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Research Article

$\mathfrak{X}$-Gorenstein Projective Modules

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We introduce and study the $\mathfrak{X}$-Gorenstein projective modules, where $\mathfrak{X}$ is a projective class. These modules are a generalization of the Gorenstein projective modules.

1. Introduction

Throughout the paper, all rings are associative with identity, and an R-module will mean right R-module. As usual, we use $\mathrm{pd}_R(M)$ to denote, respectively, the classical projective dimension of M.

An important motivation for the study of homological dimensions dates back to 1956, when Auslander and Buchsbaum [1] and Serre [2] proved the following.

Theorem A. *Let R be a commutative Noetherian local ring R with residue field k. The following conditions are equivalent:*

(1) *R is regular;*

(2) *k has finite projective dimension;*

(3) *every R-module has finite projective dimension.*

This result opened the door to the solution of two long-standing conjectures of Krull. Moreover, it introduced the theme that finiteness of a homological dimension for all modules characterizes rings with special properties. Later work has shown that modules of finite projective dimension over a general ring share many properties with modules over a regular ring. This is an incitement to study homological dimensions of individual modules.

In line with these ideas, Auslander and Bridger [3] introduced in 1969 the G-dimension. It is a homological dimension for finitely generated modules over a Noetherian ring, and it gives a characterization of the Gorenstein local rings [4, Section 3.2]. Namely,

R is Gorenstein if k has finite G-dimension, and only if every finitely generated R-module has finite G-dimension.

In Section 2, we recall the notion of the Gorenstein projective modules, and we put the point on its place in the theory of homological dimensions as a generalization of the classical projective modules. So, we recall some fundamental results about the Gorenstein projective modules and dimensions.

In Section 3, which is the main section of this paper, we show that every time we choose a projective class $\mathfrak{X}$ (Definition 3.1), we can consider a generalization of (Gorenstein) projective modules via $\mathfrak{X}$. In the general case these generalizations are different.

2. The Gorenstein Projective Modules

In the early 1990s, the G-dimension was extended beyond the realm of finitely generated modules over a Noetherian ring. This was done by Enochs and Jenda who introduced the notion of the Gorenstein projective modules [5]. The same authors and their collaborators, studied these modules in several subsequent papers. The associated dimension was studied by Christensen [6] and Holm [7].

Definition 2.1. An R-module M is called *Gorenstein projective* if there exists an exact sequence of projective R-modules

$$\mathbf{P}: \quad \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \tag{2.1}$$

such that $M \cong \operatorname{Im}(P_0 \to P^0)$ and such that the functor $\operatorname{Hom}_R(-, Q)$ leaves $\mathbf{P}$ exact whenever Q is a projective R-module. The resolution $\mathbf{P}$ is called a *complete projective resolution*.

It is evident that every projective module is Gorenstein projective. While, the converse is not true [8, Example 2.5].

Basic categorical properties are recorded in [7, Section 2]. Recall that a class $\mathfrak{X}$ of R-modules is called *projectively resolving* [7] if $\mathcal{P}(R) \subseteq \mathfrak{X}$ and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathfrak{X}$, the conditions $X' \in \mathfrak{X}$ and $X \in \mathfrak{X}$ are equivalent.

Proposition 2.2 (see [7, Theorem 2.5]). *The class of Gorenstein projective R-modules is closed under direct sums and summands.*

In [9], the authors define a particular subclass of the class of the Gorenstein projective modules.

Definition 2.3. A module M is said to be *special Gorenstein projective* if there exists an exact sequence of free modules of the form

$$\mathbf{F}: \quad \cdots \longrightarrow F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \longrightarrow \cdots \tag{2.2}$$

such that $M \cong \operatorname{Im}(f)$ and such that $\operatorname{Hom}_R(-, P)$ leaves the sequence $\mathbf{F}$ exact whenever P is projective. The resolution $\mathbf{F}$ is called a *complete free resolution*.

Every projective module is a direct summand of a free one. A parallel result for the Gorenstein projective modules holds.

