Research Article

Bivariate Poincaré Series for the Algebra of Covariants of a Binary Form

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A formula for computation of the bivariate Poincaré series $\rho_d(z, t)$ for the algebra of covariants of binary d -form is found.

1. Introduction

Let V_d be the complex vector space of binary forms of degree d endowed with the natural action of the special linear group $G = \text{SL}(2, \mathbb{C})$. Consider the corresponding action of the group G on the coordinate rings $\mathbb{C}[V_d]$ and $\mathbb{C}[V_d \oplus \mathbb{C}^2]$. Denote by $\mathcal{O}_d = \mathbb{C}[V_d]^G$ and by $\mathcal{C}_d = \mathbb{C}[V_d \oplus \mathbb{C}^2]^G$ the subalgebras of G -invariant polynomial functions. In the language of classical invariant theory, the algebras $\mathcal{O}_d$ and $\mathcal{C}_d$ are called the algebra of invariants and the algebra of covariants for the binary form of degree d , respectively. The algebra $\mathcal{C}_d$ is a finitely generated bigraded algebra:

$$\mathcal{C}_d = (\mathcal{C}_d)_{0,0} + (\mathcal{C}_d)_{1,0} + \cdots + (\mathcal{C}_d)_{i,j} + \cdots, \quad (1.1)$$

where each subspace $(\mathcal{C}_d)_{i,j}$ of covariants of degree i and order j is finite dimensional. The formal power series $\rho_d(z, t) \in \mathbb{Z}[[z, t]]$,

$$\rho_d(z, t) = \sum_{i,j=0}^{\infty} \dim((\mathcal{C}_d)_{i,j}) z^i t^j, \quad (1.2)$$

is called the bivariate Poincaré series of the algebra of covariants $\mathcal{C}_d$. It is clear that the series $\mathcal{P}_d(z, 0)$ is the Poincaré series of the algebra $\mathcal{I}_d$ and the series $\mathcal{P}_d(z, 1)$ is the Poincaré series of the algebra $\mathcal{C}_d$ with respect to the usual grading of the algebras under degree. The finitely generation of the algebra of covariants implies that its bivariate Poincaré series is the power series expansion of a rational function of two variables z, t . We consider here the problem of computing efficiently this rational function.

Calculating the Poincaré series of the algebras of invariants and covariants was an important object of research in invariant theory in the 19th century. For the cases $d \leq 10$, $d = 12$ the series $\mathcal{P}_d(z, t)$ were calculated by Sylvester, see in [1, 2] the big tables of $\mathcal{P}_d(z, t)$, named them as generating functions for covariants, reduced form. All those calculations are correct up to $d = 6$.

Relatively recently, Springer [3] found an explicit formula for computing the Poincaré series of the algebra of invariants $\mathcal{I}_d$. In the paper we have proved a Cayley-Sylvester-type formula for calculating of $\dim(\mathcal{C}_d)_{i,j}$ and a Springer-type formula for calculation of $\mathcal{P}_d(z, t)$. By using the formula, the bivariate Poincaré series $\mathcal{P}_d(z, t)$ is calculated for $d \leq 20$.

2. Cayley-Sylvester-Type Formula for $\dim(\mathcal{C}_d)_{i,j}$

To begin, we give a proof of the Cayley-Sylvester-type formula for the dimension of the graded component $(\mathcal{C}_d)_{i,j}$.

Let $V_d = \langle v_0, v_1, \dots, v_d \rangle$ and $\dim V_d = d + 1$ be standard irreducible representation of the Lie algebra $\mathfrak{sl}_2$. The basis elements $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of the algebra $\mathfrak{sl}_2$ act on V_d by the derivations D_1, D_2, E :

$$D_1(v_i) = iv_{i-1}, \quad D_2(v_i) = (d-i)v_{i+1}, \quad E(v_i) = (d-2i)v_i. \quad (2.1)$$

The action of $\mathfrak{sl}_2$ is extended to an action on the symmetrical algebra $S(V_d)$ in the natural way.

Let u_2 be the maximal unipotent subalgebra of $\mathfrak{sl}_2$. The algebra $\mathcal{S}_d$, defined by

$$\mathcal{S}_d := S(V_d)^{u_2} = \{v \in S(V_d) \mid D_1(v) = 0\}, \quad (2.2)$$

is called the *algebra of semi-invariants* of the binary form of degree d . For any element $v \in \mathcal{S}_d$, a natural number s is called *the order* of the element v if the number s is the smallest natural number such that

$$D_2^s(v) \neq 0, \quad D_2^{s+1}(v) = 0. \quad (2.3)$$

It is clear that any semi-invariant $v \in \mathcal{S}_d$ of order i is the highest weight vector for an irreducible $\mathfrak{sl}_2$ -module of the dimension $i + 1$ in $S(V_d)$.

