Research Article

Geometric Properties Solutions of a Class of Third-Order Linear Differential Equations

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We aim at investigating the geometric properties of the solutions of the initial-value problem which involves the following third-order linear differential equations:

\[ \frac{\omega'''}{z} + Q(z)\omega' = 0, \quad \omega(0) = 0, \quad \omega'(0) = 1, \quad \omega''(0) = 0, \]

where \(Q(z)\) is analytic in the open unit disk \(U\).

1. Introduction

Let \(A\) denote the class of functions \(f\) normalized by

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \]  

which are analytic in the open unit disk \(U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}\).

Also let \(S, S^*, S^*(\alpha), C,\) and \(C(\alpha)\) denote the subclasses of \(A\) consisting of functions which are, respectively, univalent, starlike with respect to the origin, starlike of order \(\alpha\) in \(U (0 \leq \alpha < 1)\), convex with respect to the origin, and convex of order \(\alpha\) in \(U (0 \leq \alpha < 1)\) (see, for details, [1–4]). Furthermore, \(SS^*(\beta)\) and \(C^C(\beta)\) denote the subclasses of \(A\) consisting of functions which are strongly starlike of order \(\beta\) and strongly convex of order \(\beta\) in \(U, 0 < \beta \leq 1\) (see, [5, 6]).

For functions \(f \in A\) with \(f'(z) \neq 0, z \in U\), we define the Schwarzian derivative of \(f\) by

\[ S(f, z) := \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2, \quad (f \in A; f'(z) \neq 0, z \in U). \]
Note that Nehari [7] had proven the quotient of the linearly independent solution of (1.2) is univalent, while Robertson [8] and Miller [9] proved that the unique solution of the equation:

$$W''(z) + a(z)W(z) = 0, \quad W(0) = 0, \quad W'(0) = 1$$  \hspace{1cm} (1.3)

is starlike.

Now, let $B_J$ denote the class of bounded functions $\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots$ analytic in the unit disk $U$ for which $|\omega(z)| < J$. If $g(z) \in B_J$, then by using the Schwarz lemma, the function $\omega(z)$ defined by

$$\omega(z) = z^{-1/2} \int_0^z g(t) t^{-1/2} dt$$  \hspace{1cm} (1.4)

is also in $B_J$. Thus, in terms of derivatives, we have

$$\left| \frac{1}{2} \omega'(z) + z \omega''(z) \right| < J, \quad (z \in U), \quad \implies |\omega'(z)| < J, \quad (z \in U).$$  \hspace{1cm} (1.5)

In 1999, Saitoh [10] proved that the differential equation

$$\omega''(z) + a(z)\omega'(z) + b(z)\omega(z) = 0,$$  \hspace{1cm} (1.6)

where $a(z)$ and $b(z)$ are analytic in the unit disc $U$, has a solution $\omega(z)$ univalent and starlike in $U$ under some conditions. Then in 2004, Owa et al. [11] studied geometric properties of the solutions of initial-value problem (1.6) and later, Saitoh [12] studied geometric properties of the solutions of the following second-order linear differential equation:

$$\omega''(z) + P_n(z)\omega(z) = 0,$$  \hspace{1cm} (1.7)

where $P_n(z)$ is nonconstant polynomial of degree $n \geq 1$.

In this work, we aim at studying certain geometric properties of the solutions of the following initial-value problem:

$$\omega''(z) + Q(z)\omega'(z) = 0, \quad \omega(0) = 0, \quad \omega'(0) = 1, \quad \omega''(0) = 0,$$  \hspace{1cm} (1.8)

where $Q(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic in $U$.

In order to prove our main results, we need the following definitions and theorems.

**Definition 1.1** (see [13]). Let $H_J$ be the set of complex functions $h(u, v)$ satisfying the following:

(i) $h(u, v)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C}$;  
(ii) $(0, 0) \in D$ and $|h(0, 0)| < J$;  
(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geq J$ when $(Je^{i\theta}, Ke^{i\theta}) \in D$, $\theta$ is real and $K \geq J$.  

Definition 1.2 (see [13]). Let $h \in H_f$ with corresponding domain $D$. We denote by $B_f(h)$ those functions $\omega(z) = \omega_1z + \omega_2z^2 + \cdots$ which are analytic in $U$ satisfying

(i) $(\omega(z), zw'(z)) \in D,$
(ii) $|h(\omega(z), zw'(z))| < J(z \in U).

