Research Article

A Proof of Constructive Version of Brouwer's Fixed Point Theorem with Uniform Sequential Continuity

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It is often said that Brouwer's fixed point theorem cannot be constructively proved. On the other hand, Sperner's lemma, which is used to prove Brouwer's theorem, can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma. They, however, assume uniform continuity of functions. We consider uniform sequential continuity of functions. In classical mathematics, uniform continuity and uniform sequential continuity are equivalent. In constructive mathematics a la Bishop, however, uniform sequential continuity is weaker than uniform continuity. We will prove a constructive version of Brouwer's fixed point theorem in an n -dimensional simplex for uniformly sequentially continuous functions. We follow the Bishop style constructive mathematics.

1. Introduction

It is often said that Brouwer's fixed point theorem cannot be constructively proved.

Reference [1] provided a constructive proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics a la Bishop. It is sufficient to say that one-dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is nonconstructive. See [2] or [3]. Brouwer's fixed point theorem can be constructively, in the sense of constructive mathematics a la Bishop, proved only approximately. The existence of an exact fixed point of a function which satisfies some property of local non-constancy may be constructively proved.

On the other hand, Sperner's lemma, which is used to prove Brouwer's theorem, can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma. See [3, 4]. They, however, assume uniform continuity of functions. We consider uniform sequential continuity of

functions according to [5]. In classical mathematics uniform continuity and uniform sequential continuity are equivalent. In constructive mathematics a la Bishop, however, uniform sequential continuity is weaker than uniform continuity. Also in constructive mathematics, sequential continuity is weaker than continuity, and uniform continuity (resp., uniform sequential continuity) is stronger than continuity (resp., sequential continuity) even in a compact space. See, for example, [6]. As stated in [7], all proofs of the equivalence between continuity and sequential continuity involve the law of excluded middle, and so they are nonconstructive. We will prove a constructive version of Brouwer's fixed point theorem in an n -dimensional simplex for uniformly sequentially continuous functions.

In the next section, we consider Sperner's lemma. In Section 3, we prove a constructive version of Brouwer's fixed point theorem for uniformly sequentially continuous functions from an n -dimensional simplex to itself using Sperner's lemma. We follow the Bishop style constructive mathematics according to [2, 8, 9].

2. Sperner's Lemma

Let Δ denote an n -dimensional simplex. n is a finite natural number. For example, a 2-dimensional simplex is a triangle. Let partition or triangulate the simplex. Figure 1 is an example of partition (triangulation) of a 2-dimensional simplex. In a 2-dimensional case, we divide each side of Δ in m equal segments and draw the lines parallel to the sides of Δ . m is a finite natural number. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ inductively for cases of higher dimension. In a 3-dimensional case, each face of Δ is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the above-mentioned way, and draw the planes parallel to the faces of Δ . Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids and similarly for cases of higher dimension.

Let K denote the set of small n -dimensional simplices of Δ constructed by partition. Vertices of these small simplices of K are labeled with the numbers $0, 1, 2, \dots, n$ subject to the following rules.

- (1) The vertices of Δ are, respectively, labeled with 0 to n . We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1, \dots, 0)$ with 2, $\dots$, and a point $(0, \dots, 0, 1)$ with n . That is, a vertex whose k th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k .
- (2) If a vertex of K is contained in an $n - 1$ -dimensional face of Δ , then this vertex is labeled with some number which is the same as the number of one of the vertices of that face.
- (3) If a vertex of K is contained in an $n - 2$ -dimensional face of Δ , then this vertex is labeled with some number which is the same as the number of one of the vertices of that face and so on for cases of lower dimension.
- (4) A vertex contained inside of Δ is labeled with an arbitrary number among $0, 1, \dots, n$.

A small simplex of K which is labeled with the numbers $0, 1, \dots, n$ is called a *fully labeled simplex*. Sperner's lemma is stated as follows.

Figure 1: Partition and labeling of 2-dimensional simplex.

