Research Article

Disconjugacy and Nonoscillation Domains for Nonlinear Singular Interface Problems on Semi-Infinite Time Scales

D. K. K. Vamsi and Pallav Kumar Baruah

Department of Mathematics and Computer Science, Sri Sathya Sai Institute of Higher Learning, Prasanthi Nilayam, Ananthapur, Andhra Pradesh, Puttaparthi 515134, India

Correspondence should be addressed to Pallav Kumar Baruah, baruahpk@gmail.com

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We define and discuss the disconjugacy ($\mathfrak{D}$) and nonoscillation ($\mathcal{N}$) domains for a pair of dynamic equations along with matching interface conditions on the semi-infinite time scale $[0, c]_{\mathbb{T}} \cup [\sigma(c), \infty]_{\mathbb{T}}$. We show that these domains are closed and convex subsets of the parameter space $\mathbb{R}^{n+m}$. The theory developed is used to discuss the oscillatory behavior of initial and boundary value problems associated with interface problems in the fields of applied elasticity, acoustic wave guides in ocean, and transverse vibrations in strings.

1. Introduction

The study of waves plays an important role in physical sciences. Waves of simple nature oscillate with a fixed frequency and wave length. The study of these simple sinusoidal waves forms the basis for the study of almost all forms of linear and nonlinear complex wave motions. The oscillation nature of waves can be modelled by differential equations specifically by ordinary Sturm-Liouville operators. Many of the eigenvalue (Sturm-Liouville) problems may be cast in the form

$$y'' + (-\alpha + \beta B(x))y = 0, \quad (1.1)$$

where α, β are real parameters and x varies over a subinterval of $\mathbb{R}$. Fixing α and allowing β to be the parameter we get weighted Sturm-Liouville equation. A survey on this is done

in [1]. In [2], Moore exploited the relation between the nonoscillation and periodicity of solutions of the equation

$$y'' + (-a + bq(x))y = 0. \quad (1.2)$$

Here a, b are real parameters and $q(x)$ is real-valued, continuous, and periodic function. A special region R a subset of the ab -plane for which the equation has nonoscillatory solutions is introduced. R is shown to be a closed, convex and is also shown to be entirely contained in the half-plane $a > 0$. In [3], Markus and Moore found the disconjugacy domain $\mathfrak{D}$ (a subset of the (a, b) -parameter plane) for which the equation

$$y'' + (-a + bp(x))y = 0 \quad (1.3)$$

has disconjugate solutions. D was shown to be closed and bounded. Influenced by this work, Mingarelli and Halvorsen [4] formalised this study. They defined the disconjugacy and non oscillation domains (denoted by $\mathfrak{D}$ and $\mathcal{N}$) for the general equation

$$y'' + (-\alpha A(x) + \beta B(x))y = 0 \quad (1.4)$$

on the closed half-line $I = [0, \infty)$, $A, B : [0, \infty) \rightarrow \mathbb{R}$. $\mathfrak{D}$ consists of all the values of $(\alpha, \beta) \in \mathbb{R}^2$ such that solutions of the general equation are disconjugate. $\mathcal{N}$ consists of all the values of $(\alpha, \beta) \in \mathbb{R}^2$ such that solutions of the general equation are non oscillatory. They discussed the properties of $\mathfrak{D}$ and $\mathcal{N}$ such as closedness and convexity. To our knowledge the concepts of disconjugacy and non oscillation domains for nonlinear equations seem to be less explored

In the late 1980s, Hilger [5], then a graduate student at the Augsburg in Germany, developed a calculus called measure chains that unifies discrete and continuous analysis. For many purposes in analysis it is sufficient to consider a special case of a measure chain, a so-called time scale, which simply is a closed subset of the real numbers. A survey of this calculus can be found in the paper by Agarwal et al. [6] and also in the books by Bohner and Peterson [7, 8]. In the literature of time scales, we see that substantial amount of work has been done on oscillation behaviour of nonlinear dynamic equations on time scales. We refer the reader to [9–14] and references therein. We see that the concept of disconjugacy and non oscillation domains for linear and nonlinear dynamic equations on time scales has not yet been defined.

In study of acoustic wave guides in ocean [15], transverse vibrations of strings [16], one-dimensional scattering in quantum theory [17], optical fiber transmission [18], and applied elasticity [19], we encounter problems wherein two different differential equations are defined on two adjacent intervals with a common point of interface, and the solutions satisfy matching conditions at the point of interface. We observe that the above problem for the regular case has been discussed in [20–25] and references therein. The problem of having singularity at the end boundary points is dealt within [20]. But the problem of having a singularity at the point of interface remained unexplored. The singularity at the point of

interface in the domain of definition could be of the following three types satisfying certain matching conditions at the singular interface:

$$\begin{array}{ll}
 \text{Interface 1: } [a, c] \cup [\sigma(c), b] & \begin{array}{c} \overline{\quad\quad\quad} \quad \overline{\quad\quad\quad} \\ a \qquad\qquad c \quad \sigma(c) \qquad\qquad b \end{array} \\
 \text{Interface 2: } [a, \rho(c)] \cup [c, b] & \begin{array}{c} \overline{\quad\quad\quad} \quad \overline{\quad\quad\quad} \\ a \qquad\qquad \rho(c) \quad c \qquad\qquad b \end{array} \\
 \text{Interface 3: } [a, \rho(c)] \cup [\sigma(c), b] & \begin{array}{c} \overline{\quad\quad\quad} \quad \overline{\quad\quad\quad} \\ a \qquad\qquad \rho(c) \quad \sigma(c) \qquad\qquad b \end{array}
 \end{array} \tag{1.5}$$

To describe the singularities in the domain of definition we take help of the terminology used on time scale [8]. The new framework of the dynamic equations on time scale with facilities of the two jump operators with various definitions of continuity and derivatives make one's job simple to study these singular interface problems. This problem of having singularity at the point of interface for linear interface problems is discussed in [26–28] and for nonlinear case is discussed in [29, 30].

From the above we see that the concepts of disconjugacy, non oscillatory domains have been less explored for nonlinear equations and has not yet been defined for linear and nonlinear dynamic equations on time scales. Also we note that the nonlinear singular interface problems and problems having singularity at the boundary are less explored.

In this paper we extend the concepts of disconjugacy, non oscillatory domains to nonlinear dynamic equations on time scales and also discuss the oscillatory behaviour of nonlinear singular interface problems on semi-infinite time scales. In brief, we study the oscillation theory for an IVP associated with nonlinear singular interface problem on the semi infinite time scale $[0, c]_{\mathbb{T}} \cup [\sigma(c), \infty)_{\mathbb{T}}$. We define and discuss the disconjugacy ($\mathfrak{D}$) and non oscillation ($\mathcal{N}$) domains for this IVP associated with nonlinear singular interface problems on semi infinite time scales. We show that these domains are closed and convex subsets of the parameter space $\mathbb{R}^{n+m}$. The theory developed is used to discuss the oscillatory nature of problems in the fields of applied elasticity, acoustic wave guides in ocean, and transverse vibrations in strings.

In Section 2, we give few mathematical preliminaries, which we use through the rest of the paper, and, in Section 3, we give few preliminary results. In Section 4, we discuss the disconjugacy domain for an IVP associated with nonlinear singular interface problems. Non oscillation domain for the IVP is discussed in Section 5. Finally, in Section 6, the oscillatory behaviour of initial and boundary value problems associated with interface problems in the fields of applied elasticity, acoustic wave guides in ocean, and transverse vibrations in strings is discussed.

2. Mathematical Preliminaries

Definition 2.1. Let $\mathbb{T}$ be a time scale (an arbitrary closed subset of real numbers). For $t \in \mathbb{T}$ one defines the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \tag{2.1}$$

while the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}. \quad (2.2)$$

If $\sigma(t) > t$, we say that t is right scattered, while if $\rho(t) < t$ we say that t is left scattered. Points that are right scattered and left scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left dense. Points that are right dense and left dense at the same time are called dense. Finally, the graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t. \quad (2.3)$$

Definition 2.2. One has

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} - \{m\}, & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty, \end{cases} \quad (2.4)$$

where m is the left-scattered maximum.

