Research Article

On the Asymptotic Behavior of p -Faber Polynomials for Domains with Piecewise Analytic Boundary

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Let $G \subset \mathbb{C}$ be a domain bounded by a piecewise analytic Jordan's curve L , and let $F_{n,p}$ denote the p -Faber polynomials associated with G . We derive estimates of the form $F_{n,p}(z) = O(1/n^\eta)$, ($n \rightarrow \infty$) for $z \in G$, where η depends on geometric properties of L and the parameter p . Also, we show that O cannot be replaced by o in the relation given above.

1. Introduction and Main Results

Let $G \subset \mathbb{C}$ be a domain bounded by a Jordan curve L , and let Φ be the conformal mapping of the domain $\Omega := \mathbb{C}_\infty \setminus \overline{G}$, where $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, onto $\Delta := \{w : |w| > 1\}$ with the usual normalization at infinity:

$$w = \Phi(z) = \alpha z + a_0 + \frac{a_1}{z} + \cdots, \quad \alpha > 0, \quad z \in \Omega. \quad (1.1)$$

Let $\Psi := \Phi^{-1} : \Delta \rightarrow \Omega$ denote the inverse conformal map. Then,

$$z = \Psi(w) = \beta w + b_0 + \frac{b_1}{w} + \cdots, \quad |w| > 1, \quad (1.2)$$

where $\beta = 1/\alpha$ gives the capacity $\text{cap}(L)$ of L .

Let $0 < p < \infty$. The p -Faber polynomials $F_{n,p}$ associated the domain G are defined as the polynomial part of the expansion of

$$\Phi^n(z)(\Phi'(z))^{1/p}, \quad n = 0, 1, 2, \dots, \quad (1.3)$$

in a neighborhood of the infinity. Therefore, from (1.1) we have

$$\Phi^n(z)(\Phi'(z))^{1/p} := F_{n,p}(z) + E_{n,p}(z), \quad z \in \Omega, \quad (1.4)$$

where

$$F_{n,p}(z) := \alpha^{n+1/p} z^n + \dots \quad (1.5)$$

is the p -Faber polynomial of degree n and

$$E_{n,p}(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \dots \quad (1.6)$$

is the singular part of $\Phi^n(z)(\Phi'(z))^{1/p}$. Indeed, the number c_0 vanishes, since $z((\Phi'(z))^{1/p} - \alpha^{1/p})$ tends to zero when z tends to infinity. Also, the p -Faber polynomials associated with G can be defined by the generating function

$$g(w) := \frac{(\Psi'(w))^{1-1/p}}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_{n,p}(z)}{w^{n+1}}, \quad z \in G, \quad |w| > 1, \quad (1.7)$$

see [1, 2].

If p tends to infinity, the p -Faber polynomials coincide with the usual Faber polynomials F_n . If $p = 1$, then it follows immediately from (1.4) that

$$F_{n,1}(z) = \frac{F'_{n+1}(z)}{n+1}, \quad (1.8)$$

where F_n is the Faber polynomials of G of degree n .

The aim of this paper is to find asymptotic behaviors of p -Faber polynomials $F_{n,p}$ in G and $E_{n,p}$ in Ω .

If L is rectifiable, then Φ' belongs to the Smirnov class $E^1(\Omega\{\infty\})$. In addition, both Ψ' and Φ' have nontangential limits almost everywhere on $\mathbb{T}$ boundary of the unit disc $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ and L ; respectively, and they are integrable with respect to the arc length measure; see [3] or [4].

Assume now that the boundary L is rectifiable. For such a boundary, Cauchy's integral formula yields the following representations for the p -Faber polynomials and their associated singular parts:

$$\begin{aligned} F_{n,p}(z) &= \frac{1}{2\pi i} \int_L \frac{\Phi^n(\zeta)(\Phi'(\zeta))^{1/p}}{\zeta - z} d\zeta, \quad z \in G, \\ E_{n,p}(z) &= \frac{1}{2\pi i} \int_L \frac{\Phi^n(\zeta)(\Phi'(\zeta))^{1/p}}{\zeta - z} d\zeta, \quad z \in G. \end{aligned} \quad (1.9)$$

Our assumption for this work is that L is a piecewise analytic Jordan's curve. That is, $L = \bigcup_{j=1}^m L_j$, $m \in \mathbb{N}$, where L_j are a finite number of analytic Jordan's arcs meeting at corners z_j where L has an exterior angle $\lambda_j\pi$ with $0 < \lambda_j < 2$. Cusps are, therefore, excluded, but $\lambda_j = 1$ may occur—a "smooth corner".