Lemma 2.1 (Sperner's lemma). *If one labels the vertices of K following the rules (1)~(4), then there are an odd number of fully labeled simplices, and so there exists at least one fully labeled simplex.*

Proof. About constructive proofs of Sperner's lemma see [10] or [11]. $\square$

Since n and partition of Δ are finite, the number of small simplices constructed by partition is also finite. Thus, we can constructively find a fully labeled n -dimensional simplex of K through finite steps.

3. Constructive Version of Brouwer's Fixed Point Theorem with Uniform Sequential Continuity

Let us consider a function f from an n -dimensional simplex Δ to itself. Denote a point in Δ by $\mathbf{p}$. Uniform continuity, sequential continuity, and uniform sequential continuity of functions are defined as follows.

Definition 3.1 (Uniform continuity). A function f is uniformly continuous in Δ if, for any $\mathbf{p}, \mathbf{p}' \in \Delta$ and $\varepsilon > 0$ there, exists $\delta > 0$ such that

$$\text{If } |\mathbf{p} - \mathbf{p}'| < \delta, \quad \text{then } |f(\mathbf{p}) - f(\mathbf{p}')| < \varepsilon. \quad (3.1)$$

δ depends on only ε .

Definition 3.2 (Sequential continuity). A function f is sequentially continuous at $\mathbf{p} \in \Delta$ in Δ if, for sequences $(\mathbf{p}_n)_{n \geq 1}$ and $(f(\mathbf{p}_n))_{n \geq 1}$ in Δ ,

$$f(\mathbf{p}_n) \longrightarrow f(\mathbf{p}) \quad \text{whenever } \mathbf{p}_n \longrightarrow \mathbf{p}. \quad (3.2)$$

Definition 3.3 (Uniform sequential continuity). A function f is uniformly sequentially continuous in Δ if, for sequences $(\mathbf{p}_n)_{n \geq 1}$, $(\mathbf{p}'_n)_{n \geq 1}$, $(f(\mathbf{p}_n))_{n \geq 1}$, and $(f(\mathbf{p}'_n))_{n \geq 1}$ in Δ ,

$$|f(\mathbf{p}_n) - f(\mathbf{p}'_n)| \rightarrow 0 \quad \text{whenever} \quad |\mathbf{p}_n - \mathbf{p}'_n| \rightarrow 0. \quad (3.3)$$

$|\mathbf{p}_n - \mathbf{p}'_n| \rightarrow 0$ means

$$\forall \varepsilon > 0 \exists N \forall n \geq N \quad (|\mathbf{p}_n - \mathbf{p}'_n| < \varepsilon), \quad (3.4)$$

where ε is a real number and n and N are natural numbers. Similarly, $|f(\mathbf{p}_n) - f(\mathbf{p}'_n)| \rightarrow 0$ means

$$\forall \varepsilon > 0 \exists N' \forall n \geq N' \quad (|f(\mathbf{p}_n) - f(\mathbf{p}'_n)| < \varepsilon). \quad (3.5)$$

N' is a natural number.

In classical mathematics, uniform continuity and uniform sequential continuity of functions are equivalent. But in constructive mathematics a la Bishop, uniform sequential continuity is weaker than uniform continuity and uniform sequential continuity is stronger than sequential continuity.

An approximate fixed point of f is defined as follows.

Definition 3.4 (Approximate fixed point). For each $\varepsilon > 0$, $\mathbf{p}^*$ is an approximate fixed point of f if we have

$$|\mathbf{p}^* - f(\mathbf{p}^*)| < \varepsilon. \quad (3.6)$$

Now, we show the following theorem.

Theorem 3.5 (Constructive version of Brouwer's fixed point theorem with uniform sequential continuity). *Any uniformly sequentially continuous function from an n -dimensional simplex Δ to itself has an approximate fixed point for each $\varepsilon > 0$.*

Proof. (1) First, we show that we can partition Δ so that the conditions for Sperner's lemma are satisfied. We partition Δ according to the method in Sperner's lemma, and label the vertices of simplices constructed by partition of Δ . It is important how to label the vertices contained in the faces of Δ . Let K be the set of small simplices constructed by partition of Δ , let $\mathbf{p} = (p_0, p_1, \dots, p_n)$ be a vertex of a simplex of K , and denote the i th component of $f(\mathbf{p})$ by f_i . Then, we label a vertex $\mathbf{p}$ according to the following rule:

$$\text{If } p_k > f_k \quad \text{or} \quad p_k + \tau > f_k, \quad \text{then we label } \mathbf{p} \text{ with } k, \quad (3.7)$$

where τ is a positive number. If there are multiple k 's which satisfy this condition, then we label $\mathbf{p}$ conveniently for the conditions for Sperner's lemma to be satisfied. We do not randomly label the vertices.