Definition 2.3. Let f be a function defined on $\mathbb{T}$. One says that f is delta differentiable at $t \in \mathbb{T}^\kappa$ provided there exists an α such that for all $\epsilon > 0$ there is a neighborhood $\mathcal{N}$ around t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s| \quad \forall s \in \mathcal{N}. \quad (2.5)$$

Remark 2.4. For a function $f : \mathbb{T} \rightarrow \mathbb{R}$ we will talk about the second derivative $f^{\Delta\Delta}$ provided f^Δ is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa^2} \rightarrow \mathbb{R}$.

Definition 2.5. One defines the parameter space

$$\mathbb{R}^{n+m} = \{(\lambda_1, \lambda_2, \dots, \lambda_{n+m}), \lambda_i \in \mathbb{R}, i = 1, 2, \dots, n+m, n, m \in \mathbb{N}\}. \quad (2.6)$$

3. Preliminary Results

Let $\mathbb{T}_1 = [0, c]_\mathbb{T}$ (a time scale with end points 0 and c), $K_1 = [\sigma(c), \infty)_\mathbb{T}$ a time scale with one end being $\sigma(c)$, $\mathbb{T}_2 = K_1^{\kappa^2}$. Also let (f_1, f_2) be nonlinear function tuple in $\mathcal{C}(\mathbb{T}_1 \times \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2 \times \mathbb{R})$. And let (f_1, f_2) be positive. In this section we consider the following IVP associated with singular interface problem (IVP-SIP-I):

$$y_1^{\Delta\Delta}(t) = f_1(t, y_1^\sigma), \quad t \in \mathbb{T}_1, \quad (3.1)$$

$$y_2^{\Delta\Delta}(t) = f_2(t, y_2^\sigma), \quad t \in \mathbb{T}_2, \quad (3.2)$$

with the initial conditions

$$y_1(0) = l, \quad (3.3)$$

$$y_1^\Delta(0) = m, \quad \text{where } l, m > 0, \quad (3.4)$$

followed by the matching interface conditions

$$\rho_1 y_1(c) = \rho_2 y_2(\sigma(c)), \quad (3.5)$$

$$\rho_3 y_1^\Delta(c) = \rho_4 y_2^\Delta(\sigma(c)), \quad \rho_i > 0, \quad i = 1, 2, 3, 4. \quad (3.6)$$

Definition 3.1. One calls a function $y : \mathbb{T}_1 \cup \mathbb{T}_2 \rightarrow \mathbb{R}$ to be a matching solution of the IVP-SIP-I if

- (i) the function $y|_{\mathbb{T}_1} = y_1$ and y_1 satisfies (3.1),
- (ii) the function $y|_{\mathbb{T}_2} = y_2$ and y_2 satisfies (3.2),
- (iii) y_1 and y_2 satisfy the initial and interface conditions (3.3)-(3.4) and (3.5)-(3.6), respectively.

Definition 3.2. One calls $t \in \mathbb{T}_1 \cup \mathbb{T}_2$ to be a zero of the IVP-SIP-I if $y(t) = 0$.

Definition 3.3. One calls an IVP associated with a singular interface problem to be disconjugate if every nontrivial solution of the IVP has at most one zero.

We denote $\mathcal{C}(\mathbb{T}_i)$, $i = 1, 2$, to be the set of continuous functions on time scales $\mathbb{T}_1$ and $\mathbb{T}_2$. Also, we denote

$$f(t) = \begin{cases} f_1(t), & t \in \mathbb{T}_1, \\ f_2(t), & t \in \mathbb{T}_2. \end{cases} \quad (3.7)$$

Definition 3.4. For a given compact subinterval $[a, c] \cup [\sigma(c), b] \subset \mathbb{T}_1 \cup \mathbb{T}_2$, one defines the space

$$\widehat{A}([a, c] \cup [\sigma(c), b]) = \{(\eta_1, \eta_2) : (\eta_1, \eta_2) \in \mathcal{C}[a, c] \times \mathcal{C}[\sigma(c), b]\}. \quad (3.8)$$

For any $(\eta_1, \eta_2) \in \widehat{A}([a, c] \cup [\sigma(c), b])$, one defines the functional

$$\begin{aligned} \zeta((\eta_1, \eta_2), (f_1, f_2); a, b) &= \int_a^c \left[\left\{ \eta_1^\Delta(t) \right\}^2 + f_1(t, y_1^\sigma) \left\{ \eta_1(t) \right\}^2 \right] \Delta t \\ &\quad + \int_{\sigma(c)}^b \left[\left\{ \eta_2^\Delta(t) \right\}^2 + f_2(t, y_2^\sigma) \left\{ \eta_2(t) \right\}^2 \right] \Delta t. \end{aligned} \quad (3.9)$$

Lemma 3.5. *If IVP-SIP-I is disconjugate on every subset of $\mathbb{T}_1 \cup \mathbb{T}_2$ (denoted by $\mathcal{D}$), then there exists a positive non trivial solution but no oscillatory solution for the IVP-SIP-I.*

Proof. It is easy to see that IVP-SIP-I is equivalent to the integral equations

$$\begin{aligned} y_1(t) &= \int_0^t \int_0^s f_1(t, y_1^\sigma) \Delta t \Delta s + mt + l, \quad t \in \mathbb{T}_1, \\ y_2(t) &= \int_{\sigma(c)}^t \int_{\sigma(c)}^s f_2(t, y_2^\sigma) \Delta t \Delta s + \int_{\sigma(c)}^t c_{21} \Delta s + c_{22}, \quad t \in \mathbb{T}_2, \end{aligned} \quad (3.10)$$

where

$$c_{21} = \frac{\rho_3}{\rho_4} \int_0^c f_1(t, y_1^\sigma) \Delta s, \quad c_{22} = \frac{\rho_1}{\rho_2} \int_0^c \int_0^s f_1(t, y_1^\sigma) \Delta t \Delta s + mc + l, \quad (3.11)$$

which give the required positive solution. Moreover, IVP-SIP-I is disconjugate implying that (y_1, y_2) is a non oscillatory solution. $\square$

Lemma 3.6. *If the IVP-SIP-I is disconjugate on $\mathcal{J}$, then, for every closed subinterval of $[a, c] \cup [\sigma(c), b]$, the functional $\zeta((\eta_1, \eta_2), (f_1, f_2); a, b)$ is positive on $\hat{A}([a, c] \cup [\sigma(c), b])$.*

Proof. Let us suppose that the IVP-SIP-I is disconjugate on $\mathcal{J}$. Then by Lemma 3.5 there is a positive solution (u_1, u_2) of the IVP-SIP-I.

Assumption 3.7. We assume that f is of the form

$$\begin{aligned} f_1(t, u_1^\sigma) &= A_1(t)B_1(u_1^\sigma), \quad t \in [a, c]_{\mathbb{T}}, \\ f_2(t, u_2^\sigma) &= A_2(t)B_2(u_2^\sigma), \quad t \in [\sigma(c), b]_{\mathbb{T}}, \end{aligned} \quad (3.12)$$

where A_i, B_i are functions on $\mathbb{T}_i, \mathcal{C}(\mathbb{T}_i)$, $i = 1, 2$. We also assume that B_1, B_2 are positive basing on the fact that (u_1, u_2) is a positive solution of the IVP-SIP-I.

From the definition of the functional $\zeta((\eta_1, \eta_2), (f_1, f_2); a, b)$ we easily see that $\zeta((\eta_1, \eta_2), (f_1, f_2); a, b) \geq 0$ whenever $A_1, A_2 \geq 0$. For the cases when $(A_1, A_2 \leq 0)$, $(A_1 \leq 0, A_2 \geq 0)$, $(A_1 \geq 0, A_2 \leq 0)$ we define the functional $\zeta((\eta_1, \eta_2), (f_1, f_2); a, b)$ in the following manner.