Theorem 1.3 (see [10]). For any $h \in H_f,$

$$B_f(h) \subset B_f, \quad (h \in H_f; J > 0). \quad (1.9)$$

Theorem 1.4 (see [10]). Let $h \in H_f$ and $b(z)$ be an analytic function in $U$ with $|b(z)| < J$. If the differential equation

$$h(\omega(z), zw'(z)) = b(z), \quad \omega(0) = 0, \quad \omega(0) = 1 \quad (1.10)$$

has a solution $\omega(z)$ analytic in $U$, then $|\omega(z)| < J$.

2. Main Results

Theorem 2.1. Let $Q(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in $U$ with

$$\sum_{n=0}^{\infty} |b_n| < J \quad (z \in U, J > 0), \quad (2.1)$$

and let $\omega(z)$ denote the solution of the initial value problem (1.8) in $U$. Then

$$1 - J < \Re \left\{ 1 + \frac{z\omega''(z)}{\omega'(z)} \right\} > 1 + J \quad (z \in U, J > 0). \quad (2.2)$$

Proof. If we let

$$u(z) = \frac{z\omega''(z)}{\omega'(z)}, \quad (2.3)$$

then $u(z)$ is analytic in $U$, such that $u(0) = 0$ and (1.8) becomes

$$[u(z)]^2 - u(z) + z u'(z) = -z^2 \sum_{n=0}^{\infty} b_n z^n \quad (2.4)$$

or, equivalently,

$$h(u(z), zu'(z)) = -z^2 \sum_{n=0}^{\infty} b_n z^n, \quad (2.5)$$

where, for convenience,

$$h(\xi, \eta) = \xi^2 - \xi + \eta. \quad (2.6)$$
From assumption, we have
\[ \sum_{n=0}^{\infty} |b_n| < J \quad (z \in \mathcal{U}, J > 0). \] (2.7)

By using Theorem 1.4, we have
\[ |u(z)| < J \] (2.8)

which, in view of the relationship (2.3), yields
\[ \left| \frac{z\omega''(z)}{\omega'(z)} \right| < J, \] (2.9)

that is,
\[ 1 - J < \Re \left\{ 1 + \frac{z\omega''(z)}{\omega'(z)} \right\} > 1 + J \quad (z \in \mathcal{U}, J > 0). \] (2.10)

Letting \( J = 1 \) in Theorem 2.1, we have the following corollary.

**Corollary 2.2.** Let \( Q(z) = \sum_{n=0}^{\infty} b_n z^n \) be analytic in \( \mathcal{U} \) with
\[ \sum_{n=0}^{\infty} |b_n| < 1. \] (2.11)

Let \( \omega(z) \) be the solution of the initial-value problem in (1.8) in \( \mathcal{U} \). Then \( \omega(z) \) is convex in \( \mathcal{U} \).

**Example 2.3.** Let \( Q(z) = 1 \) in Corollary 2.2; the solution of the following initial-value problem:
\[ \omega'''(z) + \omega'(z) = 0, \quad \omega(0) = 0, \quad \omega'(0) = 1, \quad \omega''(0) = 0 \] (2.12)

is given by
\[ \omega(z) = \sin z \in \mathbb{C}. \] (2.13)

**Theorem 2.4.** Let \( Q(z) = \sum_{n=0}^{\infty} b_n z^n \) be analytic in \( \mathcal{U} \) with
\[ \sum_{n=0}^{\infty} |b_n| < J \quad (z \in \mathcal{U}, 0 < J \leq 1). \] (2.14)
Let $\omega(z)$ be the solution of the initial-value problem in (1.8) in $U$. Then $\omega(z)$ is strongly convex of order $\alpha$, that is,

$$\left| \arg \left( 1 + \frac{z \omega''(z)}{\omega'(z)} \right) \right| < \frac{\pi}{2} \alpha$$

(2.15)

for some $\alpha$ ($0 < \alpha \leq 1$) and

$$\alpha = \frac{2}{\pi} \sin^{-1} J \quad (0 < J \leq 1).$$

(2.16)

**Proof.** By using the same technique as in the proof of Theorem 2.1, the required result is obtained.

**Remark 2.5.** Putting $\alpha = 1$ in Theorem 2.4, we have Corollary 2.2.

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**References**


