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Research Article

Geometric Properties Solutions of a Class of Third-Order Linear Differential Equations

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We aim at investigating the geometric properties of the solutions of the initial-value problem which involves the following third-order linear differential equations: $\omega'''(z) + Q(z)\omega'(z) = 0$, $\omega(0) = 0$, $\omega'(0) = 1$, $\omega''(0) = 0$, where $Q(z)$ is analytic in the open unit disk U.

1. Introduction

Let A denote the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z\colon z \in \mathbb{C} \text{ and } |z| < 1\}$.

Also let S, S^*, $S^*(\alpha)$, C, and $C(\alpha)$ denote the subclasses of A consisting of functions which are, respectively, univalent, starlike with respect to the origin, starlike of order α in U $(0 \le \alpha < 1)$, convex with respect to the origin, and convex of order α in U $(0 \le \alpha < 1)$ (see, for details, [1–4]). Furthermore, $SS^*(\beta)$ and $C^*C(\beta)$ denote the subclasses of A consisting of functions which are strongly starlike of order β and strongly convex of order β in U, $0 < \beta \le 1$ (see, [5, 6]).

For functions $f \in A$ with $f'(z) \neq 0$ $(z \in U)$, we define the Schwarzian derivative of f by

$$S(f,z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \quad (f \in A; f'(z) \neq 0, z \in U). \tag{1.2}$$

Note that Nehari [7] had proven the quotient of the linearly independent solution of (1.2) is univalent, while Robertson [8] and Miller [9] proved that the unique solution of the equation:

$$W''(z) + a(z)W(z) = 0, \qquad W(0) = 0, \qquad W'(0) = 1 \tag{1.3}$$

is starlike.

Now, let B_J denote the class of bounded functions $\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots$ analytic in the unit disk U for which $|\omega(z)| < J$. If $g(z) \in B_J$, then by using the Schwarz lemma, the function $\omega(z)$ defined by

$$\omega(z) = z^{-1/2} \int_0^z g(t) t^{-1/2} dt \tag{1.4}$$

is also in B_J. Thus, in terms of derivatives, we have

$$\left| \frac{1}{2}\omega(z) + z\omega'(z) \right| < J, \quad (z \in U), \quad \Longrightarrow |\omega(z)| < J, \quad (z \in U). \tag{1.5}$$

In 1999, Saitoh [10] proved that the differential equation

$$\omega''(z) + a(z)\omega'(z) + b(z)\omega(z) = 0, \tag{1.6}$$

where $a(z)$ and $b(z)$ are analytic in the unit disc U, has a solution $\omega(z)$ univalent and starlike in U under some conditions. Then in 2004, Owa et al. [11] studied geometric properties of the solutions of initial-value problem (1.6) and later, Saitoh [12] studied geometric properties of the solutions of the following second-order linear differential equation:

$$\omega''(z) + P_n(z)\omega(z) = 0, \tag{1.7}$$

where $P_n(z)$ is nonconstant polynomial of degree $n \geq 1$.

In this work, we aim at studying certain geometric properties of the solutions of the following initial-value problem:

$$\omega'''(z) + Q(z)\omega'(z) = 0, \qquad \omega(0) = 0, \qquad \omega'(0) = 1, \qquad \omega''(0) = 0, \tag{1.8}$$

where $Q(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic in U.

In order to prove our main results, we need the following definitions and theorems.

Definition 1.1 (see [13]). *Let H_J be the set of complex functions $h(u, v)$ satisfying the following:*

(i) $h(u, v)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C}$;

(ii) $(0, 0) \in D$ and $|h(0, 0)| < J$;

(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geq J$ when $(Je^{i\theta}, Ke^{i\theta}) \in D$, θ is real and $K \geq J$.

Definition 1.2 (see [13]). *Let $h \in H_J$ with corresponding domain D. We denote by $B_J(h)$ those functions $\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots$ which are analytic in U satisfying*

(i) $(\omega(z), z\omega'(z)) \in D$,

(ii) $|h(\omega(z), z\omega'(z))| < J (z \in U)$.