The classical theorem [4] of Roberts implies an isomorphism of the algebra of covariants and the algebra of semi-invariants. Furthermore, the order is preserved through the isomorphism. Thus, it is enough to compute the Poincaré series of the algebra $\mathcal{S}_d$.

The algebra $S(V_d)$ is $\mathbb{N}$ -graded

$$S(V_d) = S^0(V_d) + S^1(V_d) + \dots + S^i(V_d) + \dots, \quad (2.4)$$

and each $S^i(V_d)$ is a completely reducible representation of the Lie algebra $\mathfrak{sl}_2$. Thus, the following decomposition holds

$$S^i(V_d) \cong \gamma_d(i, 0)V_0 + \gamma_d(i, 1)V_1 + \cdots + \gamma_d(i, d \cdot n)V_{d \cdot i}, \quad (*)$$

here $\gamma_d(i, j)$ is the multiplicity of the representation V_j in the decomposition of $S^i(V_d)$. On the other hand, the multiplicity $\gamma_d(i, j)$ of the representation V_j is equal to the number of linearly independent homogeneous semi-invariants of degree i and order j for the binary d -form. This argument proves the following.

Lemma 2.1.

$$\dim (C_d)_{i,j} = \gamma_d(i, j). \quad (2.5)$$

The set of weights (eigenvalues of the operator E) of a representation W denote by Λ_W , in particular, $\Lambda_{V_d} = \{-d, -d+2, \dots, d-2, d\}$.

A formal sum

$$\text{Char}(W) = \sum_{k \in \Lambda_W} n_W(k) q^k, \quad (2.6)$$

is called the character of a representation W , here $n_W(k)$ denotes the multiplicity of the weight $k \in \Lambda_W$. Since, the multiplicity of any weight of the irreducible representation V_d is equal to 1, we have

$$\text{Char}(V_d) = q^{-d} + q^{-d+2} + \cdots + q^{d-2} + q^d. \quad (2.7)$$

The character $\text{Char}(S^n(V_d))$ of the representation $S^n(V_d)$ equals

$$H_n(q^{-d}, q^{-d+2}, \dots, q^d), \quad (2.8)$$

(see [5]), where $H_n(x_0, x_1, \dots, x_d)$ is the complete symmetrical function

$$H_n(x_0, x_1, \dots, x_d) = \sum_{|\alpha|=n} x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \quad |\alpha| = \sum_i \alpha_i. \quad (2.9)$$

By replacing x_k with q^{d-2k} , $k = 0, \dots, d$, we obtain the specialized expression for $\text{Char}(S^n(V_d))$:

$$\begin{aligned} \text{Char}(S^n(V_d)) &= \sum_{|\alpha|=n} (q^d)^{\alpha_0} (q^{d-2 \cdot 1})^{\alpha_1} \cdots (q^{d-2d})^{\alpha_d} \\ &= \sum_{|\alpha|=n} q^{dn-2(\alpha_1+2\alpha_2+\cdots+d\alpha_d)} = \sum_{k=-dn}^{dn} \omega_d\left(n, \frac{dn-k}{2}\right) q^k, \end{aligned} \quad (2.10)$$

here $\omega_d(n, (dn - k)/2)$ is the number nonnegative integer solutions of the equation

$$\alpha_1 + 2\alpha_2 + \cdots + d\alpha_d = \frac{dn - k}{2}, \quad (2.11)$$

on the assumption that $\alpha_0 + \alpha_1 + \cdots + \alpha_d = n$. In particular, the coefficient of q^0 (the multiplicity of zero weight) is equal to $\omega_d(n, dn/2)$, and the coefficient of q^1 is equal to $\omega_d(n, (dn - 1)/2)$.

On the other hand, the decomposition (*) implies the equality for the characters:

$$\text{Char}(S^n(V_d)) = \gamma_d(n, 0)\text{Char}(V_0) + \gamma_d(n, 1)\text{Char}(V_1) + \cdots + \gamma_d(n, dn)\text{Char}(V_{dn}). \quad (2.12)$$

We can summarize what we have shown so far in the following.

