v+\beta/2}^2(\Omega)$ and for it the inequality (5.32) is fairly.

Let us establish limitation seminorm $|u_v(x)|_{H_{2,v+\beta/2}^3(\Omega)}$. For this purpose, we will estimate its square

$$|u_v(x)|_{H_{2,v+\beta/2}^3(\Omega)}^2 = \sum_{|\lambda|=3} \int_{\Omega} \rho^{2(v+\beta/2)}(x) \left| D^\lambda u_v(x) \right|^2 dx, \quad (5.33)$$

from above on inequality

$$\sum_{|\lambda|=3} \left| D^\lambda u_v(x) \right|^2 \leq \sum_{|\lambda|=2} \left| D^\lambda \left(\frac{\partial u_v(x)}{\partial x_1} \right) \right|^2 + \sum_{|\lambda|=2} \left| D^\lambda \left(\frac{\partial u_v(x)}{\partial x_2} \right) \right|^2. \quad (5.34)$$

We will receive inequality

$$|u_v(x)|_{H_{2,v+\beta/2}^3(\Omega)}^2 \leq |u_{v1}(x)|_{H_{2,v+\beta/2}^2(\Omega)}^2 + |u_{v2}(x)|_{H_{2,v+\beta/2}^2(\Omega)}^2. \quad (5.35)$$

From inequality (5.35) and function $u_{vj}(x)$ ($j = 1, 2$) proved above, a belonging space $H_{2,v+\beta/2}^2(\Omega)$ is followed with limitation seminorm, therefore, $u_v(x) \in H_{2,v+\beta/2}^3(\Omega)$.

Let us prove now an estimation (5.7) for $k = 1$. For this purpose we will consider function $\rho^{v+\beta/2}(x)u_v(x)$: it belongs to space $H^3(\Omega)$. It follows from a belonging of function $u_v(x)$ to space $H_{2,v+\beta/2}^3(\Omega)$ and statements "A" of a lemma 1 of [6]. From statement "B" of the same lemma, the inequality

$$\|u_v(x)\|_{H_{2,v+\beta/2}^3(\Omega)} \leq C_{11} \left\| \rho^{v+\beta/2}(x)u_v(x) \right\|_{H^3(\Omega)} \quad (5.36)$$

follows. C_{11} is a positive constant independent of $u_v(x)$.

The norm for function $\rho^{v+\beta/2}(x)u_v(x)$ in Sobolev space $H^3(\Omega)$ can be entered it is equivalent (see [12, page 380])

$$\left\| \rho^{v+\beta/2}(x)u_v(x) \right\|_{H^3(\Omega)} = \left\| \rho^{v+\beta/2}(x)u_v(x) \right\|_{L_2(\Omega)} + \left| \rho^{v+\beta/2}(x)u_v(x) \right|_{H^3(\Omega)}. \quad (5.37)$$

Let us estimate from above composed the right part (5.37). For the first composed truly an inequality

$$\left\| \rho^{v+\beta/2}(x)u_v(x) \right\|_{L_2(\Omega)} \leq \max_{\forall x \in \Omega} \rho^3(x) \|u_v(x)\|_{L_{2,v+\beta/2-3}(\Omega)}. \quad (5.38)$$

For an estimation of a square of the second composed in the right part (5.37), we will take advantage of definition seminorm in space $H^3(\Omega)$, formula $D^3(g(x) \cdot h(x)) = \sum_{i=0}^3 C_3^i(D^i g(x)) \cdot (D^{3-i} h(x))$, algebraic inequality $(\sum_{q=1}^n a_q)^2 \leq n \cdot \sum_{q=1}^n a_q^2$ and conditions which derivatives of weight function possess. As a result, we will receive an inequality

$$\begin{aligned} \left| \rho^{v+\beta/2}(x) u_v(x) \right|_{H^3(\Omega)}^2 &\leq 4|u_v(x)|_{H_{2,v+\beta/2}^3(\Omega)}^2 + 12\sigma|u_v(x)|_{H_{2,v+\beta/2-1}^2(\Omega)}^2 \\ &+ 12\sigma^2|u_v(x)|_{H_{2,v+\beta/2-2}^1(\Omega)}^2 + 4\sigma^3\|u_v(x)\|_{L_{2,v+\beta/2-3}^2(\Omega)}^2. \end{aligned} \quad (5.39)$$