If $A_1, A_2 \leq 0$, then we define

$$\begin{aligned} \zeta((\eta_1, \eta_2), (f_1, f_2); a, b) &= \int_a^c \left[\left\{ \eta_1^\Delta(t) \right\}^2 - f_1(t, y_1^\sigma) \left\{ \eta_1(t) \right\}^2 \right] \Delta t \\ &\quad + \int_{\sigma(c)}^b \left[\left\{ \eta_2^\Delta(t) \right\}^2 - f_2(t, y_2^\sigma) \left\{ \eta_2(t) \right\}^2 \right] \Delta t. \end{aligned} \quad (3.13)$$

If $A_1 \leq 0, A_2 \geq 0$, then we define

$$\begin{aligned} \zeta((\eta_1, \eta_2), (f_1, f_2); a, b) &= \int_a^c \left[\left\{ \eta_1^\Delta(t) \right\}^2 - f_1(t, y_1^\sigma) \{ \eta_1(t) \}^2 \right] \Delta t \\ &+ \int_{\sigma(c)}^b \left[\left\{ \eta_2^\Delta(t) \right\}^2 + f_2(t, y_2^\sigma) \{ \eta_2(t) \}^2 \right] \Delta t. \end{aligned} \quad (3.14)$$

If $A_1 \geq 0, A_2 \leq 0$, then we define

$$\begin{aligned} \zeta((\eta_1, \eta_2), (f_1, f_2); a, b) &= \int_a^c \left[\left\{ \eta_1^\Delta(t) \right\}^2 + f_1(t, y_1^\sigma) \{ \eta_1(t) \}^2 \right] \Delta t \\ &+ \int_{\sigma(c)}^b \left[\left\{ \eta_2^\Delta(t) \right\}^2 - f_2(t, y_2^\sigma) \{ \eta_2(t) \}^2 \right] \Delta t. \end{aligned} \quad (3.15)$$

From the above definitions of $\zeta((\eta_1, \eta_2), (f_1, f_2); a, b)$, we see that $\zeta((\eta_1, \eta_2), (f_1, f_2); a, b) \geq 0$ irrespective of the sign of A_1 and A_2 . $\square$

Note 1. Through the rest of the paper we assume that $A_1, A_2 \geq 0$. Similar results can be obtained for the cases when $(A_1, A_2 < 0)$, $(A_1 < 0, A_2 > 0)$, $(A_1 > 0, A_2 < 0)$ by using the respective definitions for $\zeta((\eta_1, \eta_2), (f_1, f_2); a, b)$.

Lemma 3.8. *Let IVP-SIP-II be defined by*

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= \lambda_{11} f_1(t, y_1^\sigma), \quad t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= \lambda_{12} f_2(t, y_2^\sigma), \quad t \in \mathbb{T}_2, \\ -\infty &< \lambda_{11}, \lambda_{12} < +\infty, \end{aligned} \quad (3.16)$$

along with (3.3)–(3.6). Then IVP-SIP-II is disconjugate for every $\lambda_{11}, \lambda_{12}$ if and only if $f(t, y) = 0$ on $\mathcal{J}$.

Proof. Let $f(t, y^\sigma) = 0$, that is,

$$\begin{aligned} f_1(t, y_1^\sigma) &= 0, \quad t \in \mathbb{T}_1, \\ f_2(t, y_2^\sigma) &= 0, \quad t \in \mathbb{T}_2. \end{aligned} \quad (3.17)$$

Then we see that IVP-SIP-II reduces to the IVP-SIP-III

$$\begin{aligned}
 y_1^{\Delta\Delta}(t) &= 0, \quad t \in \mathbb{T}_1, \\
 y_2^{\Delta\Delta}(t) &= 0, \quad t \in \mathbb{T}_2, \\
 y_1(0) &= l, \\
 y_1^\Delta(0) &= m, \\
 \rho_1 y_1(c) &= \rho_2 y_2(\sigma(c)), \\
 \rho_3 y_1^\Delta(c) &= \rho_4 y_2^\Delta(\sigma(c)).
 \end{aligned} \tag{3.18}$$

Simple calculations show that

$$\begin{aligned}
 y_1(t) &= mt + l, \\
 y_2(t) &= \frac{\rho_3}{\rho_4} mt + \frac{\rho_1}{\rho_2} (cm + l) - \frac{\rho_3}{\rho_4} m\sigma(c).
 \end{aligned} \tag{3.19}$$

We observe that $y(t) = 0$ whenever

$$t = -\frac{l}{m} \quad \text{or} \quad \sigma(c) - \frac{\rho_1 \rho_4}{\rho_2 \rho_3} \left(c + \frac{l}{m} \right). \tag{3.20}$$

Clearly $t \neq -l/m$ and so $t = \sigma(c) - (\rho_1 \rho_4 / \rho_2 \rho_3)(c + (l/m))$ is the only zero possible. Hence, IVP-SIP-III is disconjugate on $\mathcal{O}$.

Now let IVP-SIP-II be disconjugate for every $\lambda_{11}, \lambda_{12}$. From Lemma 3.6, we have

$$\zeta((\eta_1, \eta_2), (f_1, f_2); a, b) \geq 0. \tag{3.21}$$

Hence

$$\int_a^c \left[\left\{ \eta_1^\Delta(t) \right\}^2 + \lambda_{11} f_1(t, y_1^\sigma) \{ \eta_1(t) \}^2 \right] \Delta t + \int_{\sigma(c)}^b \left[\left\{ \eta_2^\Delta(t) \right\}^2 + \lambda_{12} f_2(t, y_2^\sigma) \{ \eta_2(t) \}^2 \right] \Delta t \geq 0. \tag{3.22}$$

So we have

$$\begin{aligned}
 & -\lambda_{11} \int_a^c f_1(t, y_1^\sigma) \{ \eta_1(t) \}^2 \Delta t - \lambda_{12} \int_{\sigma(c)}^b f_2(t, y_2^\sigma) \{ \eta_2(t) \}^2 \Delta t \\
 & \leq \int_a^c \left\{ \eta_1^\Delta(t) \right\}^2 + \int_{\sigma(c)}^b \left\{ \eta_2^\Delta(t) \right\}^2.
 \end{aligned} \tag{3.23}$$

The previous equation is true for all $-\infty < \lambda_{11}, \lambda_{12} < +\infty$, which implies that

$$\int_a^c f_1(t, y_1^\sigma) \{ \eta_1(t) \}^2 \Delta t = 0 = \int_{\sigma(c)}^b f_2(t, y_2^\sigma) \{ \eta_2(t) \}^2 \Delta t. \tag{3.24}$$

Since the above equation holds true for all η_1 and η_2 , we must have $f_1 = 0 = f_2$, hence the proof. $\square$

Corollary 3.9. *Let IVP-SIP-IV be defined by*

$$\begin{aligned} y_1^{\Delta\Delta} &= \lambda_{11}f_{11}(t, y_1^\sigma) + \lambda_{21}f_{21}(t, y_1^\sigma) + \cdots + \lambda_{n1}f_{n1}(t, y_1^\sigma), \\ y_2^{\Delta\Delta} &= \lambda_{12}f_{12}(t, y_2^\sigma) + \lambda_{22}f_{22}(t, y_2^\sigma) + \cdots + \lambda_{n2}f_{n2}(t, y_2^\sigma), \end{aligned} \quad (3.25)$$

along with (3.3)–(3.6). Then IVP-SIP-IV is disconjugate on $\mathcal{J}$ for all $((\lambda_{11}, \lambda_{12}), (\lambda_{21}, \lambda_{22}), \dots, (\lambda_{n1}, \lambda_{n2})) \in \mathbb{R}^{2n}$,

then $f_i(t, y) = 0$, $i = 1, 2, \dots, n$, that is,

$$\begin{aligned} f_{i1}(t, y_1^\sigma) &= 0, \quad \text{for } t \in \mathbb{T}_1, \\ f_{i2}(t, y_2^\sigma) &= 0, \quad \text{for } t \in \mathbb{T}_2, \end{aligned} \quad (3.26)$$

where (f_{i1}, f_{i2}) are nonlinear function tuples in $\mathcal{C}(\mathbb{T}_1 \times \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2 \times \mathbb{R})$.

4. Disconjugacy Domain

We define IVP-SIP-V as

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= \lambda_{11}f_{11}(t, y_1^\sigma) + \lambda_{21}f_{21}(t, y_1^\sigma) + \cdots + \lambda_{n1}f_{n1}(t, y_1^\sigma), \quad t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= \lambda_{12}f_{12}(t, y_2^\sigma) + \lambda_{22}f_{22}(t, y_2^\sigma) + \cdots + \lambda_{m2}f_{m2}(t, y_2^\sigma), \quad t \in \mathbb{T}_2, \end{aligned} \quad (4.1)$$

along with (3.3)–(3.6). Without any loss of generality, let us assume that $n < m$, where $n, m \in \mathbb{N}$ and f_{i1}, f_{j2} , $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, belong to $\mathcal{C}(\mathbb{T}_1 \times \mathbb{R})$, $\mathcal{C}(\mathbb{T}_2 \times \mathbb{R})$, respectively.