Theorem 2.2.

$$\dim(\mathcal{C}_d)_{i,j} = \omega_d\left(i, \frac{di - j}{2}\right) - \omega_d\left(i, \frac{di - (j + 2)}{2}\right). \quad (2.13)$$

Proof. The weight j appears once in any representation V_k , for $k = j \bmod 2$, $k \geq j$. Therefore

$$\omega_d\left(i, \frac{di - j}{2}\right) = \gamma_d(i, j) + \gamma_d(i, j + 2) + \cdots + \gamma_d(i, j + 4) + \cdots. \quad (2.14)$$

Similarly,

$$\omega_d\left(i, \frac{di - (j + 2)}{2}\right) = \gamma_d(i, j + 2) + \gamma_d(i, j + 4) + \cdots + \gamma_d(i, j + 6) + \cdots. \quad (2.15)$$

Thus,

$$\omega_d\left(i, \frac{di - j}{2}\right) - \omega_d\left(i, \frac{di - (j + 2)}{2}\right) = \gamma_d(i, j). \quad (2.16)$$

By using Lemma 2.1 we obtain

$$\dim(\mathcal{C}_d)_{i,j} = \omega_d\left(i, \frac{di - j}{2}\right) - \omega_d\left(i, \frac{di - (j + 2)}{2}\right). \quad (2.17)$$

□

For another proof of the formula see [3].

Note that the original Cayley-Sylvester formula is

$$\dim(\mathcal{C}_d)_n = \omega_d\left(n, \frac{dn}{2}\right) - \omega_d\left(n, \frac{dn}{2} - 1\right). \quad (2.18)$$

Also, in [6] we proved that

$$\dim (C_d)_n = \omega_d\left(n, \frac{dn}{2}\right) + \omega_d\left(n, \frac{dn-1}{2}\right). \quad (2.19)$$

Here $(\mathcal{O}_d)_n, (C_d)_n$ are the components of standard grading of the algebras $\mathcal{O}_d, C_d$ under degree.

3. Calculation of $\dim(C_d)_{i,j}$

It is well known that the number $\omega_d(i, (di-j)/2)$ of nonnegative integer solutions of the following system

$$\begin{aligned} \alpha_1 + 2\alpha_2 + \cdots + d\alpha_d &= \frac{di-j}{2}, \\ \alpha_0 + \alpha_1 + \cdots + \alpha_d &= i, \end{aligned} \quad (3.1)$$

is given by the coefficient of $z^i t^{(di-j)/2}$ of the generating function

$$f_d(z, t) = \frac{1}{(1-z)(1-zt) \cdots (1-zt^d)}. \quad (3.2)$$

We will use the notation $[x^k]F(x)$ to denote the coefficient of x^k in the series expansion of $F(x) \in \mathbb{C}[[x]]$. Thus

$$\omega_d\left(i, \frac{di-j}{2}\right) = [z^i t^{(di-j)/2}] f_d(z, t). \quad (3.3)$$

It is clear that

$$\omega_d\left(i, \frac{di-j}{2}\right) = [z^i t^{di-j}] f_d(z, t^2) = \left[(zt^d)^i\right] t^j f_d(z, t^2). \quad (3.4)$$

Similarly, the number $\omega_d(i, (di-(j+2))/2)$ of nonnegative integer solutions of the following system

$$\begin{aligned} \alpha_1 + 2\alpha_2 + \cdots + d\alpha_d &= \frac{di-(j+2)}{2}, \\ \alpha_0 + \alpha_1 + \cdots + \alpha_d &= i, \end{aligned} \quad (3.5)$$

equals

$$[z^i t^{(di-(j+2))/2}] f_d(z, t) = [z^i t^{di-(j+2)}] f_d(z, t^2) = \left[(zt^d)^i\right] t^{j+2} f_d(z, t^2). \quad (3.6)$$

Therefore,

$$\begin{aligned}
 & \omega_d\left(i, \frac{di-j}{2}\right) - \omega_d\left(i, \frac{di-(j+2)}{2}\right) \\
 &= \left[(zt^d)^i \right] t^j f_d(z, t^2) - \left[(zt^d)^i \right] t^{j+2} f_d(z, t^2) \\
 &= \left[(zt^d)^i \right] (t^j - t^{j+2}) f_d(z, t^2) = \left[(zt^d)^i \right] t^j (1 - t^2) f_d(z, t^2) \\
 &= \left[z^i t^{di-j} \right] (1 - t^2) f_d(z, t^2).
 \end{aligned} \tag{3.7}$$

Thus, the following statement holds.