Let $\lambda := \min_{j=1,\dots,m} \{\lambda_j\}$. Gaier [5] proved that

$$F_n(z) = O\left(\frac{1}{n^\lambda}\right) \quad (n \rightarrow \infty), \quad (1.10)$$

for $z \in G$, uniformly on compact subsets of G . Stylianopoulos [6] indicated that one can obtain

$$F_{n,1}(z) \leq \frac{c(L, F)}{n^{\lambda+1}}, \quad z \in F \quad (1.11)$$

by using (1.8) and (1.13), where F is an arbitrary compact subset of G , and $c(L, F)$ is a constant depending on L and F . Estimates of the form (1.13) have been given by Suetin [2, 7], for boundary L has high degree of smoothness.

For simplicity of notation, we write

$$\eta := \eta(\lambda, p) := (1 - \lambda) \left(1 - \frac{1}{p}\right). \quad (1.12)$$

It is clear that, if $0 < \lambda < 2$ and $1/2 \leq p \leq \infty$, then $|\eta| < 1$.

The main results of the paper are the following.

Theorem 1.1. *Let $1/2 \leq p \leq \infty$, and let $G \subset \mathbb{C}$ be bounded by a piecewise analytic Jordan curve L . If L has exterior angles $\lambda_j\pi$ with $0 < \lambda_j < 2$ and if $\lambda := \min_j \lambda_j$, then*

$$F_{n,p}(z) = O\left(\frac{1}{n^{1-\eta}}\right) \quad (n \rightarrow \infty), \quad (1.13)$$

for $z \in G$, uniformly on compact subsets of G .

Remark 1.2. From the proof of Theorem 1.1, it follows that an estimate of the form (1.13) holds also all derivatives of the p -Faber polynomials.

Remark 1.3. If L has no smooth corners; that is, all $\lambda_j \neq 1$, O cannot be replaced by o in (1.13).

Remark 1.4. If p tends to infinity, we get the result (1.13) of Gaier.

Similar arguments in the proof of Theorem 1.1 provide the following theorem.

Theorem 1.5. *Under the assumptions of Theorem 1.1, we have, for any $n \in \mathbb{N}$,*

$$|E_{n,p}(z)| \leq \frac{c(L)}{\text{dist}(z, L)} \frac{1}{n^{1-\eta}}, \quad z \in \Omega, \quad (1.14)$$

where $c(L)$ depends on L only.

Remark 1.6. The equalities (1.13) and (1.14) are also valid for $0 < p < 1/2$ under condition $\max_{j=1,\dots,m} \lambda_j < 1/(1-p)$.

Remark 1.7. If $p = 1$, we get the result of Stylianopoulos [6]:

$$|E_{n,1}(z)| \leq \frac{c(L)}{\text{dist}(z, L)} \frac{1}{n}, \quad z \in \Omega. \quad (1.15)$$

We will prove Theorems 1.1, 1.5 and Remark 1.3 in section 3.

2. Auxiliary Results

We are going to follow the analog used by Gaier in [5]. Since the components L_j of L are assumed to be analytic Jordan's arcs, the mapping Φ from Ω to Δ can be continued analytically beyond each point of L that is not corner; this follows from Schwarz's reflection principle. Since L has no cusps, we can refine this statement. For each $j = 1, 2, \dots, m$, there are (short) line segments s_j and s'_j in the interior of L meeting L_j at z_j and z_{j+1} , respectively, at a (small) positive angle and a rectifiable arc γ_j lying entirely in the interior of L and connecting the other two endpoints of s_j and s'_j such that Φ has analytic continuation beyond L_j into the strip-like domain g_j bounded by L_j, s_j, s'_j , and γ_j (see [5] or [6] for figure). Therefore, the arc $\Phi(\gamma_j)$ is in $\mathbb{D}$ while $\Phi(s_j)$ and $\Phi(s'_j)$ are arcs in $\mathbb{D}$ except for their endpoints $\Phi(z_j)$ and $\Phi(z_{j+1})$. More can be said: since reflection preserves angles and since the exterior angle at z_j of opening $\lambda_j\pi$ is mapped onto a angle of opening π at $\Phi(z_j)$, we see that $\Phi(s_j)$ lies in a Stolz angle in $\mathbb{D}$ with corner at $\Phi(z_j)$:

$$\frac{|\Phi(z) - \Phi(z_j)|}{1 - |\Phi(z)|} \leq c_1 \quad \text{for } z \in s_j, \quad j = 1, 2, \dots, m, \quad (2.1)$$

and similarly

$$\frac{|\Phi(z) - \Phi(z_{j+1})|}{1 - |\Phi(z)|} \leq c'_1 \quad \text{for } z \in s'_j, \quad j = 1, 2, \dots, m. \quad (2.2)$$

Furthermore, we have the following inequalities for z on s_j ($j = 1, 2, \dots, m$):

$$|\Phi(z) - \Phi(z_j)| \geq \text{const} \cdot |z - z_j|^{1/\lambda_j}, \quad (2.3)$$

$$|\Phi'(z)| \leq \text{const} \cdot |z - z_j|^{1/\lambda_j - 1}, \quad (2.4)$$

$$|\Phi(z)| \leq 1 - \text{const} \cdot |z - z_j|^{1/\lambda_j}, \quad (2.5)$$

$$\text{dist}(z, L) \geq \text{const} \cdot |z - z_j|. \quad (2.6)$$

The similar inequalities of (2.3)–(2.6) can be written by replacing s_j , z_j , and λ_j with s'_j , z_{j+1} , and λ_{j+1} , respectively. The inequalities (2.3) and (2.4) emerge from Lehman's asymptotic expansions of conformal mappings near an analytic corner [8]. The inequality (2.5) follows from (2.3), because reflection preserves angles. Finally, (2.6) is a simple fact of conformal mapping geometry.

The last auxiliary facts are on the behavior of g near $\mathbb{T}$ obtained by using Lehman's asymptotic expansions of conformal mappings near an analytic corner [8]. Assume that L has a corner at ζ with exterior angle $\lambda\pi$, $0 < \lambda < 2$ and that $\zeta = \Psi(1)$. Let further $z \in G$ be fixed. Similar arguments in [5] apply to the $g(w) = (\Psi'(w))^{1-1/p} / (\Psi(w) - z)$ one can obtain that if $0 < \lambda < 1$, then

$$g(w) \cong \frac{C}{(w-1)^\eta}, \quad \text{as } w \rightarrow 1+, \text{ with } C \neq 0. \quad (2.7)$$

If $1 < \lambda < 2$, then

$$g'(w) \cong \frac{C}{(w-1)^{\eta+1}}, \quad \text{as } w \rightarrow 1+, \text{ with } C \neq 0. \quad (2.8)$$

3. Proofs of Theorems

From now on, c denote positive constants, not necessarily the same at the different places and not depending on n degree of the p -Faber polynomial and the parameter p . We recite that

$$\eta := (1 - \lambda) \left(1 - \frac{1}{p} \right). \quad (3.1)$$

Proof of Theorem 1.1. Here we use the integral representation (1.9) of the p -Faber polynomials $F_{n,p}$ in G . Given a compact subset F of G , we deform the path of integration by replacing each

component L_j of L by some $L'_j := s_j \cup L_j \cup s'_j$ ($j = 1, 2, \dots, m$), as mentioned before, with proper orientation and such that $F \subset L' := \bigcup_{j=1}^m L'_j$. This gives

$$\begin{aligned} |F_{n,p}(z)| &\leq \frac{1}{2\pi} \sum_{j=1}^m \int_{L'_j} \frac{|\Phi(\zeta)|^n |\Phi'(\zeta)|^{1/p}}{|\zeta - z|} |d\zeta| \\ &\leq C(L, F) \sum_{j=1}^m \int_{L'_j} |\Phi(\zeta)|^n |\Phi'(\zeta)|^{1/p} |d\zeta| \\ &= C(L, F) \sum_{j=1}^m \left(\int_{s_j} + \int_{s'_j} + \int_{L_j} \right) |\Phi(\zeta)|^n |\Phi'(\zeta)|^{1/p} |d\zeta|. \end{aligned} \quad (3.2)$$