Definition 4.1. One defines the disconjugacy domain $\mathfrak{D}$ of IVP-SIP-V as

$$\mathfrak{D} = \{(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \dots, \lambda_{n1}, \lambda_{n2}, \lambda_{(n+1)2}, \lambda_{(n+2)2}, \dots, \lambda_{m2}) \in \mathbb{R}^{n+m}\} \quad (4.2)$$

such that IVP-SIP-V is disconjugate on $\mathcal{J}$.

Theorem 4.2. *The disconjugacy domain $\mathfrak{D}$ of IVP-SIP-V is the whole space $\mathbb{R}^{n+m}$ if and only if*

$$\begin{aligned} f_{i1} &= 0, \quad i = 1, 2, \dots, n, \\ f_{j1} &= 0, \quad j = 1, 2, \dots, m. \end{aligned} \quad (4.3)$$

Proof. For

$$\begin{aligned} \lambda_{21} &= \lambda_{31} = \cdots = \lambda_{n1} = 0, \quad \lambda_{11} \neq 0, \\ \lambda_{22} &= \lambda_{32} = \cdots = \lambda_{m2} = 0, \quad \lambda_{12} \neq 0, \end{aligned} \quad (4.4)$$

IVP-SIP-V reduces to IVP-SIP-II, and hence from Lemma 3.8 we have $f_{11} = 0 = f_{12}$. If we choose

$$\begin{aligned}\lambda_{11} = \lambda_{31} = \cdots = \lambda_{n1} &= 0, & \lambda_{21} &\neq 0, \\ \lambda_{12} = \lambda_{32} = \cdots = \lambda_{m2} &= 0, & \lambda_{22} &\neq 0,\end{aligned}\tag{4.5}$$

we get $f_{21} = 0 = f_{22}$. In similar lines we can show that

$$\begin{aligned}f_{31} = f_{41} = \cdots = f_{n1} &= 0, \\ f_{32} = f_{42} = \cdots = f_{n2} &= 0.\end{aligned}\tag{4.6}$$

Now let us choose

$$\begin{aligned}\lambda_{11} = \lambda_{31} = \cdots = \lambda_{(n-1)1} &= 0, & \lambda_{n1} &\neq 0, \\ \lambda_{12} = \lambda_{32} = \cdots = \lambda_{n2} = 0 &= \lambda_{(n+2)2} = \cdots = \lambda_{m2}, & \lambda_{(n+1)2} &\neq 0.\end{aligned}\tag{4.7}$$

We see that from Lemma 3.8 we have $f_{(n+1)2} = 0$. Similarly we can show that

$$f_{(n+2)2} = f_{(n+3)2} = \cdots = f_{m2} = 0.\tag{4.8}$$

Hence

$$\begin{aligned}f_{i1} &= 0, & i &= 1, 2, \dots, n, \\ f_{j1} &= 0, & j &= 1, 2, \dots, m.\end{aligned}\tag{4.9}$$

whenever the disconjugacy domain $\mathfrak{D}$ of IVP-SIP-V is the whole space $\mathbb{R}^{n+m}$.

On the other hand, if

$$\begin{aligned}f_{i1} &= 0, & i &= 1, 2, \dots, n, \\ f_{j1} &= 0, & j &= 1, 2, \dots, m,\end{aligned}\tag{4.10}$$

by simple calculations, it can be seen that IVP-SIP-V is disconjugate. □

Corollary 4.3. *If at least one of the functions f_{i1} or $f_{j2} \neq 0$, then $\mathfrak{D}$ is a proper subset of $\mathbb{R}^{n+m}$.*

Proof. This is the contrapositive of Theorem 4.2. □

Theorem 4.4. *$\mathfrak{D}$ contains a proper subspace of the vector space $\mathbb{R}^{n+m}$ if the sets of functions $\{f_{11}, f_{21}, \dots, f_{n1}\}$ and $\{f_{12}, f_{22}, \dots, f_{m2}\}$ are linearly dependent.*

Proof. Let $\{f_{11}, f_{21}, \dots, f_{n1}\}$ and $\{f_{12}, f_{22}, \dots, f_{m2}\}$ be sets of linearly dependent functions. Then there exists constants $\{c_{11}, c_{21}, \dots, c_{(n-1)1}\}$, $\{c_{12}, c_{22}, \dots, c_{(m-1)2}\}$, not all zero, such that

$$\begin{aligned} f_{11}(t, y_1^\sigma) &= c_{11}f_{21}(t, y_1^\sigma) + c_{21}f_{31}(t, y_1^\sigma) + \dots + c_{(n-1)1}f_{n1}(t, y_1^\sigma), & \text{for } t \in \mathbb{T}_1, \\ f_{12}(t, y_2^\sigma) &= c_{12}f_{22}(t, y_2^\sigma) + c_{22}f_{32}(t, y_2^\sigma) + \dots + c_{(m-1)2}f_{m2}(t, y_2^\sigma), & \text{for } t \in \mathbb{T}_2. \end{aligned} \quad (4.11)$$

Therefore, the IVP-SIP-V becomes

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= \lambda_{11}[c_{11}f_{21} + c_{21}f_{31} + \dots + c_{(n-1)1}f_{n1}](t, y_1^\sigma) \\ &\quad + \lambda_{21}f_{21}(t, y_1^\sigma) + \dots + \lambda_{n1}f_{n1}(t, y_1^\sigma), \quad t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= \lambda_{12}[c_{12}f_{22} + c_{22}f_{32} + \dots + c_{(m-1)2}f_{m2}](t, y_2^\sigma) \\ &\quad + \lambda_{22}f_{22}(t, y_2^\sigma) + \dots + \lambda_{m2}f_{m2}(t, y_2^\sigma), \quad t \in \mathbb{T}_2, \end{aligned} \quad (4.12)$$

along with (3.3)–(3.6). We see that IVP-SIP-V now is to be

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= (\lambda_{11}c_{11} + \lambda_{21})f_{21}(t, y_1^\sigma) + (\lambda_{11}c_{21} + \lambda_{31})f_{31}(t, y_1^\sigma) + \dots + (\lambda_{11}c_{(n-1)1} + \lambda_{n1})f_{n1}(t, y_1^\sigma), \\ y_2^{\Delta\Delta}(t) &= (\lambda_{12}c_{12} + \lambda_{22})f_{22}(t, y_2^\sigma) + (\lambda_{12}c_{22} + \lambda_{32})f_{32}(t, y_2^\sigma) + \dots + (\lambda_{12}c_{(m-1)2} + \lambda_{m2})f_{m2}(t, y_2^\sigma), \end{aligned} \quad (4.13)$$

along with (3.3)–(3.6). Hence $\mathfrak{D}$ contains the subspace

$$\mathcal{S} = \{(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \dots, \lambda_{n1}, \lambda_{n2}, \lambda_{(n+1)2}, \lambda_{(n+2)2}, \dots, \lambda_{m2}) \in \mathbb{R}^{n+m}\}, \quad (4.14)$$

where

$$\begin{aligned} \lambda_{21} &= -\lambda_{11}c_{11}, & \lambda_{31} &= -\lambda_{11}c_{21}, \dots, & \lambda_{n1} &= -\lambda_{11}c_{(n-1)1}, \\ \lambda_{22} &= -\lambda_{12}c_{12}, & \lambda_{32} &= -\lambda_{12}c_{22}, \dots, & \lambda_{m2} &= -\lambda_{12}c_{(m-1)2}, \end{aligned} \quad (4.15)$$

hence the proof. $\square$

Corollary 4.5. *If the sets of functions $\{f_{11}, f_{21}, \dots, f_{n1}\}$ and $\{f_{12}, f_{22}, \dots, f_{m2}\}$ are linearly independent, then $\mathfrak{D}$ cannot contain a proper subspace of $\mathbb{R}^{n+m}$.*

Proof. This is the contrapositive of Theorem 4.4. $\square$

Lemma 4.6. *Let us consider the following two IVP-SIPs which are disconjugate on $\mathcal{O}$:*

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= f_{11}(t, y_1^\sigma), \quad t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= f_{12}(t, y_2^\sigma), \quad t \in \mathbb{T}_2, \text{ along with (3.3)–(3.6),} \\ y_1^{\Delta\Delta}(t) &= f_{21}(t, y_1^\sigma), \quad t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= f_{22}(t, y_2^\sigma), \quad t \in \mathbb{T}_2, \text{ along with (3.3)–(3.6).} \end{aligned} \tag{4.16}$$

Then the IVP-SIP-VI defined by

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= (1 - \gamma)f_{11}(t, y_1^\sigma) + \gamma f_{21}(t, y_1^\sigma), \quad t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= (1 - \gamma)f_{12}(t, y_2^\sigma) + \gamma f_{22}(t, y_2^\sigma), \quad t \in \mathbb{T}_2, \end{aligned} \tag{4.17}$$

along with (3.3)–(3.6) is also disconjugate on $\mathcal{O}$, for each $\gamma \in [0, 1]$.