Theorem 3.1. *The number $\dim (C_d)_{i,j}$ of linearly independent covariants of degree i and order j for the binary d -form is given by the formula*

$$\dim (C_d)_{i,j} = \left[z^i t^{di-j} \right] \left(\frac{1 - t^2}{(1 - z)(1 - zt^2) \cdots (1 - zt^{2d})} \right). \tag{3.8}$$

It is clear that

$$\begin{aligned}
 & \left[z^i t^{di-j} \right] \left(\frac{1 - t^2}{(1 - z)(1 - zt^2) \cdots (1 - zt^{2d})} \right) \\
 &= \left[z^i t^{(di-j)/2} \right] \left(\frac{1 - t}{(1 - z)(1 - zt) \cdots (1 - zt^d)} \right).
 \end{aligned} \tag{3.9}$$

By using the decomposition

$$\frac{1}{(1 - z)(1 - zt) \cdots (1 - zt^d)} = \sum_{k=0}^{\infty} \begin{bmatrix} d \\ i \end{bmatrix}_t z^i, \tag{3.10}$$

where $\begin{bmatrix} d \\ n \end{bmatrix}_q$ is the q -binomial coefficient

$$\begin{bmatrix} d \\ n \end{bmatrix}_q := \frac{(1 - q^{d+1})(1 - q^{d+2}) \cdots (1 - q^{d+n})}{(1 - q)(1 - q^2) \cdots (1 - q^n)}, \tag{3.11}$$

one obtains the well-known formula

$$\dim (C_d)_{i,j} = \left[t^{(di-j)/2} \right] (1 - t) \begin{bmatrix} d \\ i \end{bmatrix}_t, \tag{3.12}$$

for instance, see [3].

4. Explicit Formula for $\rho_d(z, t)$

Let us prove Springer-type formula for the bivariate Poincaré series $\rho_d(z, t)$ of the algebra covariants of the binary d -form. Consider the $\mathbb{C}$ -algebra $\mathbb{Z}[[t, z]]$ of formal power series. For an integer $d \in \mathbb{N}$ define the $\mathbb{C}$ -linear function

$$\Psi_d : \mathbb{Z}[[z, t]] \longrightarrow \mathbb{Z}[[z, t]], \quad (4.1)$$

in the following way

$$\Psi_d(z^i t^j) = \begin{cases} z^i t^{di-j}, & \text{if } di - j \geq 0, \\ 0, & \text{if } di - j < 0. \end{cases} \quad (4.2)$$

The main idea of the ensuing calculations is that the Poincaré series $\rho_d(z, t)$ can be expressed in terms of function Ψ_d . The following simple but important statement holds.

Lemma 4.1.

$$\rho_d(z, t) = \Psi_d \left(\frac{1 - t^2}{(1 - zt)(1 - zt^2) \cdots (1 - zt^{2d})} \right). \quad (4.3)$$

Proof. Theorem 2.2 implies that $\dim (C_d)_{i,j} = [z^i t^{di-j}] f_d(z, t^2)$. Then

$$\begin{aligned} \rho_d(z, t) &= \sum_{i,j=0}^{\infty} \dim (C_d)_{i,j} z^i t^j = \sum_{i,j=0}^{\infty} \left([z^i t^{di-j}] f_d(z, t^2) \right) z^i t^j \\ &= \Psi_d \left(f_d(z, t^2) \right). \end{aligned} \quad (4.4)$$

□

Let $\varphi_n : \mathbb{Z}[[t]] \rightarrow \mathbb{Z}[[t, z]]$, $n \in \mathbb{Z}$ be a $\mathbb{C}$ -linear function defined by

$$\varphi_n(t^m) := z^i t^j, \quad \text{where } i := \min\{k' \mid nk' - m \geq 0\}, \quad j = ni - m, \quad (4.5)$$

for $i, j, m, n \in \mathbb{N}$. Note that $\varphi_n(t^0) = 1$ and $\varphi_1(t^m) = z^m$, $\varphi_0(t^m) = 1$. Also, put $\varphi_n(t^m) = 0$ for $n < 0$. It is clear that $\varphi_n(t^{ni-j}) = z^i t^j$ if $ni - j \geq 0$, $j < n$.

In important special cases, calculating the functions Ψ can be reduced to calculating the functions φ . The following statements hold.

