Research Article

On Paranormed Ideal Convergent Generalized Difference Strongly Summable Sequence Spaces Defined over n -Normed Spaces

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We introduce a new class of ideal convergent (shortly I -convergent) sequence spaces using an Orlicz function and difference operator of order $s \geq 1$ defined over the n -normed spaces. We investigate these spaces for some linear topological structures. These investigations will enhance the acceptability of the notion of n -norm by giving a way to construct different sequence spaces with elements in n -normed spaces. We also give some relations related to these sequence spaces.

1. Introduction

Throughout the paper w , ℓ_∞ , c , c_0 , and ℓ_p denote the classes of *all*, *bounded*, *convergent*, *null*, and *p -absolutely summable* sequences of complex numbers.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical sequences/matrices (double sequences) through the concept of density. It was first introduced by Fast [1], and Schoenberg [2], independently for the real sequences. Later on, it was further investigated from sequence point of view and linked with the summability theory by Fridy [3] and many others. The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of positive integers, which is defined as follows: the natural density of a subset E of $\mathbb{N}$ is denoted by $\delta(E)$ and is defined by

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \in E : k \leq n\}|, \quad (1.1)$$

where the vertical bar denotes the cardinality of the respective set.

The notion of I -convergence (I denotes the ideal of subsets of $\mathbb{N}$), which is a generalization of statistical convergence, was introduced by Kastyko et al. [4] and further studied by many other authors.

The notion of difference sequence space was introduced by Kizmaz [5], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [6] by introducing the sequence spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$, and $c_0(\Delta^s)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [7], who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$, and $c_0(\Delta_m)$. Tripathy et al. [8] generalized the above notion and unified these as follows.

For nonnegative integers m, s , the generalized difference sequence spaces are defined as follows: for a given sequence space Z , we have

$$Z(\Delta_m^s) = \{x = (x_k) \in w : (\Delta_m^s x_k) \in Z\}, \quad (1.2)$$

where $\Delta_m^s x = (\Delta_m^s x_k) = (\Delta_m^{s-1} x_k - \Delta_m^{s-1} x_{k+m})$, $\Delta_m^0 x_k = x_k$, for all $k \in \mathbb{N}$, the difference operator is equivalent to the following binomial representation:

$$\Delta_m^s x_k = \sum_{v=0}^s (-1)^v \binom{s}{v} x_{k+mv}. \quad (1.3)$$

Taking $s = 1$, we get the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$, and $c_0(\Delta_m)$, introduced and studied by Tripathy and Et [7]. Taking $m = 1$, we get the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$, and $c_0(\Delta^s)$ studied by Et and Çolak [6]. Taking $s = m = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ introduced and studied by Kizmaz [5].

Let m, s be nonnegative integers, then for a given sequence space Z , we introduced

$$Z\left(\Delta_m^{(s)}\right) = \left\{x = (x_k) \in w : \left(\Delta_m^{(s)} x_k\right) \in Z\right\}, \quad (1.4)$$

where $\Delta_m^{(s)} x = (\Delta_m^{(s)} x_k) = (\Delta_m^{(s-1)} x_k - \Delta_m^{(s-1)} x_{k-m})$, $\Delta_m^{(0)} x_k = x_k$, for all $k \in \mathbb{N}$, the difference operator is equivalent to the following binomial representation:

$$\Delta_m^{(s)} x_k = \sum_{v=0}^s (-1)^v \binom{s}{v} x_{k-mv}, \quad (1.5)$$

where $x_k = 0$, for $k < 0$.

The concept of 2-normed space was initially introduced by Gähler [9], in the mid of 1960s, while that of n -normed spaces can be found in Misiak [10]. Since then, many others authors have used this concept and obtained various results, see, for instance, Gunawan [11] and Gunawan and Mashadi [12, 13]. Recently, a lot of activities have started to study summability, sequence spaces, and related topics in these spaces (see [14, 15]).

The notion of ideal-convergence in 2-normed spaces was initially introduced by Gürdal [16]. Later on, it was extended to n -normed spaces by Gürdal and Sahnier [17].

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$, which is continuous, nondecreasing and convex with $M(0) = 0$, $M(0) > 0$, as $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$.

An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int_0^x p(t)dt, \quad (1.6)$$

where p is the known kernel of M , right differentiable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$, for $t > 0$ and $p(t) \rightarrow \infty$, as $t \rightarrow \infty$.