From estimations (5.36)–(5.39), the inequality

$$\|u_v(x)\|_{H_{2,v+\beta/2}^3(\Omega)}^2 \leq C_{12}^2|u_v(x)|_{H_{2,v+\beta/2}^3(\Omega)}^2 + C_{13}^2\|u_v(x)\|_{H_{2,v+\beta/2-1}^2(\Omega)}^2 \quad (5.40)$$

follows. $C_{12}^2 = 8C_{11}^2$, $C_{13}^2 = 2C_{11}^2 \cdot \max\{\max_{x \in \bar{\Omega}} \rho^3(x) + 4\sigma^3; 12\sigma^2; 12\sigma\}$.

From this estimation if to apply inequality $\sqrt{a^2 + b^2} \leq a + b$ which is true for nonnegative a and b , we will receive

$$\|u_v(x)\|_{H_{2,v+\beta/2}^3(\Omega)} \leq C_{12}|u_v(x)|_{H_{2,v+\beta/2}^3(\Omega)} + C_{13}\|u_v(x)\|_{H_{2,v+\beta/2-1}^2(\Omega)}. \quad (5.41)$$

From estimations (5.29)–(5.31), (5.41), and estimation (5.32) which is written down for $u_{vj}(x) \equiv u^*(x)$ ($j = 1, 2$), we will receive an inequality

$$\|u_v(x)\|_{H_{2,v+\beta/2}^3(\Omega)} \leq C_8 \left(\|f(x)\|_{H_{2,\mu}^1(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{5/2}(\partial\Omega)} \right). \quad (23.1)$$

So, $u_v(x) \in H_{2,v+\beta/2}^3(\Omega)$, the estimation (5.7) for $k = 1$ is carried out. Therefore, the statement of the Theorem 5.1 is true for $k = 1$.

Stage 2. Let us assume that under conditions (3.6), (3.7), (4.1), and (4.4), and conditions

$$a_{ls}(x) \in H_{\infty,-\beta}^m(\Omega, C_1) \quad (l, s = 1, 2), \quad (5.1_2)$$

$$a_l(x) \in H_{\infty,-(\beta-1)}^{m-1}(\Omega, C_2) \quad (l = 1, 2) \quad (5.2_2)$$

$$a(x) \in H_{\infty,-(\beta-2)}^{m-1}(\Omega, C_3), \quad (5.3_2)$$

$$f(x) \in H_{2,\mu}^{m-1}(\Omega), \quad (5.4_2)$$

$$\varphi(x) \in H_{2,\mu+\beta}^{m+1/2}(\partial\Omega), \quad (5.5_2)$$

$$v + \frac{\beta}{2} > m + 1 \quad (5.6_2)$$

the R_ν -generalized solution $u_\nu(x)$ of the Dirichlet problem with coordinated degeneration of the input data belongs to space $H_{2,\nu+\beta/2}^{m+1}(\Omega)$ and the estimation

$$\|u_\nu(x)\|_{H_{2,\nu+\beta/2}^{m+1}(\Omega)} \leq C_8 \left(\|f(x)\|_{H_{2,\mu}^{m-1}(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{m+1/2}(\partial\Omega)} \right) \quad (5.7_2)$$

is fair.

At the put-forward assumption of validity of the statement of the Theorem 5.1 for $k = m - 1$, under conditions (3.6), (3.7), (4.1), and (4.4) and under conditions

$$a_{ls}(x) \in H_{\infty,-\beta}^{m+1}(\Omega, C_1) \quad (l, s = 1, 2), \quad (5.1_3)$$

$$a_l(x) \in H_{\infty, -(\beta-1)}^m(\Omega, C_2) \quad (l = 1, 2), \quad (5.2_3)$$

