Research Article

Some New Identities on the $q$-Genocchi Numbers and Polynomials with Weight $\alpha$

Seog-Hoon Rim and Joohee Jeong

Department of Mathematics Education, Kyungpook National University, Daegu 702-701, Republic of Korea

Correspondence should be addressed to Joohee Jeong, jhjeong@knu.ac.kr

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We construct a new type of $q$-Genocchi numbers and polynomials with weight $\alpha$. From these $q$-Genocchi numbers and polynomials with weight $\alpha$, we establish some interesting identities and relations.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively denote the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a $p$-adic number $q$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. As a definition of $q$-numbers, we use the notation of $q$-number of

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (q)^x}{1 + q} \quad (1.1)$$

(cf. [1–11]). Note that $\lim_{q \to 1} [x]_q = x$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim [1, 3],

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x. \quad (1.2)$$
From (1.2), we have the well-known integral equation

\[
q^n L_1(f_n) + (-1)^{n-1} I_q(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l)
\]

(1.3)

(see [1, 3]), where \( f_n(x) = f(x + n), (n \in \mathbb{N}). \)

For \( \alpha \in \mathbb{N}, \) in [11], the \( q \)-Genocchi polynomials with weight \( \alpha \) are introduced by

\[
t \int_{\mathbb{Z}_p} e^{[x+y]_{q^\alpha}} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \tilde{G}_{n,q}^{(\alpha)}(x) \frac{t^n}{n!}.
\]

(1.4)

By comparing the coefficients of both sides of (1.4), we have

\[
\tilde{G}_{0,q}^{(\alpha)}(x) = 0,
\]

\[
\frac{\tilde{G}_{n+1,q}^{(\alpha)}(x)}{(n + 1)} = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_{-q}(y), \quad \text{for } n \in \mathbb{N}.
\]

(1.5)

In the special case, \( x = 0, \tilde{G}_{n,q}^{(\alpha)}(0) = \tilde{G}_{n,q}^{(\alpha)} \) are called the \( n \)th \( q \)-Genocchi numbers with weight \( \alpha. \)

2. \( q \)-Genocchi Numbers and Polynomials with Weight \( \alpha \)

In this section, we show some new identities on the \( q \)-Genocchi numbers and polynomials with weight \( \alpha. \) And we establish the distribution relation for \( q \)-Genocchi polynomials with weight \( \alpha. \)

From (1.5), we can easily see that

\[
\frac{\tilde{G}_{n+1,q}^{(\alpha)}(x)}{n + 1} = \frac{[2]_q}{[\alpha]_q (1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{alx} \frac{q_1}{1 + q_1l+1}.
\]

(2.1)

From (1.5) and (2.1), we note that

\[
\frac{\tilde{G}_{n+1,q}^{(\alpha)}(x)}{n + 1} = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_{-q}(y)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^l} q^{alx} \int_{\mathbb{Z}_p} [y]_q^l d\mu_{-q}(y)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^l} q^{alx} \tilde{G}_{n+1,q}^{(\alpha)}(x) \frac{q_1}{1 + q_1l+1}
\]

(2.2)
Theorem 2.1. For $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has

$$q^{ax}\tilde{G}_{n+1,q}^{(a)}(x) = (n + 1)q^{ax}(x)_q^{n+1}$$

$$= \sum_{l=0}^{n+1} \binom{n+1}{l} (x)_q^n q^{alx} \tilde{G}_{l+1,q}^{(a)}$$

with the usual convention of replacing $(\tilde{G}_q^{(a)})^n$ by $(\tilde{G}_q^{(a)})$.

Thus, by (2.3), we have a theorem.

Theorem 2.1. For $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has

$$q^{ax}\tilde{G}_{n+1,q}^{(a)}(x) = (n + 1)q^{ax}(x)_q^{n+1}$$

In (1.3), if we take $n = 1$,

$$qI_{-1}(f_1) + I_{-1}(f) = [2]_q.$$  

We apply $f(x) = e^{[x]_q}a^x$ with (1.5), and we have the following:

$$[2]_q = \sum_{n=0}^{\infty} \left( q \int_{\mathbb{Z}_q} [x + 1]_q^n d\mu_{-q}(x) + \int_{\mathbb{Z}_q} [x]_q^n d\mu_{-q}(x) \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left( q^{\tilde{G}_{n+1,q}^{(a)}(1)} + \frac{\tilde{G}_{n+1,q}^{(a)}}{n+1} \right) \frac{t^n}{n!}.$$  

By comparing the coefficients on both the sides in (2.6), we get

$$q^{\tilde{G}_{n+1,q}^{(a)}(1)} + \frac{\tilde{G}_{n+1,q}^{(a)}}{n+1} = \begin{cases} [2]_q & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$
From (2.2) and (2.7), we can derive the following:

\[
\tilde{G}_{1,q}^{(\alpha)}(1) = 1, \\
q^{1-a} \left( q^n\tilde{G}_{q}^{(\alpha)} + 1 \right)^n + \tilde{G}_{n,q}^{(\alpha)} = 0 \quad \text{if } n \in \mathbb{N},
\]

(2.8)

with the usual convention of replacing \((\tilde{G}_{q}^{(\alpha)})^n\) by \(\tilde{G}_{n,q}^{(\alpha)}\).