If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called Modulus function, which was presented and discussed by Maddox [18].

Lindenstrauss and Tzafriri [19] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}. \quad (1.7)$$

The space ℓ_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\} \quad (1.8)$$

becomes a Banach space which is called an Orlicz sequence space.

The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$, for $1 \leq p < \infty$.

Subsequently, Orlicz function was used to define sequence spaces by Parashar and Choudhary [20] and others (see [10, 15, 21–23]).

Remark 1.1. It is well known if M is a convex function and $M(0) = 0$, then

$$M(\lambda x) \leq \lambda M(x), \quad \forall \lambda \text{ with } 0 < \lambda < 1. \quad (1.9)$$

In this paper, we introduce some new ideal convergent sequences over n -normed spaces by using Orlicz functions and generalized difference sequence spaces.

2. Definitions and Preliminaries

Let n be a nonnegative integer and X a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following conditions:

- (1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,
 - (2) $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation,
 - (3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$, for any $\alpha \in \mathbb{R}$,
 - (4) $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$,
- is called an n -norm on X and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n -normed space.

A trivial example of an n -normed space is $X = \mathbb{R}^n$, equipped with the Euclidean n -norm $\|x_1, x_2, \dots, x_n\|_E =$ the volume of the n -dimensional parallelopiped spanned by the vectors $x_1, x_2, \dots, x_n$, which may be given explicitly by the formula

$$\|x_1, x_2, \dots, x_n\|_E = |\det(x_{ij})| = \text{abs} \left(\begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix} \right), \quad (2.1)$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$, for each $i = 1, 2, \dots, n$.

Let $(X, \|\cdot, \dots, \cdot\|)$ be an n -normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \dots, a_n\}$ a linearly independent set in X . Then, the function $\|\cdot, \dots, \cdot\|_\infty$ on X^{n-1} is defined by

$$\|x_1, x_2, \dots, x_n\|_\infty = \max_{1 \leq i \leq n} \{\|x_1, x_2, \dots, x_{n-1}, a_i\|\} \quad (2.2)$$

defines as $(n-1)$ -norm on X with respect to $\{a_1, a_2, \dots, a_n\}$, and this is known as the derived $(n-1)$ -norm (see [12] for details).

The standard n -norm on X , a real inner product space of dimension $d \geq n$, is as follows:

$$\|x_1, x_2, \dots, x_n\|_S = \left| \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix} \right|^{1/2}, \quad (2.3)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on X . If we take $X = \mathbb{R}^n$, then this n -norm is exactly the same as the Euclidean n -norm $\|x_1, x_2, \dots, x_n\|_E$ mentioned earlier.

For $n = 1$, this n -norm is the usual norm $\|x_1\| = \sqrt{\langle x_1, x_1 \rangle}$.

Definition 2.1. A sequence (x_k) in an n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to *converge* to some $l \in X$ with respect the n -norm if for each $\varepsilon > 0$, there exists an positive integer n_0 such that

$$\|x_k - l, z_1, z_2, \dots, z_{n-1}\| < \varepsilon, \quad \forall k \geq n_0 \text{ for every } z_1, z_2, \dots, z_{n-1} \in X. \quad (2.4)$$

Definition 2.2. A sequence (x_k) in an n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be *Cauchy* with respect the n -norm if for each $\varepsilon > 0$, there exists an positive integer $n_0 = n_0(\varepsilon)$ such that

$$\|x_k - x_m, z_1, z_2, \dots, z_{n-1}\| < \varepsilon, \quad \forall k, m \geq n_0 \text{ for every } z_1, z_2, \dots, z_{n-1} \in X. \quad (2.5)$$

If every Cauchy sequence in X converges to some $l \in X$, then X is said to be *complete* with respect to the n -norm. Any complete n -normed space is said to be *n -Banach space*.

Definition 2.3. Let S be a nonempty set. A nonempty family of sets $I \subseteq P(S)$ (power set of S) is called an *ideal* in S if (i) for each $A, B \in I$, we have $A \cup B \in I$; (ii) for each $A \in I$ and $B \subseteq A$, we have $B \in I$.