For example, let $\mathbf{p}$ be a point contained in an $n - 1$ -dimensional face of Δ such that $p_i = 0$ for one i among $0, 1, 2, \dots, n$ (its i th coordinate is 0). With $\tau > 0$, we have $f_i > 0$ or $f_i < \tau$.

In constructive mathematics, for any real number x , we cannot prove that $x \geq 0$ or $x < 0$, that $x > 0$ or $x = 0$ or $x < 0$. But for any distinct real numbers x , y , and z such that $x > z$, we can prove that $x > y$ or $y > z$.

When $f_i > 0$, from $\sum_{j=0}^n p_j = 1$, $\sum_{j=0}^n f_j = 1$, and $p_i = 0$,

$$\sum_{j=0, j \neq i}^n p_j > \sum_{j=0, j \neq i}^n f_j. \quad (3.8)$$

Then, for at least one j (denote it by k), we have $p_k > f_k$ and we label $\mathbf{p}$ with k , where k is one of the numbers which satisfy $p_k > f_k$. Since $f_i > p_i = 0$, i does not satisfy this condition. Assume that $f_i < \tau \cdot p_i = 0$ implies $\sum_{j=0, j \neq i}^n p_j = 1$. Since $\sum_{j=0, j \neq i}^n f_j \leq 1$, we obtain

$$\sum_{j=0, j \neq i}^n p_j \geq \sum_{j=0, j \neq i}^n f_j. \quad (3.9)$$

Then, for a positive number τ , we have

$$\sum_{j=0, j \neq i}^n (p_j + \tau) > \sum_{j=0, j \neq i}^n f_j. \quad (3.10)$$

There is at least one $j (\neq i)$ which satisfies $p_j + \tau > f_j$. Denote it by k , and we label $\mathbf{p}$ with k . k is one of the numbers other than i such that $p_k + \tau > f_k$ is satisfied. i itself satisfies this condition ($p_i + \tau > f_i$). But, since there is a number other than i which satisfies this condition, we can select a number other than i . We have proved that we can label the vertices contained in an $n - 1$ -dimensional face of Δ such that $p_i = 0$ for one i among $0, 1, 2, \dots, n$ with the numbers other than i . By similar procedures, we can show that we can label the vertices contained in an $n - 2$ -dimensional face of Δ such that $p_i = 0$ for two i 's among $0, 1, 2, \dots, n$ with the numbers other than those i 's, and so on.

Consider the case where $p_i = p_{i+1} = 0$. We see that, when $f_i > 0$ or $f_{i+1} > 0$,

$$\sum_{j=0, j \neq i, i+1}^n p_j > \sum_{j=0, j \neq i, i+1}^n f_j, \quad (3.11)$$

and so for at least one j (denote it by k), we have $p_k > f_k$, and we label $\mathbf{p}$ with k . On the other hand, when $f_i < \tau$ and $f_{i+1} < \tau$, we have

$$\sum_{j=0, j \neq i, i+1}^n p_j \geq \sum_{j=0, j \neq i, i+1}^n f_j. \quad (3.12)$$

Then, for a positive number τ , we have

$$\sum_{j=0, j \neq i, i+1}^n (p_j + \tau) > \sum_{j=0, j \neq i, i+1}^n f_j. \quad (3.13)$$

Thus, there is at least one $j (\neq i, i+1)$ which satisfies $p_j + \tau > f_j$. Denote it by k , and we label $\mathbf{p}$ with k .

Next, consider the case where $p_i = 0$ for all i other than n . If, for some i , $f_i > 0$, then we have $p_n > f_n$ and label $\mathbf{p}$ with n . On the other hand, if $f_j < \tau$ for all $j \neq n$, then we obtain $p_n \geq f_n$. It implies $p_n + \tau > f_n$. Thus, we can label $\mathbf{p}$ with n .