For a fixed odd positive integer \(d\) with \((p,d) = 1\), we set

\[
X = X_d = \lim_{\longrightarrow} \frac{\mathbb{Z}}{dp^N \mathbb{Z}}, \quad X_1 = \mathbb{Z}_p, \\
X^* = \bigcup_{0 < a < dp, (a,p) = 1} (a + dp \mathbb{Z}_p),
\]

(2.9)

\[
a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},
\]

where \(a \in \mathbb{Z}\) satisfies the condition \(0 \leq a < dp^N\). For the distribution relation for the \(q\)-Genocchi polynomials with weight \(\alpha\), we consider the following:

\[
\int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(y) = \int_{\mathbb{Z}_p} [n + y]_q^n d\mu_q(y) \\
= \frac{[d]^n}{[d]_q} \sum_{a=0}^{d-1} (-1)^a q^a \int_{\mathbb{Z}_p} \left[ \frac{x + a}{d} + y \right]_q^n d\mu_q(y).
\]

(2.10)

By (1.5) and (2.10), we get a theorem.

**Theorem 2.2.** For \(\alpha \in \mathbb{N}\) and \(n \in \mathbb{Z}_+\), \(d \in \mathbb{N}\) with \(d \equiv 1(\mod 2)\), one has

\[
\tilde{G}_{n+1,q}^{(\alpha)}(\chi) = \frac{[d]^n}{[d]_q} \sum_{a=0}^{d-1} (-1)^a q^a \tilde{G}_{n+1,q}^{(\alpha)} \left( \frac{x + a}{a} \right). 
\]

(2.11)

### 3. Higher-Order \(q\)-Genocchi Numbers and Polynomials with Weight \(\alpha\)

In this section, we define higher-order \(q\)-Genocchi polynomials \(\tilde{G}_{n+1,q}^{(\alpha)}(h,k \mid x)\) and numbers \(\tilde{G}_{n+1,q}^{(\alpha)}(h,k)\) with weight \(\alpha\). We find an integral equation for higher-order \(q\)-Genocchi numbers with weight \(\alpha\). And we establish a combination property.
Let $\alpha \in \mathbb{Z}$ and $h, k \in \mathbb{Z}_+$, for $n \in \mathbb{Z}_+$, then we define higher-order $q$-Genocchi polynomials with weight $\alpha$ as follows:

$$
\tilde{G}_{n+1,q}(h, k | x) = \left[ \sum_{j=0}^{k} \left( \frac{2}{[\alpha]_q^n (1-q^n)_{[n]}} \sum_{l=0}^{\infty} \frac{(n)_l (-1)^{l \cdot n} q^{n \cdot l}}{(1 + q^{l+1}) (1 + q^{l+1})} \right) \right]
$$

$$
= \left[ \frac{2}{[\alpha]_q^n (1-q^n)} \sum_{l=0}^{\infty} \frac{(n)_l (-1)^{l \cdot n} q^{n \cdot l}}{(1 + q^{l+1}) (1 + q^{l+1})} \right]
$$

$$
= \left[ \frac{2}{[\alpha]_q^n (1-q^n)} \sum_{l=0}^{\infty} \frac{(n)_l (-1)^{l \cdot n} q^{n \cdot l}}{(1 + q^{l+1}) (1 + q^{l+1})} \right]
$$

where $(x: q)_n = \prod_{j=0}^{n-1} (1 - x q^j)$.

In the special case, $x = 0$, $\tilde{G}_{n+1,q}(h, k | 0) = \tilde{G}_{n+1,q}(h, k)$ are called the $(n+1)$th $(h, k)$-Genocchi numbers with weight $\alpha$.

In (3.1), apply the following identity:

$$
[x_1 + x_2 + \cdots + x_k]_q (1 - q^n) + q^n (x_1 + x_2 + \cdots + x_k) = 1,
$$

and we have a theorem.

**Theorem 3.1.** For $\alpha \in \mathbb{N}$ and $h, k \in \mathbb{Z}_+$, one has

$$
\frac{\tilde{G}_{n+1,q}(h, k)}{n+1} = \frac{\tilde{G}_{n+2,q}(h, k)}{n+2} + \frac{\tilde{G}_{n+1,q}(h + \alpha, k)}{n+1}.
$$

We consider, for $\alpha \in \mathbb{N}$ and $h, k \in \mathbb{Z}_+$,

$$
\sum_{j=0}^{i} \binom{i}{j} (q^a - 1)^j \frac{\tilde{G}_{n+1,q}(h, k)}{n+j-i+1}
$$

$$
= \sum_{j=0}^{i} \binom{i}{j} (q^a - 1)^j \left[ \sum_{\ell=1}^{k} \left[ \sum_{\ell=1}^{k} x_{\ell} \right]^{n+j-i} q^{\sum_{\ell=1}^{k} (h-a+c) x_{\ell}} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \right]
$$

$$
= \sum_{j=0}^{i} \binom{i}{j} (q^a - 1)^j \left[ \sum_{\ell=1}^{k} \left[ \sum_{\ell=1}^{k} x_{\ell} \right]^{n+j-i} q^{\sum_{\ell=1}^{k} (h-c) x_{\ell}} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \right]
$$

$$
= \sum_{j=0}^{i} \binom{i}{j} (q^a - 1)^j \frac{\tilde{G}_{n+1,q}(h, k)}{n+j-i+1}.
$$


Therefore, we obtain the following combinatorial property.

**Theorem 3.2.** For $\alpha \in \mathbb{N}$ and $h, k \in \mathbb{Z}_+$, one has

\[
\sum_{j=0}^{i-1} \binom{i}{j} (q^a - 1)^j \frac{\tilde{C}^{(a)}_{n+j-i+1}(h - \alpha, k)}{n + j - i + 1} = \sum_{j=0}^{i-1} \binom{i-1}{j} (q^a - 1)^j \frac{\tilde{C}^{(a)}_{n+j-i+1}(h, k)}{n + j - i + 1}.
\] (3.5)

**References**


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