Definition 2.4. Let S be a nonempty set. A family $F \subseteq P(S)$ (power set of S) is called a *filter* on S if (i) $\emptyset \notin F$; (ii) for each $A, B \in F$, we have $A \cap B \in F$; (iii) for each $A \in F$ and $B \supseteq A$, we have $B \in F$.

An ideal I is called *nontrivial* if $I \neq \emptyset$ and $S \notin I$. It is clear that $I \subseteq P(S)$ is a nontrivial ideal if and only if the class $F = F(I) = \{S - A : A \in I\}$ is a filter on S . The filter $F(I)$ is called the filter associated with the ideal I .

A nontrivial ideal $I \subseteq P(S)$ is called an *admissible ideal* in S if and only if it contains all singletons, *that is*, if it contains $\{\{x\} : x \in S\}$.

Definition 2.5. A sequence (x_k) in a normed space $(X, \|\cdot\|)$ is said to be *statistically convergent* to $x_0 \in X$ if for each $\varepsilon > 0$, the set $E(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x_0\| \geq \varepsilon\}$ has natural density zero.

Definition 2.6. A sequence (x_k) in a normed space $(X, \|\cdot\|)$ is said to be *I-convergent* to $x_0 \in X$ if for each $\varepsilon > 0$, the set

$$E(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x_0\| \geq \varepsilon\} \text{ belongs to } I. \quad (2.6)$$

Definition 2.7. A sequence (x_k) in a normed space $(X, \|\cdot\|)$ is said to be *I-bounded* if there exists $M > 0$ such that the set $\{k \in \mathbb{N} : \|x_k\| > M\}$ belongs to I .

Definition 2.8. A sequence (x_k) in a normed space $(X, \|\cdot\|)$ is said to be *I-Cauchy* if for each $\varepsilon > 0$, there exists a positive integer $m = m(\varepsilon)$ such that the set

$$E(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x_m\| \geq \varepsilon\} \text{ belongs to } I. \quad (2.7)$$

Definition 2.9. A sequence (x_k) in an n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be *I-convergent* to $x_0 \in X$ with respect the n -norm, if for each $\varepsilon > 0$, the set

$$E(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x_0, z_1, z_2, \dots, z_{n-1}\| \geq \varepsilon, \text{ for every } z_1, z_2, \dots, z_{n-1} \in X\} \text{ belongs to } I. \quad (2.8)$$

Definition 2.10. A sequence (x_k) in an n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be *I-Cauchy* with respect to the n -norm if for each $\varepsilon > 0$, there exists a positive integer $m = m(\varepsilon)$ such that the set

$$E(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x_m, z_1, z_2, \dots, z_{n-1}\| \geq \varepsilon, \text{ for every } z_1, z_2, \dots, z_{n-1} \in X\} \text{ belongs to } I. \quad (2.9)$$

Definition 2.11. A sequence space E is said to be *normal* (or *solid*) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

Let $K = \{k_1 < k_2 < \dots\} \subseteq \mathbb{N}$ and E a sequence space. A *K-step space* of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}$.

A canonical preimage of a sequence $\{x_{k_n}\} \in \lambda_K^E$ is a sequence $\{y_k\} \in w$ defined as

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases} \quad (2.10)$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E ; that is, y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

Definition 2.12. A sequence space E is said to be *monotone* if it contains the canonical preimages of all its step spaces.

Example 2.13. If we take $I = I_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset}\}$. Then, I_f is a nontrivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with usual convergence of sequences in n -normed spaces.

Example 2.14. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ denote the asymptotic density of the set A then I_δ is a nontrivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with statistical convergence in n -normed spaces.

Now, we state the following lemmas (see [12] for details).

Lemma 2.15. *Every n -normed space is an $(n-r)$ -normed space for all $r = 1, 2, \dots, n-1$. In particular, every n -normed space is a normed space.*

Lemma 2.16. *A standard n -normed space is complete if and only if it is complete with respect to the usual norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.*

Lemma 2.17. *On a standard n -normed space X , the derived $(n-1)$ -norm $\|\cdot, \dots, \cdot\|_\infty$ defined with respect to the orthogonal set $\{e_1, e_2, \dots, e_n\}$ is equivalent to the standard $(n-1)$ -norm $\|\cdot, \dots, \cdot\|_S$. To be precise, one has*

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty \leq \|x_1, x_2, \dots, x_{n-1}\|_S \leq \sqrt{n} \|x_1, x_2, \dots, x_{n-1}\|_\infty, \quad (2.11)$$

for all $x_1, x_2, \dots, x_{n-1} \in X$, where $\|x_1, x_2, \dots, x_{n-1}\|_\infty = \max_{1 \leq i \leq n} \{\|x_1, x_2, \dots, x_{n-1}, e_i\|_S\}$.