Therefore, the conditions for Sperner's lemma are satisfied and there exists an odd number of fully labeled simplices in K .

(2) Consider a sequence $(\Delta_m)_{m \geq 1}$ of partitions of Δ and a sequence of fully labeled simplices $(\delta_m)_{m \geq 1}$. The larger m , the finer partition. The larger m , the smaller the diameter of a fully labeled simplex. Let $\mathbf{p}_m^0, \mathbf{p}_m^1, \dots, \mathbf{p}_m^n$ be the vertices of a fully labeled simplex δ_m . We name these vertices so that $\mathbf{p}_m^0, \mathbf{p}_m^1, \dots, \mathbf{p}_m^n$ are labeled, respectively, with $0, 1, \dots, n$. The values of f at these vertices are $f(\mathbf{p}_m^0), f(\mathbf{p}_m^1), \dots$ and $f(\mathbf{p}_m^n)$. We can consider sequences of vertices of fully labeled simplices. Denote them by $(\mathbf{p}_m^0)_{m \geq 1}, (\mathbf{p}_m^1)_{m \geq 1}, \dots$, and $(\mathbf{p}_m^n)_{m \geq 1}$. And consider sequences of the values of f at vertices of fully labeled simplices. Denote them by $(f(\mathbf{p}_m^0))_{m \geq 1}, (f(\mathbf{p}_m^1))_{m \geq 1}, \dots$, and $(f(\mathbf{p}_m^n))_{m \geq 1}$. By the uniform sequential continuity of f ,

$$\left| (f(\mathbf{p}_m^i))_{m \geq 1} - (f(\mathbf{p}_m^j))_{m \geq 1} \right| \rightarrow 0 \quad \text{whenever} \quad \left| (\mathbf{p}_m^i)_{m \geq 1} - (\mathbf{p}_m^j)_{m \geq 1} \right| \rightarrow 0, \quad (3.14)$$

for $i \neq j$. $|(\mathbf{p}_m^i)_{m \geq 1} - (\mathbf{p}_m^j)_{m \geq 1}| \rightarrow 0$ means

$$\forall \varepsilon > 0 \exists M \forall m \geq M \quad \left(|\mathbf{p}_m^i - \mathbf{p}_m^j| < \varepsilon \right) \quad i \neq j, \quad (3.15)$$

and $|(f(\mathbf{p}_m^i))_{m \geq 1} - (f(\mathbf{p}_m^j))_{m \geq 1}| \rightarrow 0$ means

$$\forall \varepsilon > 0 \exists M' \forall m \geq M' \quad \left(|f(\mathbf{p}_m^i) - f(\mathbf{p}_m^j)| < \varepsilon \right) \quad i \neq j. \quad (3.16)$$

Consider a fully labeled simplex δ_l in partition of Δ such that $l \geq \max(M, M')$. Denote vertices of δ_l by $\mathbf{p}^0, \mathbf{p}^1, \dots, \mathbf{p}^n$. We name these vertices so that $\mathbf{p}^0, \mathbf{p}^1, \dots, \mathbf{p}^n$ are labeled, respectively, with $0, 1, \dots, n$. Then, $|\mathbf{p}^i - \mathbf{p}^j| < \varepsilon$ and $|f(\mathbf{p}^i) - f(\mathbf{p}^j)| < \varepsilon$.

About $\mathbf{p}^0$, from the labeling rules, we have $\mathbf{p}_0^0 + \tau > f(\mathbf{p}^0)_0$. About $\mathbf{p}^1$, also from the labeling rules, we have $\mathbf{p}_1^1 + \tau > f(\mathbf{p}^1)_1$ which implies $\mathbf{p}_1^1 > f(\mathbf{p}^1)_1 - \tau$. $|f(\mathbf{p}^0) - f(\mathbf{p}^1)| < \varepsilon$ means $f(\mathbf{p}^1)_1 > f(\mathbf{p}^0)_1 - \varepsilon$. On the other hand, $|\mathbf{p}^0 - \mathbf{p}^1| < \varepsilon$ means $\mathbf{p}_1^0 > \mathbf{p}_1^1 - \varepsilon$. Thus, from