$$a(x) \in H_{\infty, -(\beta-2)}^m(\Omega, C_3), \quad (5.3_3)$$

$$f(x) \in H_{2,\mu}^m(\Omega), \quad (5.4_3)$$

$$\varphi(x) \in H_{2,\mu+\beta}^{m+3/2}(\partial\Omega), \quad (5.5_3)$$

$$\nu + \frac{\beta}{2} > m + 2, \quad (5.6_3)$$

we will prove that the R_ν -generalized solution $u_\nu(x)$ of the Dirichlet problem with coordinated degeneration of the input data belongs to space $H_{2,\nu+\beta/2}^{m+2}(\Omega)$ and the estimation

$$\|u_\nu(x)\|_{H_{2,\nu+\beta/2}^{m+2}(\Omega)} \leq C_8 \left(\|f(x)\|_{H_{2,\mu}^m(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{m+3/2}(\partial\Omega)} \right) \quad (5.7_3)$$

is fair.

The plan of the proof is the same as at a Stage 1, therefore, we will consider only the most essential details of the proof.

Obviously at performance of conditions (5.1₃)–(5.6₃) conditions (5.1₂)–(5.6₂) are carried out automatically, that is, $u_\nu(x) \in H_{2,\nu+\beta/2}^{m+1}(\Omega)$ and the estimation (5.1₂)–(5.6₂) is fair.

Again, we will take advantage of designations $u_{\nu j}(x) = \partial u_\nu(x) / \partial x_j$ for $(j = 1, 2)$. To similarly how it has been made at a Stage 1, it is possible to show that function $u_{\nu j}(x)$ at fixed j is the R_ν -generalized solution of a boundary value problem (5.21), (5.22) and belongs to space $H_{2,\nu+\beta/2}^m(\Omega)$. The last follows from an inequality

$$|u_{\nu j}(x)|_{H_{2,\nu+\beta/2}^m(\Omega)} \leq |u_\nu(x)|_{H_{2,\nu+\beta/2}^{m+1}(\Omega)} \quad (5.4_2)$$

and belongings of function $u_\nu(x)$ to space $H_{2,\nu+\beta/2}^{m+1}(\Omega)$.

Let us prove that right part $F_j(x)$ of the differential equation (5.21) belongs to space $H_{2,\mu}^{m-1}(\Omega)$. For this purpose, we will estimate composed the right part in equality $F_j(x)$. The first composed we will estimate in norm of space $H_{2,\mu}^{m-1}(\Omega)$

$$\left\| \frac{\partial f(x)}{\partial x_j} \right\|_{H_{2,\mu}^{m-1}(\Omega)} \leq \|f(x)\|_{H_{2,\mu}^m(\Omega)}. \quad (5.43)$$

The second, the third, and the fourth composed we will estimate in norm of space $H_{2,\nu-\beta/2}^{m-1}(\Omega)$ with use of conditions (5.1₃)–(5.3₃), formulas $D^\alpha(g(x) \cdot h(x)) = \sum_{i=0}^{|\alpha|} C_{|\alpha|}^i(D^i g(x)) \cdot (D^{\alpha-i} h(x))$, and algebraic inequalities $(\sum_{q=1}^n a_q)^2 \leq n \cdot \sum_{q=1}^n a_q^2$. As a result, we will receive inequalities

$$\begin{aligned} \left\| \sum_{l,s=1}^2 \frac{\partial a_{ls}(x)}{\partial x_j} \frac{\partial^2 u_\nu(x)}{\partial x_l \partial x_s} \right\|_{H_{2,\nu-\beta/2}^{m-1}(\Omega)} &\leq 2(m-1)\sqrt{m}C_1 \|u_\nu(x)\|_{H_{2,\nu+\beta/2-1}^{m+1}(\Omega)}, \\ \left\| \sum_{s=1}^2 \frac{\partial a_s(x)}{\partial x_j} \frac{\partial u_\nu(x)}{\partial x_s} \right\|_{H_{2,\nu-\beta/2}^{m-1}(\Omega)} &\leq \sqrt{2}(m-1)\sqrt{m}C_2 \|u_\nu(x)\|_{H_{2,\nu+\beta/2-2}^m(\Omega)}, \\ \left\| \frac{\partial a(x)}{\partial x_j} u_\nu(x) \right\|_{H_{2,\nu-\beta/2}^{m-1}(\Omega)} &\leq (m-1)\sqrt{m}C_3 \|u_\nu(x)\|_{H_{2,\nu+\beta/2-3}^{m-1}(\Omega)}. \end{aligned} \quad (5.44)$$