Lemma 2.18 (Kamthan and Gupta [24]). *Every normal space is monotone.*

3. Main Results

In this section, we define some new ideal convergent sequence spaces and investigate their linear topological structures. Also, we find out some relations related to these sequence spaces.

Let I be an admissible ideal of $\mathbb{N}$, M an Orlicz function, and $(X, \|\cdot, \dots, \cdot\|)$ an n -normed space. Further, let $p = (p_k)$ be any bounded sequence of positive real numbers, and let $w(n - X)$, denote the spaces of all X -valued sequences. For each $\varepsilon > 0$, we define the following sequence spaces:

$$\begin{aligned} & W^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|) \\ &= \left\{ (x_k) \in w(n - X) : \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k - 1}{\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I, \right. \\ & \quad \left. \text{for some } \rho > 0, l \in X \text{ and } \forall z_1, z_2 \dots z_{n-1} \in X \right\}, \end{aligned} \quad (3.1)$$

$$\begin{aligned} & W^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|) \\ &= \left\{ (x_k) \in w(n - X) : \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I, \right. \\ & \quad \left. \text{for some } \rho > 0 \text{ and } \forall z_1, z_2 \dots z_{n-1} \in X \right\}, \end{aligned} \quad (3.2)$$

$$\begin{aligned} & W^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|) \\ &= \left\{ (x_k) \in w(n - X) : \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} < \infty, \right. \\ & \quad \left. \text{for some } \rho > 0 \text{ and } \forall z_1, z_2 \dots z_{n-1} \in X \right\}, \end{aligned} \quad (3.3)$$

$$\begin{aligned} & W^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|) \\ &= \left\{ (x_k) \in w(n - X) : \exists K > 0 \text{ s.t. } \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq K \right\} \in I, \right. \\ & \quad \left. \text{for some } \rho > 0 \text{ and } \forall z_1, z_2 \dots z_{n-1} \in X \right\}. \end{aligned} \quad (3.4)$$

The following well-known inequality will be used throughout the article. Let $p = (p_k)$ be any sequence of positive real numbers with $0 \leq p_k \leq \sup_k p_k = G, D = \max\{1, 2^{G-1}\}$, then $|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k})$, for all $k \in \mathbb{N}$ and $a_k, b_k \in \mathbb{C}$. Also, $|a|^{p_k} \leq \max\{1, |a|^G\}$, for all $a \in \mathbb{C}$.

Theorem 3.1. *The spaces $W^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$, $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ and $W_\infty^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ are linear.*

Proof. We will prove the result for the space $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ only and the others can be proved similarly.

Let $x = (x_k)$ and $y = (y_k)$ be any two elements of the space $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$. Then, there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that the sets

$$\begin{aligned} A_{\varepsilon/2}(x) &= \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2}, \right. \\ &\quad \left. \text{for every } z_1, z_2, \dots, z_{n-1} \in X \right\}, \\ B_{\varepsilon/2}(y) &= \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s y_k}{\rho_2}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2}, \right. \\ &\quad \left. \text{for every } z_1, z_2, \dots, z_{n-1} \in X \right\} \end{aligned} \quad (3.5)$$

belong to I .

Let $\alpha, \beta \in \mathbb{R}$ be two scalars. Since $\|\cdot, \dots, \cdot\|$ is an n -norm, the operator Δ^s is linear and the Orlicz function M is continuous, the following inequality holds:

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s (\alpha x_k + \beta y_k)}{(|\alpha|\rho_1 + |\beta|\rho_2)}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ &\leq D \frac{1}{n} \sum_{k=1}^n \left[\frac{|\alpha|}{(|\alpha|\rho_1 + |\beta|\rho_2)} M \left(\left\| \frac{\Delta_m^s x_k}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ &\quad + D \frac{1}{n} \sum_{k=1}^n \left[\frac{|\beta|}{(|\alpha|\rho_1 + |\beta|\rho_2)} M \left(\left\| \frac{\Delta_m^s y_k}{\rho_2}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ &\leq DK \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ &\quad + DK \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s y_k}{\rho_2}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k}, \end{aligned} \quad (3.6)$$

where $K = \max\{1, (|\alpha|/(|\alpha|\rho_1 + |\beta|\rho_2)), (|\beta|/(|\alpha|\rho_1 + |\beta|\rho_2))\}$.