Lemma 4.2. (i) For $R(t), H(t) \in \mathbb{C}[[t]]$ holds $\varphi_n(R(t^n)H(t)) = R(z)\varphi_n(H(t))$.
(ii) For $R(t) \in \mathbb{C}[[t]]$ and for $n, k \in \mathbb{N}$ holds

$$\Psi_n \left(\frac{R(t)}{1 - zt^k} \right) = \begin{cases} \frac{\varphi_{n-k}(R(t))}{1 - zt^{n-k}}, & n \geq k, \\ 0, & \text{if } n < k. \end{cases} \quad (4.6)$$

Proof. (i) The statement follows from the linearity of the function ψ_n and from the following simple observation:

$$\psi_n(t^{nk}t^{ni-j}) = \psi_n(t^{n(k+i)-j}) = z^{k+i}t^j = z^k z^i t^j = z^k \psi_n(t^{ni-j}), \quad (4.7)$$

for $ni - j \geq 0$ and $j < n$.

(ii) Let $R(t) = \sum_{m=0}^{\infty} a_m t^m$. Then for $k < n$ we have

$$\begin{aligned} \Psi_n\left(\frac{R(t)}{1-zt^k}\right) &= \Psi_n\left(\sum_{m,s=0}^{\infty} a_m t^m (zt^k)^s\right) = \Psi_n\left(\sum_{m,s=0}^{\infty} a_m z^s t^{ks+m}\right) \\ &= \sum_{(n-k)s-m \geq 0} a_m z^s t^{(n-k)s-m} = \sum_{m,s=0}^{\infty} a_m \psi_{n-k}(t^m) (zt^{n-k})^s \\ &= \sum_{m=0}^{\infty} a_m \psi_{n-k}(t^m) \frac{1}{1-zt^{n-k}} = \frac{\psi_{n-k}(R(t))}{1-zt^{n-k}}. \end{aligned} \quad (4.8)$$

□

Now we can present Springer-type formula for calculating of the bivariate Poincaré series $\rho_d(z, t)$.

Theorem 4.3.

$$\rho_d(z, t) = \sum_{0 \leq k < d/2} \psi_{d-2k} \left(\frac{(-1)^k t^{k(k+1)} (1-t^2)}{(t^2, t^2)_k (t^2, t^2)_{d-k}} \right) \frac{1}{1-zt^{d-2k}}, \quad (4.9)$$

here $(a, q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$ is q -shifted factorial.

Proof. Consider the partial fraction decomposition of the rational function $f_d(z, t^2)$:

$$f_d(z, t^2) = \sum_{k=0}^d \frac{R_k(z)}{1-tz^{2k}}. \quad (4.10)$$

It is easy to see, that

$$\begin{aligned} R_k(t) &= \lim_{z \rightarrow t^{-2k}} \left(f_d(z, t^2) (1-zt^{2k}) \right) = \lim_{z \rightarrow t^{-2k}} \left(\frac{(1-t^2)}{(z, t)_{d+1}} (1-zt^{2k}) \right) \\ &= \frac{1-t^2}{(1-t^{-2k})(1-t^{2-2k}) \cdots (1-t^{2(k-1)-2k})(1-t^{2(k+1)-2k}) \cdots (1-t^{2d-2k})} \end{aligned}$$

$$\begin{aligned}
&= \frac{t^{2k+(2k-2)+\dots+2}(1-t^2)}{(t^{2k}-1)(t^{2k-2}-1)\dots(t^2-1)(1-t^2)\dots(1-t^{2d-2k})} \\
&= \frac{(-1)^k t^{k(k+1)}(1-t^2)}{(t^2, t^2)_k (t^2, t^2)_{d-k}}.
\end{aligned} \tag{4.11}$$

Using the above Lemmas we obtain

$$\begin{aligned}
\rho_d(z, t) &= \Psi_d(f_d(z, t^2)) = \Psi_d\left(\sum_{k=0}^n \frac{R_k(t^2)}{1-zt^{2k}}\right) \\
&= \sum_{0 \leq k < d/2} \varphi_{d-2k}\left(\frac{(-1)^k t^{k(k+1)}(1-t^2)}{(t^2, t^2)_k (t^2, t^2)_{d-k}}\right) \frac{1}{1-zt^{d-2k}}.
\end{aligned} \tag{4.12}$$

□

Corollary 4.4. *A denominator of the bivariate Poincaré series $\rho_d(z, t)$, $d > 2$ can be written in the form*

$$\prod_{k=0}^{[(d+1)/2]} (1-zt^{d-2k}) \prod_{i=1}^{d-2} (1-z^{k_i}), \tag{4.13}$$

where $k_1, k_2, \dots, k_{d-2}$ are the degrees of elements of homogeneous system of parameters for the algebra of invariants $\mathcal{O}_d$.