The received estimations are fair for all values of parameter ν which satisfy to condition (4.4). Therefore, they will be fair and for $\nu = \mu + \beta/2$, hence $\nu - \beta/2 = \mu$. From this fact and the four previous estimations, we will receive inequalities

$$\begin{aligned} \|F_j(x)\|_{H_{2,\mu}^{m-1}(\Omega)} &\leq \left\| \frac{\partial f(x)}{\partial x_j} \right\|_{H_{2,\mu}^{m-1}(\Omega)} + \left\| \sum_{l,s=1}^2 \frac{\partial a_{ls}(x)}{\partial x_j} \frac{\partial^2 u_\nu(x)}{\partial x_l \partial x_s} \right\|_{H_{2,\mu}^{m-1}(\Omega)} + \left\| \sum_{s=1}^2 \frac{\partial a_s(x)}{\partial x_j} \frac{\partial u_\nu(x)}{\partial x_s} \right\|_{H_{2,\mu}^{m-1}(\Omega)} \\ &\quad + \left\| \frac{\partial a(x)}{\partial x_j} u_\nu(x) \right\|_{H_{2,\mu}^{m-1}(\Omega)} \\ &\leq \|f(x)\|_{H_{2,\mu}^m(\Omega)} + (m-1)\sqrt{m} \cdot \max\{2C_1; \sqrt{2}C_2; C_3\} \\ &\quad \times \left(\|u_\nu(x)\|_{H_{2,\nu+\beta/2-1}^{m+1}(\Omega)} + \|u_\nu(x)\|_{H_{2,\nu+\beta/2-2}^m(\Omega)} + \|u_\nu(x)\|_{H_{2,\nu+\beta/2-3}^{m-1}(\Omega)} \right). \end{aligned} \quad (5.45)$$

To strengthen this estimation, we will receive an inequality

$$\|F_j(x)\|_{H_{2,\mu}^{m-1}(\Omega)} \leq \|f(x)\|_{H_{2,\mu}^m(\Omega)} + (m-1)\sqrt{m} \cdot C_9 \|u_\nu(x)\|_{H_{2,\nu+\beta/2-1}^{m+1}(\Omega)}. \quad (5.46)$$

The first composed the right part of the received inequality is limited at performance of a condition (5.4₃). Limitation of the second composed follows from a belonging of function $u_\nu(x)$ to space $H_{2,\nu+\beta/2-1}^{m+1}(\Omega)$ (under the Theorem 4.2 and to integrated identity (5.13) which is fair for function $u_\nu(x)$ at ν and at $\nu - 1$). Thus, the estimation (5.7₂) takes the form

$$\|u_\nu(x)\|_{H_{2,\nu+\beta/2-1}^{m+1}(\Omega)} \leq C_8 \left(\|f(x)\|_{H_{2,\mu}^{m-1}(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{m+1/2}(\partial\Omega)} \right). \quad (5.47)$$

From here by virtue of validity of inequalities

$$\begin{aligned} \|f(x)\|_{H_{2,\mu}^{m-1}(\Omega)} &\leq \max_{x \in \bar{\Omega}} \rho(x) \|f(x)\|_{H_{2,\mu}^m(\Omega)}, \\ \|\varphi(x)\|_{H_{2,\mu+\beta}^{m+1/2}(\partial\Omega)} &\leq \max_{x \in \bar{\Omega}} \rho(x) \|\varphi(x)\|_{H_{2,\mu+\beta}^{m+3/2}(\partial\Omega)}, \end{aligned} \quad (5.48)$$

the estimation

$$\|u_\nu(x)\|_{H_{2,\nu+\beta/2-1}^{m+1}(\Omega)} \leq C_8 \max_{x \in \bar{\Omega}} \rho(x) \left(\|f(x)\|_{H_{2,\mu}^m(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{m+3/2}(\partial\Omega)} \right) \quad (5.49)$$