Proof. On account of Lemma 3.6 it is sufficient if we show that

$$\zeta((\eta_1, \eta_2), ((1 - \gamma)f_{11} + \gamma f_{21}, (1 - \gamma)f_{12} + \gamma f_{22}); a, b) \geq 0 \tag{4.18}$$

on $A([a, c] \cup [\sigma(c), b])$ for every compact subinterval $[a, c] \cup [\sigma(c), b] \subset \mathbb{T}_1 \cup \mathbb{T}_2$. We see that

$$\begin{aligned} &\zeta((\eta_1, \eta_2), ((1 - \gamma)f_{11} + \gamma f_{21}, (1 - \gamma)f_{12} + \gamma f_{22}); a, b) \\ &= \int_a^c \left[\left\{ \eta_1^\Delta(t) \right\}^2 + [(1 - \gamma)f_{11}(t, y_1^\sigma) + \gamma f_{21}(t, y_1^\sigma)] \left\{ \eta_1(t) \right\}^2 \right] \Delta t \\ &\quad + \int_{\sigma(c)}^b \left[\left\{ \eta_2^\Delta(t) \right\}^2 + [(1 - \gamma)f_{12}(t, y_2^\sigma) + \gamma f_{22}(t, y_2^\sigma)] \left\{ \eta_2(t) \right\}^2 \right] \Delta t. \end{aligned} \tag{4.19}$$

Now

$$\begin{aligned} &\int_a^c \left[\left\{ \eta_1^\Delta(t) \right\}^2 + [(1 - \gamma)f_{11}(t, y_1^\sigma) + \gamma f_{21}(t, y_1^\sigma)] \left\{ \eta_1(t) \right\}^2 \right] \Delta t \\ &= \int_a^c \left[(1 - \gamma) \left\{ \eta_1^\Delta(t) \right\}^2 + \gamma \left\{ \eta_1^\Delta(t) \right\}^2 + [(1 - \gamma)f_{11}(t, y_1^\sigma) + \gamma f_{21}(t, y_1^\sigma)] \left\{ \eta_1(t) \right\}^2 \right] \Delta t \\ &= (1 - \gamma) \int_a^c \left[\left\{ \eta_1^\Delta(t) \right\}^2 + f_{11}(t, y_1^\sigma) \left\{ \eta_1(t) \right\}^2 \right] \Delta t + \gamma \int_a^c \left[\left\{ \eta_1^\Delta(t) \right\}^2 + f_{21}(t, y_1^\sigma) \left\{ \eta_1(t) \right\}^2 \right] \Delta t. \end{aligned} \tag{4.20}$$

Similarly it can be shown that

$$\begin{aligned} & \int_{\sigma(c)}^b \left[\left\{ \eta_2^\Delta(t) \right\}^2 + [(1-\gamma)f_{12}(t, y_1^\sigma) + \gamma f_{22}(t, y_1^\sigma)] \left\{ \eta_2(t) \right\}^2 \right] \Delta t \\ &= (1-\gamma) \int_{\sigma(c)}^b \left[\left\{ \eta_2^\Delta(t) \right\}^2 + f_{12}(t, y_2^\sigma) \left\{ \eta_2(t) \right\}^2 \right] \Delta t + \gamma \int_{\sigma(c)}^b \left[\left\{ \eta_2^\Delta(t) \right\}^2 + f_{22}(t, y_2^\sigma) \left\{ \eta_2(t) \right\}^2 \right] \Delta t. \end{aligned} \quad (4.21)$$

Hence we have

$$\begin{aligned} & \zeta((\eta_1, \eta_2), ((1-\gamma)f_{11} + \gamma f_{21}, (1-\gamma)f_{12} + \gamma f_{22}); a, b) \\ &= (1-\gamma) \zeta((\eta_1, \eta_2), (f_{11}, f_{12}); a, b) + \gamma \zeta((\eta_1, \eta_2), (f_{21}, f_{22}); a, b) \geq 0, \end{aligned} \quad (4.22)$$

hence the proof. $\square$

Definition 4.7. Let $(f_{n1}, f_{n2}), (f_1, f_2), n \in \mathbb{N}$ be function tuples in $\mathcal{C}(\mathbb{T}_1 \times \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2 \times \mathbb{R})$. One says that $(f_{n1}, f_{n2}), (f_1, f_2)$ are close in uniform norm whenever

$$\sup_{t \in \mathbb{T}_1} |f_{n1}(t) - f_1(t)| + \sup_{t \in \mathbb{T}_2} |f_{n2}(t) - f_2(t)| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (4.23)$$

Definition 4.8. Let $(y_{n1}, y_{n2}), (y_1, y_2), n \in \mathbb{N}$ be function tuples in $\mathcal{C}(\mathbb{T}_1) \times \mathcal{C}(\mathbb{T}_2)$. One says that $(y_{n1}, y_{n2}), (y_1, y_2)$ are close in uniform norm whenever

$$\sup_{t \in \mathbb{T}_1} |y_{n1}(t) - y_1(t)| + \sup_{t \in \mathbb{T}_2} |y_{n2}(t) - y_2(t)| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (4.24)$$

Lemma 4.9. Let $f_n = (f_{n1}, f_{n2}), f = (f_1, f_2), n \in \mathbb{N}$. Let the sequence of the IVP-SIPs defined by

$$\begin{aligned} y_{n1}^{\Delta\Delta}(t) &= f_{n1}(t, y_1^\sigma), \quad t \in \mathbb{T}_1, \\ y_{n2}^{\Delta\Delta}(t) &= f_{n2}(t, y_2^\sigma), \quad t \in \mathbb{T}_2, \end{aligned} \quad (4.25)$$

along with (3.3)–(3.6) be disconjugate on $\mathcal{O}$ for each $n = 1, 2, \dots$. If $f_n \rightarrow f$ uniformly, then the IVP-SIP

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= f_1(t, y_1^\sigma), \quad t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= f_2(t, y_2^\sigma), \quad t \in \mathbb{T}_2, \end{aligned} \quad (4.26)$$

is disconjugate on $\mathcal{O}$.

Proof. Now let us assume that (4.26) along with (3.3)–(3.6) is not disconjugate. Let us assume that it has a solution $y(t)$ with two zeros t_1, t_2 . Let us assume that $t_1 \in \mathbb{T}_1$ and $t_2 \in \mathbb{T}_2$. The other cases when $t_1, t_2 \in \mathbb{T}_1, t_1, t_2 \in \mathbb{T}_2$ can be worked in similar lines. Now let (y_{n1}, y_{n2}) be a solution such that $y_{n1}(t_1) = 0$.

Claim 1. The solutions of (4.25), (4.26) satisfying (3.3)–(3.6) are close in the uniform norm whenever (f_{n1}, f_{n2}) and (f_1, f_2) are close in uniform norm on $\mathcal{O}$.