Proposition 2.4 (see [9, Corollary 2.4]). *A module M is Gorenstein projective if and only if it is a direct summand of a special Gorenstein projective module.*

An (augmented) Gorenstein projective resolution of a module M is an exact sequence $\cdots \to G_i \to G_{i-1} \to \cdots \to G_0 \to M \to 0$, where each module G_i is Gorenstein projective. Note that every module has a Gorenstein projective resolution, as a free resolution is trivially a Gorenstein projective one.

Definition 2.5. The Gorenstein projective dimension of a module $M \neq 0$, denoted by $\mathrm{Gpd}_R(M)$, is the least integer $n \geq 0$ such that there exists a Gorenstein projective resolution of M with $G_i = 0$ for all $i > n$. If no such n exists, then $\mathrm{Gpd}_R(M)$ is infinite. By convention, set $\mathrm{Gpd}_R(M) = \infty$.

In [7], Holm gave the following fundamental functorial description of the Gorenstein dimension.

Theorem 2.6 (see [7, Theorem 2.22]). *Let M be an R-module of finite Gorenstein projective dimension. For every integer $n \geq 0$, the following conditions are equivalent:*

(1) $\mathrm{Gpd}_R(M) \leq n$;

(2) $\mathrm{Ext}^i_R(M, P) = 0$ *for all* $i > 0$ *and all projective module* P;

(3) $\mathrm{Ext}^i_R(M, Q) = 0$ *for all* $i > 0$ *and all R-modules Q with finite* $\mathrm{pd}_R(Q)$;

(4) *for every exact sequence* $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, *if* $G_0; \dots ; G_{n-1}$ *are Gorenstein projective, then also K_n is Gorenstein projective.*

The Gorenstein projective dimension is a refinement of the projective dimension; this follows from [7, Proposition 2.27].

Proposition 2.7. *For every R-module M,* $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$ *with equality if* $\mathrm{pd}_R(M)$ *is finite.*

3. $\mathfrak{X}$-Gorenstein Projective Modules

Notation. By $\mathcal{P}(R)$ and $\mathcal{GP}(R)$ we denote the classes of all projective and Gorenstein projective R-modules, respectively. Given a class $\mathfrak{X}$ of R-modules, we set:

$$^{\perp\infty}\mathfrak{X} = \left\{ M \mid \mathrm{Ext}^i_R(M, X) = 0 \ \ \forall X \in \mathfrak{X} \text{ and all } i > 0 \right\}. \tag{3.1}$$

We define the projective class as follows.

Definition 3.1. Let R be a ring. A class $\mathfrak{X}$ of R-modules is a *projective class*, if it is projectively resolving and closed under direct sum.

Remark 3.2. For any ring R, any projective class is closed under direct summands (by [7, Proposition 1.4]).

The main purpose of this section, which is the main section of this paper, is to see that every time we chose a projective class $\mathfrak{X}$, we can consider a generalization of (Gorenstein) projective modules via $\mathfrak{X}$. In the general case these generalizations are different.

Definition 3.3. Let R be a ring, and let $\mathfrak{X}$ be a projective class over R. The *$\mathfrak{X}$-projective dimension* of an R-module M, $\mathfrak{X}-\mathrm{pd}_R(M)$, is defined by declaring that $\mathfrak{X}-\mathrm{pd}_R(M)\leq n$ if and only if M has an $\mathfrak{X}$-resolution of length n.

Note that, by the definition of the $\mathfrak{X}$-projective dimension, for each module M we have $\mathfrak{X}-\mathrm{pd}_R(M)\leq \mathrm{pd}_R(M)$. On the other hand, we have the following.

Proposition 3.4. *Let R be a ring and $\mathfrak{X}$ a projective class over R. Then, $\mathfrak{X}-\mathrm{pd}_R(M)\leq n$ if and only if for each exact sequence*

$$0\longrightarrow X_n\longrightarrow X_{n-1}\longrightarrow\cdots\longrightarrow X_0\longrightarrow M\longrightarrow 0, \tag{3.2}$$

where $X_i\in\mathfrak{X}$ for each $i=0,\dots,n-1$, the module X_n belongs in $\mathfrak{X}$.