Proof. The formula of Theorem 4.3 implies that the bivariate Poincaré series has the form

$$\rho_d(z, t) = \frac{P_d(z, t)}{\prod_{k=0}^{[(d+1)/2]} (1-zt^{d-2k}) R_d(z)}, \tag{4.14}$$

for some polynomials $P_d(z, t), R_d(z)$. Thus, the Poincaré series for the algebra of invariants $\mathcal{O}_d$ has the form

$$\rho_d(z, 0) = \frac{P_d(z, 0)}{R_d(z)}. \tag{4.15}$$

The algebra of invariants $\mathcal{O}_d$ is Cohen-Macaulay, and its transcendence degree for $d > 2$ equals $d - 2$, see [3]. Therefore it has a homogeneous system of $d - 2$ parameters and the denominator $R_d(z)$ of its Poincaré series can be written in the following way $R_d(z) = (1 - z^{k_1})(1 - z^{k_2}) \dots (1 - z^{k_{d-2}})$, where $k_1, k_2, \dots, k_{d-2}$ are the degrees of elements of this homogeneous system of parameters. □

5. Examples

For direct computations we use the following technical lemma.

Lemma 5.1. *For $R(t) \in \mathbb{C}[[t]]$ one has*

$$\psi_n \left(\frac{R(t)}{(1-t^{k_1})(1-t^{k_2}) \cdots (1-t^{k_m})} \right) = \frac{\psi_n(R(z)Q_n(t^{k_1})Q_n(t^{k_2})Q_n(t^{k_m}))}{(1-z^{k_1})(1-z^{k_2}) \cdots (1-z^{k_m})}, \quad (5.1)$$

here $Q_n(t) = 1 + t + t^2 + \cdots + t^{n-1}$, and k_i are natural numbers.

Proof. Taking into account Lemma 4.2 we get

$$\begin{aligned} \psi_n \left(\frac{g(t)}{1-t^m} \right) &= \psi_n \left(\frac{g(t)}{1-t^{nm}} \frac{1-t^{nm}}{1-t^m} \right) = \frac{1}{1-t^m} \psi_n \left(g(t) \frac{1-t^{nm}}{1-t^m} \right) \\ &= \frac{1}{1-t^m} \psi_n \left(g(t) (1 + t^m + (t^m)^2 + \cdots + (t^m)^{n-1}) \right) \\ &= \frac{1}{1-t^m} \psi_n (g(t)Q_n(t^m)). \end{aligned} \quad (5.2)$$

In a similar fashion we prove the general case. □

By using Lemma 5.1 the bivariate Poincaré series $\rho_d(z, t)$ for $d \leq 20$ are found. All these results agree with Sylvester's calculations up to $d = 6$, see [1, 2].

Below is the list of several series:

$$\begin{aligned} \rho_1(z, t) &= \frac{1}{1-zt}, & \rho_2(z, t) &= \frac{1}{(1-zt^2)(1-z^2)}, \\ \rho_3(z, t) &:= \frac{z^2t^2 - zt + 1}{(1-zt)(1-zt^3)(1-z^4)}, \\ \rho_4(z, t) &= \frac{z^2t^4 - zt^2 + 1}{(1-zt^2)(1-t^4z)(1-z^2)(1-z^3)}, \\ \rho_5(z, t) &= \frac{p_5(z, t)}{(1-zt)(1-t^3z)(1-zt^5)(1-z^8)(1-z^6)(1-z^4)}, \\ p_5(z, t) &= 1 + z^7t^3 - z^6t^4 + z^2t^2 + 2z^7t - z^5t^5 - z^8t^2 - 2z^8t^6 - z^8t^4 + z^5t^3 + z^5t + z^9t^7 - z^{10}t^6 \\ &\quad + z^{10}t^2 - z^{10}t^4 - z^{11}t^3 + z^9t^3 - t^3z - z^6 + z^4t^4 - zt + z^2t^6 + z^2t^4 + z^{12} + z^{14}t^6 - z^{13}t \\ &\quad - z^{13}t^5 - z^{13}t^3 - z^{15}t^7 + z^{14}t^4 - z^3t^7 + z^7t^5. \end{aligned} \quad (5.3)$$

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