follows. Therefore, function $F_j(x)$ is limited in norm of space $H_{2,\mu}^{m-1}(\Omega)$, that is, $F_j(x) \in H_{2,\mu}^{m-1}(\Omega)$. Besides, for the right part of the differential equation (5.21) on the basic of inequalities (5.46) and (5.49) the estimation

$$\|F_j(x)\|_{H_{2,\mu}^{m-1}(\Omega)} \leq C_{14} \left(\|f(x)\|_{H_{2,\mu}^m(\Omega)} + \|\varphi(x)\|_{H_{2,\mu+\beta}^{m+3/2}(\partial\Omega)} \right) \quad (5.50)$$

will be true. Constant C_{14} is positive and does not depend from $F_j(x)$, $f(x)$, and $\varphi(x)$.

Right part $\varphi_j(x)$ of a boundary conditions (5.22) belongs to space $H_{2,\mu+\beta}^{m+1/2}(\partial\Omega)$. This fact at performance of a condition (5.4₂) directly follows from an inequality

$$\|\varphi_j(x)\|_{H_{2,\mu+\beta}^{m+1/2}(\partial\Omega)} \leq \|\varphi(x)\|_{H_{2,\mu+\beta}^{m+3/2}(\partial\Omega)}. \quad (5.51)$$

Therefore, for a boundary value problem (5.21), (5.22) conditions at which function $u_{\nu j}(x)$ belongs to space $H_{2,\nu+\beta/2}^{m+1}(\Omega)$ are satisfied all and the estimation

$$\|u_{\nu j}(x)\|_{H_{2,\nu+\beta/2}^{m+1}(\Omega)} \leq C_8 \left(\|F_j(x)\|_{H_{2,\mu}^{m-1}(\Omega)} + \|\varphi_j(x)\|_{H_{2,\mu+\beta}^{m+1/2}(\partial\Omega)} \right) \quad (5.52)$$

is fair (according to the assumption of validity of the statement of the Theorem 5.1 at $k = m - 1$).

Let us establish limitation seminorm $|u_v(x)|_{H_{2,v+\beta/2}^{m+2}(\Omega)}$. For this purpose, we will estimate its square

$$|u_v(x)|_{H_{2,v+\beta/2}^{m+2}(\Omega)}^2 = \sum_{|\lambda|=m+2} \int_{\Omega} \rho^{2(v+\beta/2)}(x) \left| D^{\lambda} u_v(x) \right|^2 dx, \quad (5.53)$$

from above on inequality

$$\sum_{|\lambda|=m+2} \left| D^{\lambda} u_v(x) \right|^2 \leq \sum_{|\lambda|=m+1} \left| D^{\lambda} \left(\frac{\partial u_v(x)}{\partial x_1} \right) \right|^2 + \sum_{|\lambda|=m+1} \left| D^{\lambda} \left(\frac{\partial u_v(x)}{\partial x_2} \right) \right|^2. \quad (5.54)$$

We will receive inequality

$$|u_v(x)|_{H_{2,v+\beta/2}^{m+2}(\Omega)}^2 \leq |u_{v1}(x)|_{H_{2,v+\beta/2}^{m+1}(\Omega)}^2 + |u_{v2}(x)|_{H_{2,v+\beta/2}^{m+1}(\Omega)}^2. \quad (5.55)$$

From an inequality (5.55) and the function $u_{vj}(x)$ ($j = 1, 2$) proved above a belonging space $H_{2,v+\beta/2}^{m+1}(\Omega)$ is followed with limitation seminorm, that is, $u_v(x)$ belongs to space $H_{2,v+\beta/2}^{m+2}(\Omega)$.

Let us prove now an estimation (5.73). For this purpose, we will consider function $\rho^{v+\beta/2}(x)u_v(x)$. This function belongs to space $H^{m+2}(\Omega)$, that follows from a belonging of function $u_v(x)$ to space $H_{2,v+\beta/2}^{m+2}(\Omega)$ and statement "A" a lemma 1 of [6]. From statement "B" of the same lemma the inequality

$$\|u_v(x)\|_{H_{2,v+\beta/2}^{m+2}(\Omega)} \leq C_{15} \left\| \rho^{v+\beta/2}(x)u_v(x) \right\|_{H^{m+2}(\Omega)} \quad (5.56)$$

follows. Constant C_{15} is positive and does not depend from $u_v(x)$.