Simple calculations show that the solution of (4.25), (4.26) satisfying (3.3)–(3.6) are

$$\begin{aligned} y_{n1}(t) &= \int_0^t \int_0^k f_{n1}(s, y_1^\sigma) \Delta s \Delta k + lt + m, \\ y_{n2}(t) &= \int_{\sigma(c)}^t \int_{\sigma(c)}^{k'} f_{n2}(s, y_2^\sigma) \Delta s \Delta k' + \int_{\sigma(c)}^t \frac{\rho_3}{\rho_4} \left[\int_0^c f_{n1}(s, y_1^\sigma) \Delta s + m \right] \Delta k' \\ &\quad + \frac{\rho_1}{\rho_2} \int_0^c \int_0^k f_{n1}(s, y_1^\sigma) \Delta s \Delta k + lc + m, \end{aligned} \quad (4.27)$$

$$\begin{aligned} y_1(t) &= \int_0^t \int_0^k f_1(s, y_1^\sigma) \Delta s \Delta k + lt + m, \\ y_2(t) &= \int_{\sigma(c)}^t \int_{\sigma(c)}^{k'} f_2(s, y_2^\sigma) \Delta s \Delta k' + \int_{\sigma(c)}^t \frac{\rho_3}{\rho_4} \left[\int_0^c f_1(s, y_1^\sigma) \Delta s + m \right] \Delta k' \\ &\quad + \frac{\rho_1}{\rho_2} \int_0^c \int_0^k f_1(s, y_1^\sigma) \Delta s \Delta k + lc + m. \end{aligned} \quad (4.28)$$

From the above equations we clearly see that the solutions of (4.25), (4.26) along with (3.3)–(3.6) are close in the uniform norm whenever (f_{n1}, f_{n2}) and (f_1, f_2) are close in uniform norm on $\mathcal{O}$, so the claim. Hence for $\beta > 0$ we can find $\epsilon(> 0)$ and $\delta(> 0)$ such that

$$\sup_{t \in [t_1, c]} |y_{n1}(t) - y_1(t)| + \sup_{t \in [\sigma(c), t_2 + \beta]} |y_{n2}(t) - y_2(t)| < \epsilon \quad (4.29)$$

whenever

$$\sup_{t \in [t_1, c]} |f_{n1}(t) - f_1(t)| + \sup_{t \in [\sigma(c), t_2 + \beta]} |f_{n2}(t) - f_2(t)| < \delta. \quad (4.30)$$

Since $y_2(t)$ must change sign at $t = t_2$, it follows that $y_{n2}(t)$ must also change sign near $t = t_2$ for sufficiently large n . Thus, for such n , (4.25) along with (3.3)–(3.6) is not disconjugate which leads us to a contradiction. $\square$

Theorem 4.10. *The disconjugacy domain $\mathfrak{D}$ of IVP-SIP-V is a closed set of the parameter space $\mathbb{R}^{n+m}$.*

Proof. Let $(\lambda_1, \lambda_2, \dots, \lambda_{n+m})$ be a limit point of the sequence $(\lambda_{k1}, \lambda_{k2}, \dots, \lambda_{k(n+m)}) \in D$, $k = 1, 2, \dots$. Then for every $\epsilon > 0$ there exists sufficiently large K such that for all $k > K$ we have $|\lambda_{k1} - \lambda_1| < \epsilon, |\lambda_{k2} - \lambda_2| < \epsilon, \dots, |\lambda_{k(n+m)} - \lambda_{n+m}| < \epsilon$, and IVP-SIP-VII defined by

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= \lambda_{k1}f_{11} + \lambda_{k3}f_{21} + \dots + \lambda_{k(2n-1)}f_{n1}, & t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= \lambda_{k2}f_{12} + \lambda_{k4}f_{22} + \dots + \lambda_{k(2n)}f_{n2} \\ &\quad + \lambda_{k(2n+1)}f_{(n+1)2} + \dots + \lambda_{k(n+m)}f_{m2}, & t \in \mathbb{T}_2, \end{aligned} \quad (4.31)$$

along with (3.3)–(3.6) is disconjugate. Let $y = (y_1, y_2)$ be a non trivial solution of IVP-SIP-V for

$$(\lambda_{11}, \lambda_{21}, \dots, \lambda_{n1}, \lambda_{12}, \lambda_{22}, \dots, \lambda_{m2}) = (\lambda_1, \lambda_2, \dots, \lambda_{n+m}). \quad (4.32)$$

Then we see that either $y(t)$ never vanishes in which case $y(t)$ is disconjugate or let $y(t_0) = 0$ for some t_0 . In the latter case we let (y_{n1}, y_{n2}) be a solution of IVP-SIP-VII such that $y_{n1}(t_0) = 0$. From assumption we have that $y_{n1}(t) \neq 0$ for $t \neq t_0$. From Claim 1 we have (y_{n1}, y_{n2}) uniformly approximating (y_1, y_2) . Hence $y(t)$ can change sign only at $t = t_0$ and so $y(t) \neq 0$ for all $t \neq t_0$. Hence $y(t)$ is disconjugate, so the result. $\square$

Theorem 4.11. *The disconjugacy domain $\mathfrak{D}$ of IVP-SIP-V is a convex set in the parameter space $\mathbb{R}^{n+m}$.*

Proof. We need to show that for $(\lambda_{11}, \lambda_{12}, \dots, \lambda_{1(n+m)}), (\lambda_{21}, \lambda_{22}, \dots, \lambda_{2(n+m)}) \in D$ the convex combination

$$(1 - \gamma)(\lambda_{11}, \lambda_{12}, \dots, \lambda_{1(n+m)}) + \gamma(\lambda_{21}, \lambda_{22}, \dots, \lambda_{2(n+m)}) \in D, \quad (4.33)$$

that is,

$$[(1 - \gamma)\lambda_{11} + \gamma\lambda_{21}, (1 - \gamma)\lambda_{12} + \gamma\lambda_{22}, \dots, (1 - \gamma)\lambda_{1(n+m)} + \gamma\lambda_{2(n+m)}] \in D. \quad (4.34)$$

Since $(\lambda_{11}, \lambda_{12}, \dots, \lambda_{1(n+m)}), (\lambda_{21}, \lambda_{22}, \dots, \lambda_{2(n+m)}) \in D$, we have the IVP-SIPs

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= \lambda_{11}f_{11} + \lambda_{13}f_{21} + \dots + \lambda_{1(2n-1)}f_{n1}, & t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= \lambda_{12}f_{12} + \lambda_{14}f_{22} + \dots + \lambda_{1(2n)}f_{n2} + \lambda_{1(2n+1)}f_{(n+1)2} + \dots + \lambda_{1(n+m)}f_{m2}, & t \in \mathbb{T}_2, \\ y_1^{\Delta\Delta}(t) &= \lambda_{21}f_{11} + \lambda_{23}f_{21} + \dots + \lambda_{2(2n-1)}f_{n1}, & t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= \lambda_{22}f_{12} + \lambda_{24}f_{22} + \dots + \lambda_{2(2n)}f_{n2} + \lambda_{2(2n+1)}f_{(n+1)2} + \dots + \lambda_{2(n+m)}f_{m2}, & t \in \mathbb{T}_2, \end{aligned} \quad (4.35)$$

along with (3.3)–(3.6) to be disconjugate. Now from Lemma 4.6 we have the IVP-SIP-VIII defined by

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= (1 - \gamma)(\lambda_{11}f_{11} + \lambda_{13}f_{21} + \cdots + \lambda_{1(2n-1)}f_{n1}) \\ &\quad + \gamma(\lambda_{21}f_{11} + \lambda_{23}f_{21} + \cdots + \lambda_{2(2n-1)}f_{n1}), \quad t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= (1 - \gamma)(\lambda_{12}f_{12} + \lambda_{14}f_{22} + \cdots + \lambda_{1(2n)}f_{n2} + \lambda_{1(2n+1)}f_{(n+1)2} + \cdots + \lambda_{1(n+m)}f_{m2}) \\ &\quad + \gamma(\lambda_{22}f_{12} + \lambda_{24}f_{22} + \cdots + \lambda_{2(2n)}f_{n2} + \lambda_{2(2n+1)}f_{(n+1)2} + \cdots + \lambda_{2(n+m)}f_{m2}), \quad t \in \mathbb{T}_2, \end{aligned} \quad (4.36)$$

along with (3.3)–(3.6) to be disconjugate; that, is we have the IVP-SIP

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= (1 - \gamma)\lambda_{11}f_{11} + \gamma\lambda_{21}f_{11} + (1 - \gamma)\lambda_{13}f_{21} + \gamma\lambda_{23}f_{21} \\ &\quad + \cdots + (1 - \gamma)\lambda_{1(2n-1)}f_{n1} + \gamma\lambda_{2(2n-1)}f_{n1}, \quad t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= (1 - \gamma)\lambda_{12}f_{12} + \gamma\lambda_{22}f_{12} + (1 - \gamma)\lambda_{14}f_{22} + \gamma\lambda_{24}f_{22} \\ &\quad + \cdots + (1 - \gamma)\lambda_{1(n+m)}f_{m2} + \gamma\lambda_{2(n+m)}f_{m2}, \quad t \in \mathbb{T}_2, \end{aligned} \quad (4.37)$$

along with (3.3)–(3.6) to be disconjugate. Hence

$$(1 - \gamma)(\lambda_{11}, \lambda_{12}, \dots, \lambda_{1(n+m)}) + \gamma(\lambda_{21}, \lambda_{22}, \dots, \lambda_{2(n+m)}) \in D, \quad (4.38)$$

so the proof. $\square$

5. Nonoscillation Domain

Definition 5.1. One calls an IVP associated with a singular interface problem to be non oscillatory if every non trivial solution of the IVP has at most finite number of zeros.