Proof. The condition "if" is clear. So, we have to prove the "only if" condition. Assume that $\mathfrak{X}-\mathrm{pd}_R(M)\leq n$, and let

$$0\longrightarrow X_n\longrightarrow X_{n-1}\longrightarrow\cdots\longrightarrow X_0\longrightarrow M\longrightarrow 0 \tag{3.3}$$

be an exact sequence, where $X_i\in\mathfrak{X}$ for each $i=0,\dots,n-1$. Since $\mathfrak{X}-\mathrm{pd}_R(M)\leq n$, there exists an exact sequence

$$0\longrightarrow X'_n\longrightarrow X'_{n-1}\longrightarrow\cdots\longrightarrow X'_0\longrightarrow M\longrightarrow 0, \tag{3.4}$$

where $X'_i\in\mathfrak{X}$ for each $i=0,\dots,n$. Since the class $\mathfrak{X}$ is projectively resolving and closed under arbitrary sums and under direct summands, by using [3, Lemma 3.12], $X_n\in\mathfrak{X}$ since $\mathfrak{X}'_n\in\mathfrak{X}$. $\square$

We introduce the $\mathfrak{X}$-Gorenstein projective modules as follows.

Definition 3.5. Let R be a ring, and let $\mathfrak{X}$ be a projective class over R. An R-module M is called *$\mathfrak{X}$-Gorenstein projective* if there exists an exact sequence of projective modules

$$\mathbf{P}:\quad \cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow P^0\longrightarrow P^1\longrightarrow\cdots \tag{3.5}$$

such that $M\cong\mathrm{Im}(P_0\to P^0)$ and such that the functor $\mathrm{Hom}_R(-,X)$ leaves $\mathbf{P}$ exact whenever $X\in\mathfrak{X}$. The complex $\mathbf{P}$ is called an *$\mathfrak{X}$-complete projective resolution*.

Remark 3.6. Let $\mathfrak{X}$ be a projective class over a ring R. Then, we have the following:

(1) every projective module is $\mathfrak{X}$-Gorenstein projective;

(2) every $\mathfrak{X}$-Gorenstein projective module is Gorenstein projective;

(3) the class of all $\mathfrak{X}$-Gorenstein projective module is closed under direct sums by definition.

(4) if $\mathbf{P}: \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ is an $\mathcal{X}$-complete projective resolution then, by symmetry, all the images, all the kernels, and all the cokernels of $\mathbf{P}$ are $\mathcal{X}$-Gorenstein projective;

(5) if $\mathcal{X} = \mathcal{P}(R)$ then, the $\mathcal{X}$-Gorenstein projective modules are just the Gorenstein projective modules.

The next example shows that there exists a Gorenstein projective module which is not $\mathcal{X}$-Gorenstein projective for a given projective class $\mathcal{X}$.

Example 3.7. Consider the local quasi-Frobenius ring $R = k[X]/(X^2)$, where k is a field. Let $(\overline{X})$ be the residue class of X in R. Let $\mathcal{X}$ be any projective class which contain $(\overline{X})$. Then, $(\overline{X})$ is a Gorenstein projective module which is not $\mathcal{X}$-Gorenstein projective.

Proof. First, note that there is always a projective class which contain any module M. A trivial case is $\mathcal{X} = {}_R\mathrm{Mod}$.

Since R is quasi-Frobenius, $(\overline{X})$ is Gorenstein projective [10, Theorem 2.2]. Moreover, the short sequence $0 \to (\overline{X}) \hookrightarrow R \xrightarrow{f} (\overline{X}) \to 0$, where f is the multiplication by $\overline{X}$ is exact. Then, if we suppose that $(\overline{X})$ is $\mathcal{X}$-Gorenstein projective, then $\mathrm{Ext}^1_R((\overline{X}),(\overline{X})) = 0$. Thus, $(\overline{X})$ is a direct summand of R and so projective. Then, $(\overline{X})$ is free since R is local. However, $\overline{X}^2 = 0$. Then, $(\overline{X})$ cannot be free. $\square$

The next result is a direct consequence of the definition of $\mathcal{X}$-Gorenstein projective modules.