Theorem 1.3 (see [10]). *For any $h \in H_J$,*

$$B_J(h) \subset B_J, \quad \left(h \in H_J; J > 0\right). \tag{1.9}$$

Theorem 1.4 (see [10]). *Let $h \in H_J$ and $b(z)$ be an analytic function in U with $|b(z)| < J$. If the differential equation*

$$h\left(\omega(z), z\omega'(z)\right) = b(z), \qquad \omega(0) = 0, \qquad \omega(0) = 1 \tag{1.10}$$

has a solution $\omega(z)$ analytic in U, then $|\omega(z)| < J$.

2. Main Results

Theorem 2.1. *Let $Q(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in U with*

$$\sum_{n=0}^{\infty} |b_n| < J \quad (z \in U, J > 0), \tag{2.1}$$

and let $\omega(z)$ denote the solution of the initial value problem (1.8) *in U. Then*

$$1 - J < \Re\left\{1 + \frac{z\omega''(z)}{\omega'(z)}\right\} > 1 + J \quad (z \in U, J > 0). \tag{2.2}$$

Proof. If we let

$$u(z) = \frac{z\omega''(z)}{\omega'(z)}, \tag{2.3}$$

then $u(z)$ is analytic in U, such that $u(0) = 0$ and (1.8) becomes

$$[u(z)]^2 - u(z) + zu'(z) = -z^2 \sum_{n=0}^{\infty} b_n z^n \tag{2.4}$$

or, equivalently,

$$h\left(u(z), zu'(z)\right) = -z^2 \sum_{n=0}^{\infty} b_n z^n, \tag{2.5}$$

where, for convenience,

$$h(\xi, \eta) = \xi^2 - \xi + \eta. \tag{2.6}$$

From assumption, we have

$$\sum_{n=0}^{\infty}|b_n| < J \quad (z \in U, J > 0). \tag{2.7}$$

By using Theorem 1.4, we have

$$|u(z)| < J \tag{2.8}$$

which, in view of the relationship (2.3), yields

$$\left|\frac{z\omega''(z)}{\omega'(z)}\right| < J, \tag{2.9}$$

that is,

$$1 - J < \Re\left\{1 + \frac{z\omega''(z)}{\omega'(z)}\right\} > 1 + J \quad (z \in U, J > 0). \tag{2.10}$$

□

Letting $J = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $Q(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in U with*

$$\sum_{n=0}^{\infty}|b_n| < 1. \tag{2.11}$$

Let $\omega(z)$ be the solution of the initial-value problem in (1.8) *in U. Then $\omega(z)$ is convex in U.*

Example 2.3. Let $Q(z) = 1$ in Corollary 2.2; the solution of the following initial-value problem:

$$\omega'''(z) + \omega'(z) = 0, \qquad \omega(0) = 0, \qquad \omega'(0) = 1, \qquad \omega''(0) = 0 \tag{2.12}$$

is given by

$$\omega(z) = \sin z \in C. \tag{2.13}$$

Theorem 2.4. *Let $Q(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in U with*

$$\sum_{n=0}^{\infty}|b_n| < J \quad (z \in U,\ 0 < J \leq 1). \tag{2.14}$$

Let $\omega(z)$ be the solution of the initial-value problem in (1.8) *in U. Then $\omega(z)$ is strongly convex of order α, that is,*

$$\left|\arg\left\{1+\frac{z\omega''(z)}{\omega'(z)}\right\}\right| < \frac{\pi}{2}\alpha \tag{2.15}$$

for some α $(0 < \alpha \leq 1)$ and

$$\alpha = \frac{2}{\pi}\sin^{-1} J \quad (0 < J \leq 1). \tag{2.16}$$

Proof. By using the same technique as in the proof of Theorem 2.1, the required result is obtained. □

Remark 2.5. Putting $\alpha = 1$ in Theorem 2.4, we have Corollary 2.2.

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