Definition 5.2. One defines the non oscillation domain $\mathcal{N}$ of IVP-SIP-V as

$$\mathcal{N} = \{(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \dots, \lambda_{n1}, \lambda_{n2}, \lambda_{(n+1)2}, \lambda_{(n+2)2}, \dots, \lambda_{m2}) \in \mathbb{R}^{n+m}\} \quad (5.1)$$

such that IVP-SIP-V is non oscillatory on $\mathcal{O}$.

Lemma 5.3. *If IVP-SIP-I is non oscillatory on $\mathcal{O}$, then there exists at least one positive or negative non trivial solution.*

Proof. Let $y(= (y_1, y_2))$ be a solution of IVP-SIP-I. Let $t_1, t_2, \dots, t_n$ be the zeros of the IVP-SIP-I (since $y(t)$ is non oscillatory). By the nature of $y(t)$ we see that either $y(t) < 0$ or $y(t) > 0$ for $t > t_n$. We now define

$$K(t) = \begin{cases} 0, & t \leq t_n, \\ y(t), & t > t_n. \end{cases} \quad (5.2)$$

From Assumption 3.7 we see that $K(t)$ is a solution of IVP-SIP-I. We also see that either $K(t) \leq 0$ or $K(t) \geq 0$. $\square$

Note 2. We assume $K(t) \geq 0$. Similar results can be developed when $K(t) \leq 0$.

All the results discussed in Section 5 regarding $\mathfrak{D}$ can be easily extended to $\mathcal{N}$. We just state two theorems without proof.

Theorem 5.4. *The non oscillatory domain $\mathcal{N}$ of IVP-SIP-V is a closed set of the parameter space $\mathbb{R}^{n+m}$.*

Theorem 5.5. *The non oscillatory domain $\mathcal{N}$ of IVP-SIP-V is a convex set in the parameter space $\mathbb{R}^{n+m}$.*

6. Applications

We define IVP-SIP-X by

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= \lambda_{11}f_{11}(t, y_1^\sigma) + \chi_{11}f_{21}(t, y_1^\sigma) + \cdots + \chi_{(n-1)1}f_{n1}(t, y_1^\sigma), & t \in \mathbb{T}_1, \\ y_2^{\Delta\Delta}(t) &= \lambda_{12}f_{12}(t, y_2^\sigma) + \chi_{12}f_{22}(t, y_2^\sigma) + \cdots + \chi_{(m-1)2}f_{m2}(t, y_2^\sigma), & t \in \mathbb{T}_2, \end{aligned} \quad (6.1)$$

where $\chi_{11}, \chi_{21}, \dots, \chi_{(n-1)1}, \chi_{12}, \chi_{22}, \dots, \chi_{(m-1)2} \in \mathbb{R}$, along with (3.3)–(3.6). We discuss the oscillatory behaviour of the IVP-SIP-X using the non oscillatory domain for IVP-SIP-V.

Theorem 6.1. *One of the following cases can occur for IVP-SIP-X for every $(\lambda_{11}, \lambda_{12}) \in \mathbb{R}^2$.*

- (i) IVP-SIP-X is oscillatory for every $(\lambda_{11}, \lambda_{12})$.
- (ii) IVP-SIP-X is oscillatory for every $(\lambda_{11}, \lambda_{12})$ except at some unique $(\lambda_{11}, \lambda_{12}) = (\lambda_{01}, \lambda_{02})$.
- (iii) There exists a finite interval $(\lambda_{a1}, \lambda_{b1}) \in \mathbb{R}$ or $(\lambda_{a2}, \lambda_{b2}) \in \mathbb{R}$ such that IVP-SIP-X is non oscillatory for either

$$\lambda_{11} \in (\lambda_{a1}, \lambda_{b1}), \quad \lambda_{12} = c_1, \quad (6.2)$$

or

$$\lambda_{11} = c_2, \quad \lambda_{12} \in (\lambda_{a2}, \lambda_{b2}), \quad c_1, c_2 \in \mathbb{R}. \quad (6.3)$$

For every other combinations of $(\lambda_{11}, \lambda_{12}) \in \mathbb{R}^2$, IVP-SIP-X is oscillatory.

- (iv) There exists either λ_1 or $\lambda_2 \in \mathbb{R}$ such that IVP-SIP-X is non oscillatory (resp., oscillatory) for either

$$\lambda_{11} \in (\lambda_1, \infty), \quad \lambda_{12} = c_1, \quad (6.4)$$

or

$$\lambda_{11} = c_2, \quad \lambda_{12} \in (\lambda_2, \infty), \quad (6.5)$$

and oscillatory (resp., non oscillatory) for either

$$\lambda_{11} \in (-\infty, \lambda_1), \quad \lambda_{12} = c_1, \quad (6.6)$$

or

$$\lambda_{11} = c_2, \quad \lambda_{12} \in (-\infty, \lambda_2). \quad (6.7)$$

(v) IVP-SIP-X is non oscillatory for

either

$$\lambda_{11} \in \mathbb{R}, \quad \lambda_{12} = c_1, \quad (6.8)$$

or

$$\lambda_{11} = c_2, \quad \lambda_{12} \in \mathbb{R}. \quad (6.9)$$

(vi) The non oscillatory domain for IVP-SIP-X can be a finite, semi infinite, or infinite plane in one of the following ways:

finite plane

$$c_3 \leq \lambda_{11} \leq c_4, \quad c_3' \leq \lambda_{12} \leq c_4' \quad (6.10)$$

Semi-infinite plane

$$c_3 \leq \lambda_{11} \leq c_4, \quad -\infty \leq \lambda_{12} \leq c_5'$$

$$c_3 \leq \lambda_{11} \leq c_4, \quad c_6' \leq \lambda_{12} \leq +\infty$$

$$c_3 \leq \lambda_{11} \leq c_4, \quad -\infty \leq \lambda_{12} \leq +\infty$$

$$c_6 \leq \lambda_{11} \leq +\infty, \quad c_3' \leq \lambda_{12} \leq c_4'$$

$$c_6 \leq \lambda_{11} \leq +\infty, \quad -\infty \leq \lambda_{12} \leq c_5'$$

$$c_6 \leq \lambda_{11} \leq +\infty, \quad c_6' \leq \lambda_{12} \leq +\infty$$

$$c_6 \leq \lambda_{11} \leq +\infty, \quad -\infty \leq \lambda_{12} \leq +\infty$$

$$-\infty \leq \lambda_{11} \leq c_5, \quad c_3' \leq \lambda_{12} \leq c_4'$$

$$-\infty \leq \lambda_{11} \leq c_5, \quad -\infty \leq \lambda_{12} \leq c_5'$$

$$\begin{aligned}
-\infty &\leq \lambda_{11} \leq c_5, & c_6' &\leq \lambda_{12} \leq +\infty \\
-\infty &\leq \lambda_{11} \leq c_5, & -\infty &\leq \lambda_{12} \leq +\infty \\
-\infty &\leq \lambda_{11} \leq +\infty, & c_3' &\leq \lambda_{12} \leq c_4' \\
-\infty &\leq \lambda_{11} \leq +\infty, & -\infty &\leq \lambda_{12} \leq c_5' \\
-\infty &\leq \lambda_{11} \leq +\infty, & c_6' &\leq \lambda_{12} \leq +\infty
\end{aligned} \tag{6.11}$$

infinite plane

$$-\infty \leq \lambda_{11} \leq +\infty, \quad -\infty \leq \lambda_{12} \leq +\infty, \tag{6.12}$$

where $c_3, c_4, c_5, c_6, c_3', c_4', c_5', c_6' \in \mathbb{R}$.