Proposition 3.8. *Given a projective class $\mathcal{X}$, an R-module M is $\mathcal{X}$-Gorenstein projective if and only if*

(1) $\mathrm{Ext}^i_R(M, X) = 0$ *for all $X \in \mathcal{X}$ and all $i > 0$ (i.e., $M \in {}^{\perp\infty}\mathcal{X}$) and*

(2) *there exists an exact sequence $0 \to M \to P^0 \to P^1 \to \cdots$, where all P^i are projectives and* $\mathrm{Hom}_R(-, X)$ *leaves this sequence exact whenever $X \in \mathcal{X}$.*

The next result shows that an $\mathcal{X}$-projective module with finite $\mathcal{X}-\mathrm{pd}_R(-)$ is projective.

Proposition 3.9. *Let $\mathcal{X}$ be a projective class over R, and let M be an $\mathcal{X}$-Gorenstein projective module. Then,*

(1) $\mathrm{Ext}^i_R(M, G) = 0$ *for all module G with $\mathcal{X}-\mathrm{pd}_R(G) < \infty$ and all $i > 0$.*

(2) *either M is projective or $\mathcal{X}-\mathrm{pd}_R(M) = \infty$.*

Proof. (1) Since M is $\mathcal{X}$-Gorenstein projective, $\mathrm{Ext}^i_R(M, X) = 0$ for all $X \in \mathcal{X}$ and all $i > 0$. Thus, by dimension shifting,we obtain the desired result.

(2) Suppose that $\mathcal{X}-\mathrm{pd}_R(M) < \infty$ and consider a short exact sequence $0 \to K \to P \to M \to 0$ where P is projective. It is clear that $\mathcal{X}-\mathrm{pd}_R(K) < \infty$. Then, by (1), $\mathrm{Ext}^1_R(M, K) = 0$. Hence, this short exact sequence splits, and then M is a direct summand of P. Hence, it is projective. $\square$

Proposition 3.10. *Let R be a ring. If $\mathcal{X} = \mathcal{GP}(R)$, then an R-module M is $\mathcal{X}$-Gorenstein projective if and only if it is projective.*

Proof. First note that $\mathcal{X} = \mathcal{GP}(R)$ is a projective class [7, Theorem 2.5] and it clear that every projective module is $\mathcal{X}$-Gorenstein projective. Now, suppose that M is an $\mathcal{X}$-Gorenstein projective module. It is trivial that M is also a Gorenstein projective module. Now, consider an exact sequence $0 \to M' \to P \to M \to 0$, where P is projective. Since $\mathcal{GP}(R)$ is resolving, M' is also Gorenstein projective. Then, $M' \in \mathcal{X}$. Thus, $\mathrm{Ext}^1_R(M, M') = 0$. So, the short exact sequence splits and so, M is a direct summand of P and therefore projective.

The converse implication is immediate. □

Next we set out to investigate how $\mathcal{X}$-Gorenstein projective modules behave in short exact sequences.

Theorem 3.11. *Let $\mathcal{X}$ be a projective class over a ring R. The class of all $\mathcal{X}$-Gorenstein projective modules is projectively resolving. Furthermore, it is closed under arbitrary direct sums and under direct summands.*

Proof. It is clear that every projective module is $\mathcal{X}$-Gorenstein projective. So, consider any short exact sequence of R-modules $0 \to M' \to M \to M'' \to 0$, where M'' is $\mathcal{X}$-Gorenstein projective.

First suppose that M' is $\mathcal{X}$-Gorenstein projective. We claim that M is also $\mathcal{X}$-Gorenstein projective. Since ${}^{\perp\infty}\mathcal{X}$ is projectively resolving (by [11, Lemma 2.2.9]) and by Proposition 3.8, we get that M belongs to ${}^{\perp\infty}\mathcal{X}$. Thus, to show that M is $\mathcal{X}$-Gorenstein projective, we only have to prove the existence of an exact sequence $0 \to M \to P^0 \to P^1 \to \cdots$, where all P^i are projectives and $\mathrm{Hom}_R(-, X)$ leaves this sequence exact whenever $X \in \mathcal{X}$ (by Proposition 3.8). By assumption, there exist exact projective resolutions

$$\mathbf{M}' = 0 \longrightarrow M' \longrightarrow P_0' \to P_1' \longrightarrow \cdots, \qquad \mathbf{M}'' = 0 \longrightarrow M'' \longrightarrow P_0'' \longrightarrow P_1'' \longrightarrow \cdots, \tag{3.6}$$