The contributions of the arcs L_j to these integrals are obviously $O(\rho^n)$ for some $\rho < 1$, so that typical integrals

$$I_j = \int_{s_j} |\Phi(\zeta)|^n |\Phi'(\zeta)|^{1/p} |d\zeta| \quad (3.3)$$

extended over the line segments s_j (and s'_j) remain to be estimated. If s is arc length of s_j measured from z_j , by using (2.4) and (2.5),

$$|\Phi(\zeta)| \leq 1 - cs^{1/\lambda_j} < \exp(-cs^{1/\lambda_j}), \quad |\Phi'(\zeta)| \leq cs^{1/\lambda_j - 1}, \quad (3.4)$$

and, therefore,

$$\begin{aligned} I_j &< c \int_0^\infty \exp(-cns^{1/\lambda_j}) s^{(1/\lambda_j - 1)/p} ds \\ &= \frac{c}{n^{(1-\lambda_j)/p + \lambda_j}} \int_0^\infty e^{-t} t^{(1/p - 1)(1-\lambda_j)} ds. \end{aligned} \quad (3.5)$$

If $1/2 \leq p \leq \infty$, then the integral in the last line is convergent; that is,

$$I_j < \frac{c}{n^{(1-\lambda_j)/p + \lambda_j}}. \quad (3.6)$$

From this we obtain the Theorem 1.1.

In the case $0 < p < 1/2$, similar arguments show that, under the condition $\lambda_j < 1/(1 - p)$, the inequality (3.6) is valid. This proves Remark 1.4. $\square$

Proof of Remark 1.3. We now show that O in Theorem 1.1 cannot be replaced by o . Let L be any piecewise analytic Jordan's curve that has no smooth corners; that is, all $\lambda_j \neq 1$. We assume that one of them is at $\Psi(1)$ with exterior angle λ . Assume that $F_{n,p}(z) = o(1/n^{1-\eta})$ ($n \rightarrow \infty$) for some fixed $z \in G$.

(a) If $0 < \lambda < 1$ and $p > 1$, we consider for $|w| > 1$

$$g(w) = \frac{(\Psi'(w))^{1-1/p}}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_{n,p}(z)}{w^{n+1}}. \quad (3.7)$$

Since

$$\sum_{n=1}^{\infty} \frac{x^n}{n^\alpha} \sim \frac{\Gamma(1-\alpha)}{(1-x)^{1-\alpha}}, \quad \text{when } x \rightarrow 1- \quad (3.8)$$

for $\alpha < 1$, where Γ is Gamma Function (see [9, page 225]), then

$$|g(w)| = o\left(\sum_{n=1}^{\infty} \frac{1}{n^{1-\eta} w^n}\right) = o(1) \frac{1}{(w-1)^\eta}, \quad \text{as } w \rightarrow 1+. \quad (3.9)$$

This contradicts (2.7).

(b) If $1 < \lambda < 2$, then, by using (3.8) and

$$g'(w) = -\sum_{n=0}^{\infty} \frac{(n+1)F_{n,p}(z)}{w^{n+2}}, \quad (3.10)$$

we get

$$|g'(w)| = o\left(\sum_{n=1}^{\infty} \frac{n}{n^{1-\eta} w^n}\right) = o(1) \frac{1}{(w-1)^{\eta+1}}, \quad \text{as } w \rightarrow 1+ \quad (3.11)$$

which is contradiction to (2.8). $\square$

Proof of Theorem 1.5. The proof goes along similar lines as those taken in the proof of Theorem 1.1 with one significant difference: here, z lies in Ω , instead of G , and, thus, can tend to L without need altering the curve L' . As a consequence, the set F defined above does not depend on z , and; thus, $\text{dist}(z, F) > \text{dist}(L, F) = c(L)$. $\square$

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