Proof. Clearly we see that IVP-SIP-X is a special case of IVP-SIP-V with

$$\begin{aligned}
\lambda_{21} &= \chi_{11}, \lambda_{31} = \chi_{21}, \dots, \lambda_{n1} = \chi_{(n-1)1}, \\
\lambda_{22} &= \chi_{12}, \lambda_{32} = \chi_{22}, \dots, \lambda_{m2} = \chi_{(m-1)2}.
\end{aligned} \tag{6.13}$$

Let $\mathcal{L} = \{\lambda_{11}, \chi_{11}, \chi_{21}, \dots, \chi_{(n-1)1}, \lambda_{12}, \chi_{12}, \chi_{22}, \dots, \chi_{(m-1)2}\}$ be a subset of the parameter space $\mathbb{R}^{n+m}$. The above claims are consequences of the intersection of $\mathcal{L}$ with $\mathcal{N}$, the non oscillation domain for IVP-SIP-V. We recall that $\mathcal{N}$ is convex in $\mathbb{R}^{n+m}$. We see that $\mathcal{L}$ intersects $\mathcal{N}$ in one of the following ways:

- (i) $\mathcal{L} \cap \mathcal{N} = \emptyset$,
- (ii) $\mathcal{L} \cap \mathcal{N}$ = a single point,
- (iii) $\mathcal{L} \cap \mathcal{N}$ = a line segment,
- (iv) $\mathcal{L} \cap \mathcal{N}$ = a one-sided ray,
- (v) $\mathcal{L} \cap \mathcal{N}$ = a full ray,
- (vi) $\mathcal{L} \cap \mathcal{N}$ = a finite plane in $\mathbb{R}^2$,
- (vii) $\mathcal{L} \cap \mathcal{N}$ = a semi infinite plane in $\mathbb{R}^2$,
- (viii) $\mathcal{L} \cap \mathcal{N}$ = an infinite plane in $\mathbb{R}^2$,

hence the proof. □

Application I (see, Wang [19]) Applied Elasticity

In the branch of applied elasticity, we encounter the problem of buckling of columns of variable cross-sections given by

$$\begin{aligned}
L_1 u_1 &= \frac{d^2 u_1}{dx^2} + k_1^2 u_1^2 = 0, & 0 \leq x \leq l_1, \\
L_2 u_2 &= \frac{d^2 u_2}{dx^2} + k_2^2 u_2^2 = 0, & l_1 \leq x \leq l_2,
\end{aligned} \tag{6.14}$$

where $k_i^2 = P/EI_i$, E is the modulus of elasticity, P is the load applied, I_i are moments of inertia, $i = 1, 2$, and u_1, u_2 are the displacements of cross-sections for the thinner and the thicker portions of the column, respectively. The physical conditions are given by

$$u_1(0) = u_1'(0) = 0, \quad u_1(l_1) = u_2(l_1), \quad u_1'(l_1) = u_2'(l_1). \quad (6.15)$$

Here we see that $\mathbb{T}_1 = [0, l_1]$, $\mathbb{T}_2 = [l_1, l_2]$, $\rho_1 = 1 = \rho_2$, $\rho_3 = 1 = \rho_4$. We see that

$$\begin{aligned} \lambda_{11} &= -k_1^2, & \lambda_{12} &= -k_2^2, \\ \chi_{11}, \chi_{21}, \dots, \chi_{(n-1)1} &= 0 = \chi_{12}, \chi_{22}, \dots, \chi_{(m-1)2}. \end{aligned} \quad (6.16)$$

Hence from Theorem 6.1 we see that if the point $\{-k_1^2, 0, 0, \dots, 0, -k_2^2, 0, 0, \dots, 0\}$ intersects $\mathcal{N}$ of IVP-SIP-V then we have the problem to be non oscillatory otherwise it will be oscillatory. The set of λ 's for which the problem is non oscillatory can be one of the sets discussed in Theorem 6.1.

Application II (see, Allan Boyles [15]) Acoustic Wave Guides in Oceans

In the study of acoustic wave guides in ocean we encounter the following problem. The ocean is considered to be consisting of two homogeneous layers bounded by a pressure-release surface above and a rigid bottom below. Let d_1, c_1 and d_2, c_2 be the constant density and sound velocity in layers 1 and 2, respectively. Let k_1 and k_2 be the wave vectors which are given by $k_1 = \omega/c_1, k_2 = \omega/c_2$, where ω is the angular frequency. The governing problem is given by

$$\begin{aligned} L_1 u_1 &= \frac{d}{dz} \left(\frac{1}{d_1} \frac{du_1}{dz} \right) + \left(\frac{k_1^2}{d_1} - \frac{\lambda}{d_1} \right) u_1 = 0, & 0 \leq z \leq a, \\ L_2 u_2 &= \frac{d}{dz} \left(\frac{1}{d_2} \frac{du_2}{dz} \right) + \left(\frac{k_2^2}{d_2} - \frac{\lambda}{d_2} \right) u_2 = 0, & a \leq z \leq b, \end{aligned} \quad (6.17)$$

together with the mixed boundary conditions given by

$$\begin{aligned} u_1(0) &= u_2'(b) = 0, \\ u_1(a) &= u_2(a), \\ \frac{1}{d_1} u_1'(a) &= \frac{1}{d_2} u_2'(a), \end{aligned} \quad (6.18)$$

u_1 and u_2 denote the depth eigenfunctions corresponding to the eigenvalue λ .

Here we see that $\mathbb{T}_1 = [0, a]$, $\mathbb{T}_2 = [a, b]$, $\rho_1 = 1 = \rho_2$, $\rho_3 = 1/d_1$, $\rho_4 = 1/d_2$. Though we have an extra boundary condition at b , similar theory can be developed for the above problem. We see that

$$\begin{aligned} \lambda_{11} &= \lambda, & \chi_{11} &= -k_1^2, & \lambda_{12} &= \lambda, & \chi_{12} &= -k_2^2, \\ \chi_{21}, \chi_{31}, \dots, \chi_{(n-1)1} &= 0 = \chi_{22}, \chi_{32}, \dots, \chi_{(m-1)2}. \end{aligned} \quad (6.19)$$

Hence we see that the set of λ 's for which the problem is non oscillatory can be one of the sets discussed in Theorem 6.1.

Application III (see, Ghosh [16]) Transverse Vibrations in Strings

We encounter the following problem in the study of transverse vibrations of strings consisting of two portions of lengths a and b and different uniform densities d_1 and d_2 , respectively, having a tension T stretched between the points $x = 0$ and $x = a + b$:

$$\begin{aligned} L_1 u_1 &= c_1^2 (-u_1'') = \lambda u_1, & 0 \leq x \leq a, \\ L_2 u_2 &= c_2^2 (-u_2'') = \lambda u_2, & a \leq x \leq a + b, \end{aligned} \quad (6.20)$$

with the mixed boundary conditions given by

$$\begin{aligned} u_1(a) &= u_2(a), \\ u_1'(a) &= u_2'(a), \end{aligned} \quad (6.21)$$

and $u_1(0) = u_2(a+b) = 0$, where $c_i^2 = T/d_i$, $i = 1, 2$. u_1 and u_2 are eigenfunctions corresponding to the eigenvalue λ . Here we see that $\mathbb{T}_1 = [0, a]$, $\mathbb{T}_2 = [a, a + b]$, $\rho_1 = 1 = \rho_2$, $\rho_3 = 1 = \rho_4$. Though we have an extra boundary condition at $a + b$, similar theory can be developed for the above problem. We see that

$$\begin{aligned} \lambda_{11} &= -\frac{\lambda}{c_1^2}, & \lambda_{12} &= -\frac{\lambda}{c_2^2}, \\ \chi_{11}, \chi_{21}, \dots, \chi_{(n-1)1} &= 0 = \chi_{12}, \chi_{22}, \dots, \chi_{(m-1)2}. \end{aligned} \quad (6.22)$$

Hence we see that the set of λ 's for which the problem is non oscillatory can be one of the sets discussed in Theorem 6.1.

Remark 6.2. The results presented here are generalization for the nonlinear problems of corresponding linear problems studied in [20–25] and references therein. A pair of nonlinear ordinary differential equations with matching interface conditions is a special case of the problem considered here, and our results hold true by considering $\rho(c) = \sigma(c) = c$ and the delta derivative becomes the ordinary derivative.

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