where $\mathrm{Hom}(-, X)$ keeps the exactness of these sequences whenever $X \in \mathcal{X}$ and all the cokernels of $\mathbf{M}'$ and $\mathbf{M}''\mathcal{X}$-Gorenstein projectives (such a sequences exists by the definition of $\mathcal{X}$-Gorenstein projective modules). Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, "f'"] & M \arrow[r, "\beta"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & P_0' \arrow[r] & P_0' \oplus P''_0 \arrow[r] & P''_0 \arrow[r] & 0
\end{tikzcd} \tag{3.7}$$

Since M'' is $\mathcal{X}$-Gorenstein projective, we have $\mathrm{Ext}^1_R(M'', P_0') = 0$. Hence, the following sequence is exact.

$$0 \longrightarrow \mathrm{Hom}_R(M'', P_0') \xrightarrow{\mathrm{Hom}_R(\beta, P_0')} \mathrm{Hom}_R(M, P_0') \xrightarrow{\mathrm{Hom}_R(\alpha, P_0')} \mathrm{Hom}_R(M', P_0') \longrightarrow 0. \tag{3.8}$$

Thus, there exists an R-morphism $\gamma : M \to P_0'$ such that $f' = \gamma \circ \alpha$.

It is easy to check that the morphism $f : M \to P'_0 \oplus P''_0$ defined by setting $f(m) = (\gamma(m), f'' \circ \beta(m))$ for each $m \in M$ completes the above diagram and makes it commutative. Then, using the snake lemma, we get the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, "f'"] & M \arrow[r, "\beta"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & P'_0 \arrow[r] \arrow[d] & P'_0 \oplus P''_0 \arrow[r] \arrow[d] & P''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \text{Coker }(f') \arrow[r] \arrow[d] & \text{Coker }(f) \arrow[r] \arrow[d] & \text{Coker }(f'') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{3.9}$$

Since coker (f') and coker (f'') are $\mathfrak{X}$-Gorenstein projectives, they belong to ${}^{\perp\infty}\mathfrak{X}$ which is projectively resolving (by [11, Lemma 2.2.9]). Then, coker (f) belongs also to ${}^{\perp\infty}\mathfrak{X}$. Accordingly, $\mathrm{Hom}_R(-, X)$ keeps the exactness of the short exact sequence $0 \to M \to P'_0 \oplus P''_0 \to \text{coker }(f) \to 0$ whenever $X \in \mathfrak{X}$. By induction, we can construct a commutative diagram with the form:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{M'} \arrow[r] \arrow[d, equal] & \mathbf{M} \arrow[r] \arrow[d, equal] & \mathbf{M''} \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_0 \arrow[r] \arrow[d] & P'_0 \oplus P''_0 \arrow[r] \arrow[d] & P''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_1 \arrow[r] \arrow[d] & P'_1 \oplus P''_1 \arrow[r] \arrow[d] & P''_1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{3.10}$$

such that $\mathrm{Hom}_R(-, X)$ leaves $\mathbf{M}$ exact whenever $X \in \mathfrak{X}$. Consequently, M is $\mathfrak{X}$-Gorenstein projective.

Now suppose that M is $\mathfrak{X}$-Gorenstein projective, and we claim that M' is $\mathfrak{X}$-Gorenstein projective. As above M belongs to ${}^{\perp\infty}\mathfrak{X}$. Hence,we have to prove that M

satisfies condition (2) of Proposition 3.8. To do it, pick a short exact sequence $0 \to M \to P \to X \to 0$ where P is projective and X is $\mathfrak{X}$-Gorenstein projective (such a sequence exists by Remark 3.6(4)), and consider the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M' \arrow[r, equal] \arrow[d] & M' \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, dotted] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M'' \arrow[r, dotted] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{3.11}$$