$$\mathbf{p}_1^0 > \mathbf{p}_1^1 - \varepsilon, \quad \mathbf{p}_1^1 > f(\mathbf{p}^1)_1 - \tau, \quad f(\mathbf{p}^1)_1 > f(\mathbf{p}^0)_1 - \varepsilon, \quad (3.17)$$

we obtain

$$\mathbf{p}_1^0 > f(\mathbf{p}^0)_1 - 2\varepsilon - \tau. \quad (3.18)$$

By similar arguments, for each i other than 0,

$$\mathbf{p}_i^0 > f(\mathbf{p}^0)_i - 2\varepsilon - \tau. \quad (3.19)$$

For $i = 0$, we have $\mathbf{p}_0^0 + \tau > f(\mathbf{p}^0)_0$. Then,

$$\mathbf{p}_0^0 > f(\mathbf{p}^0)_0 - \tau. \quad (3.20)$$

Adding (3.19) and (3.20) side by side except for some i (denote it by k) other than 0,

$$\sum_{j=0, j \neq k}^n \mathbf{p}_j^0 > \sum_{j=0, j \neq k}^n f(\mathbf{p}^0)_j - 2(n-1)\varepsilon - n\tau. \quad (3.21)$$

From $\sum_{j=0}^n \mathbf{p}_j^0 = 1$, $\sum_{j=0}^n f(\mathbf{p}^0)_j = 1$, we have $1 - \mathbf{p}_k^0 > 1 - f(\mathbf{p}^0)_k - 2(n-1)\varepsilon - n\tau$, which is rewritten as

$$\mathbf{p}_k^0 < f(\mathbf{p}^0)_k + 2(n-1)\varepsilon + n\tau. \quad (3.22)$$

Since (3.19) implies $\mathbf{p}_k^0 > f(\mathbf{p}^0)_k - 2\varepsilon - \tau$, we have

$$f(\mathbf{p}^0)_k - 2\varepsilon - \tau < \mathbf{p}_k^0 < f(\mathbf{p}^0)_k + 2(n-1)\varepsilon + n\tau. \quad (3.23)$$

Thus,

$$|\mathbf{p}_k^0 - f(\mathbf{p}^0)_k| < 2(n-1)\varepsilon + n\tau \quad (3.24)$$

is derived. On the other hand, adding (3.19) from 1 to n yields

$$\sum_{j=1}^n \mathbf{p}_j^0 > \sum_{j=1}^n f(\mathbf{p}^0)_j - 2n\varepsilon - n\tau. \quad (3.25)$$

From $\sum_{j=0}^n \mathbf{p}_j^0 = 1$, $\sum_{j=0}^n f(\mathbf{p}^0)_j = 1$, we have

$$1 - \mathbf{p}_0^0 > 1 - f(\mathbf{p}^0)_0 - 2n\varepsilon - n\tau. \quad (3.26)$$

Then, from (3.20) and (3.26), we get

$$|\mathbf{p}_0^0 - f(\mathbf{p}^0)_0| < 2n\varepsilon + n\tau. \quad (3.27)$$

From (3.24) and (3.27), we obtain the following result:

$$\left| \mathbf{p}_i^0 - f(\mathbf{p}^0) \right| < 2n\varepsilon + n\tau \quad \forall i. \quad (3.28)$$

Thus,

$$\left| \mathbf{p}^0 - f(\mathbf{p}^0) \right| < n(n+1)(2\varepsilon + \tau). \quad (3.29)$$

Since n is finite, $\mathbf{p}^0$ is an approximate fixed point of f . Similarly, we can prove that every other vertex, $\mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^n$, and all points in a fully-labeled simplex of K are approximate fixed points. $\square$

4. Concluding Remarks

There are some themes to which we can apply the result of this paper. In [11], we studied a proof of the existence of an approximate equilibrium in a competitive economy with uniformly continuous excess demand functions by Sperner's lemma. Using the result of this paper, we can prove the existence of an approximate equilibrium with *uniformly sequentially* continuous excess demand functions.

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