The norm of function $\rho^{v+\beta/2}(x)u_v(x)$ in Sobolev space $H^{m+2}(\Omega)$ can be determined the formula

$$\left\| \rho^{v+\beta/2}(x)u_v(x) \right\|_{H^{m+2}(\Omega)} = \left\| \rho^{v+\beta/2}(x)u_v(x) \right\|_{L_2(\Omega)} + \left| \rho^{v+\beta/2}u_v(x) \right|_{H^{m+2}(\Omega)}. \quad (5.57)$$

Let us estimate from above composed which enter into the right part of equality (5.57). For the first composed the estimation

$$\left\| \rho^{v+\beta/2}(x)u_v(x) \right\|_{L_2(\Omega)} \leq \max_{\forall x \in \Omega} \rho^{m+2}(x) \|u_v(x)\|_{L_{2,v+\beta/2-m-2}(\Omega)} \quad (5.58)$$

is fair. For an estimation of a square of the second composed in the right part (5.57), we will take advantage of definition of seminorm in space $H^{m+2}(\Omega)$, the formula $D^{m+2}(g(x) \cdot h(x)) = \sum_{i=0}^{m+2} C_{m+2}^i (D^i g(x)) \cdot (D^{m+2-i} h(x))$, an algebraic inequality $(\sum_{q=1}^n a_q)^2 \leq n \cdot \sum_{q=1}^n a_q^2$, and conditions which derivatives of weight function possess. The inequality

$$\begin{aligned} \left| \rho^{v+\beta/2}(x) u_v(x) \right|_{H^{m+2}(\Omega)}^2 &\leq (m+3) |u_v(x)|_{H_{2,v+\beta/2}^{m+2}(\Omega)}^2 \\ &\quad + (m+3)(m+2) \sigma |u_v(x)|_{H_{2,v+\beta/2-1}^{m+1}(\Omega)}^2 + (m+3)(m+2) \sigma^2 \\ &\quad \times |u_v(x)|_{H_{2,v+\beta/2-2}^m(\Omega)}^2 + \cdots + (m+3) \sigma^{m+2} \|u_v(x)\|_{L_{2,v+\beta/2-m-2}(\Omega)}^2 \end{aligned} \quad (5.59)$$

will be as a result received. From inequalities (5.56)–(5.59), the estimation

$$\|u_v(x)\|_{H_{2,v+\beta/2}^{m+2}(\Omega)}^2 \leq C_{16}^2 |u_v(x)|_{H_{2,v+\beta/2}^{m+2}(\Omega)}^2 + C_{17}^2 \|u_v(x)\|_{H_{2,v+\beta/2-1}^{m+1}(\Omega)}^2 \quad (5.60)$$

follows

$$\begin{aligned} C_{16}^2 &= 2(m+3)C_{15}^2, \\ C_{17}^2 &= 2C_{15}^2 \max \left\{ \max_{\forall x \in \Omega} \rho^{m+2}(x) + (m+3)\sigma^{m+2}; (m+3)(m+2)\sigma^{m+1}; \cdots; (m+3)(m+2)\sigma \right\}. \end{aligned} \quad (5.61)$$

From it on an inequality $\sqrt{a^2 + b^2} \leq a + b$ which is fair for nonnegative values a and b , the inequality

$$\|u_v(x)\|_{H_{2,v+\beta/2}^{m+2}(\Omega)} \leq C_{16} |u_v(x)|_{H_{2,v+\beta/2}^{m+2}(\Omega)} + C_{17} \|u_v(x)\|_{H_{2,v+(\beta/2)-1}^{m+1}(\Omega)} \quad (5.62)$$

follows. From estimation (5.49)–(5.52), (5.62), the proved inequality (5.7₃) directly follows. Therefore, for $k = m$ the statement of the Theorem 5.1 is proved.

So, the statement of the Theorem 5.1 is proved for $k = 1$ and also for $k = m$ in the assumption at which it is true for $k = m - 1$. From these facts on the basic of a method of a mathematical induction validity of the statement of the Theorem 5.1 for any natural value k follows. *The Theorem 5.1 is proved.*

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