The first part of this proof, applying to the short exact sequence $0 \to M'' \to Y \to X \to 0$, shows that Y is $\mathfrak{X}$-Gorenstein projective. Hence, it admits a right projective resolution $\mathbf{Y}$:$0 \to Y \to F^0 \to F^1 \to \cdots$ which remains exact by $\operatorname{Hom}_R(-, X)$ whenever $X \in \mathfrak{X}$. In addition, the short exact sequence $0 \to M' \to P \to Y \to 0$ remains exact by $\operatorname{Hom}_R(-, X)$ whenever $X \in \mathfrak{X}$ since Y is $\mathfrak{X}$-Gorenstein projective. Finally, it is easy to check that:

$$\mathbf{M}'\colon\quad \begin{tikzcd}[column sep=small]
0 \arrow[r] & M' \arrow[r] & P \arrow[rr] \arrow[dr] & & F^0 \arrow[r] & F^1 \arrow[r] & \cdots \\
 & & & Y \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{3.12}$$

is also exact by $\operatorname{Hom}_R(-, X)$ whenever $X \in \mathfrak{X}$.

The closing of the class of $\mathfrak{X}$-Gorenstein projective modules under direct sums is clear by the definition of these modules, while its closing under direct summands is deduced from [7, Proposition 1.4]. □

Corollary 3.12. *Let $\mathfrak{X}$ be a projective class over a ring R. Let $0 \to G' \to G \to M \to 0$ be an exact sequence where G' and G are $\mathfrak{X}$-Gorenstein projective modules and where $\operatorname{Ext}^1_R(M, P) = 0$ for all projective modules P. Then, M is $\mathfrak{X}$-Gorenstein projective.*

Proof. Pick a short exact sequence $0 \to G' \to P \to G'' \to 0$ where P is projective and G'' is $\mathfrak{X}$-Gorenstein projective. Consider the following push-out diagram.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, dotted] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & G'' \arrow[r, equal] \arrow[d] & G'' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
\tag{3.13}
$$

□

Using Theorem 3.11, X is $\mathfrak{X}$-Gorenstein projective. On the other hand, since $\mathrm{Ext}^1_R(M, P) = 0$, the short exact sequence $0 \to P \to X \to M \to 0$ splits. Thus, M is a direct summand of X. Consequently, M is $\mathfrak{X}$-Gorenstein projective, as a direct summand of an X-Gorenstein projective module (by Theorem 3.11).

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References

[1] M. Auslander and D. A. Buchsbaum, "Homological dimension in Noetherian rings," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 42, pp. 36–38, 1956.
[2] J. P. Serre, "Sur la dimension homologique des anneaux et des modules noethériens," in *Proceedings of the International Symposium on Algebraic Number Theory*, pp. 175–189, Science Council of Japan, Tokyo, Japan, 1956.
[3] M. Auslander and M. Bridger, *Stable Module Theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, RI, USA, 1969.
[4] M. Auslander, *Anneaux de Gorenstein, et Torsion en Algèbre Commutative*, Séeminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des expos és de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro. Ecole Normale Supérieure de Jeunes Filles, Secrétariat Mathématique, Paris, France, 1976, http://www.numdam.org/.
[5] E. E. Enochs and O. M. G. Jenda, "Gorenstein injective and projective modules," *Mathematische Zeitschrift*, vol. 220, no. 4, pp. 611–633, 1995.
[6] L. W. Christensen, *Gorenstein Dimensions*, vol. 1747 of *Lecture Notes in Mathematics*, Springer, Berlin, 2000.
[7] H. Holm, "Gorenstein homological dimensions," *Journal of Pure and Applied Algebra*, vol. 189, no. 1–3, pp. 167–193, 2004.
[8] D. Bennis and N. Mahdou, "Strongly Gorenstein projective, injective, and flat modules," *Journal of Pure and Applied Algebra*, vol. 210, no. 2, pp. 437–445, 2007.
[9] M. Tamekkante, "Special gorenstein projective modules via a matrix approach," *Arabian Journal for Science and Engineering*, vol. 35, no. 2, pp. 241–250, 2010.

[10] D. Bennis, N. Mahdou, and K. Ouarghi, “Rings over which all modules are strongly Gorenstein projective,” *The Rocky Mountain Journal of Mathematics*, vol. 40, no. 3, pp. 749–759, 2010.
[11] R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, vol. 41 of *De Grutyter Expositions in Mathematics*, de Gruyter, Berlin, Germany, 2006.


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