From the above relation, we get

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s (\alpha x_k + \beta y_k)}{(|\alpha|\rho_1 + |\beta|\rho_2)}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \\ &\subseteq \left\{ n \in \mathbb{N} : DK \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \\ &\cup \left\{ n \in \mathbb{N} : DK \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s y_k}{\rho_2}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\}. \end{aligned} \quad (3.7)$$

Since both the sets on the right hand side of the relation (3.7) belong to I , this completes the proof. $\square$

Note 1. It is easy to verify that the space $W_\infty(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ is a linear space.

Theorem 3.2. *The space $W^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$, $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$, $W_\infty^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$, and $W_\infty(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ are paranormed spaces (not totally paranormed) with respect to the paranorm g_Δ defined by*

$$g_\Delta(x) = \sum_{k=1}^{ms} \|x_k, z_1, z_2, \dots, z_{n-1}\|$$

$$+ \inf \left\{ \rho^{p_k/H} : \sup_k M \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \leq 1, \right. \quad (3.8)$$

$$\left. \text{for some } \rho > 0 \text{ and } \forall z_1, z_2, \dots, z_{n-1} \in X \right\}$$

where $H = \max\{1, \sup p_k\}$.

Proof. Clearly, $g_\Delta(-x) = g_\Delta(x)$ and $g_\Delta(\theta) = 0$. Let $x = (x_k)$ and $y = (y_k)$ be any two elements of the space $Z(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$, where $Z = W^I, W_0^I, W_\infty^I$ and W_∞ .

Then, for $\rho > 0$, we set

$$A_1 = \left\{ \rho > 0 : \sup_k M \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \leq 1, \forall z_1, z_2, \dots, z_{n-1} \in X \right\},$$

$$A_2 = \left\{ \rho > 0 : \sup_k M \left(\left\| \frac{\Delta_m^s y_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \leq 1, \forall z_1, z_2, \dots, z_{n-1} \in X \right\}. \quad (3.9)$$

Let $\rho_1 \in A_1$ and $\rho_2 \in A_2$. If $\rho = \rho_1 + \rho_2$, then we have

$$M \left(\left\| \frac{\Delta_m^s (x_k + y_k)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right)$$

$$\leq \frac{\rho_1}{\rho_1 + \rho_2} M \left(\left\| \frac{\Delta_m^s x_k}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) + \frac{\rho_2}{\rho_1 + \rho_2} M \left(\left\| \frac{\Delta_m^s y_k}{\rho_2}, z_1, z_2, \dots, z_{n-1} \right\| \right). \quad (3.10)$$

Thus, we have

$$\begin{aligned}
& \sup_k \left[M \left(\left\| \frac{\Delta_m^s(x_k + y_k)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \leq 1, \\
g_\Delta(x + y) &= \sum_{k=1}^{ms} \|x_k + y_k, z_1, z_2, \dots, z_{n-1}\| + \inf \left\{ (\rho_1 + \rho_2)^{p_k/H} : \rho_1 \in A_1, \rho_2 \in A_2 \right\} \\
&\leq \sum_{k=1}^{ms} \|x_k, z_1, z_2, \dots, z_{n-1}\| + \inf \left\{ (\rho_1)^{p_k/H} : \rho_1 \in A_1 \right\} \\
&\quad + \sum_{k=1}^{ms} \|y_k, z_1, z_2, \dots, z_{n-1}\| + \inf \left\{ (\rho_2)^{p_k/H} : \rho_2 \in A_2 \right\} = g_\Delta(x) + g_\Delta(y).
\end{aligned} \tag{3.11}$$

Let $\lambda^s \rightarrow \lambda$, where $\lambda^s, \lambda \in \mathbb{C}$, and let $g_\Delta(x^s - x) \rightarrow 0$ as $s \rightarrow \infty$. We have to show that $g_\Delta(\lambda^s x^s - \lambda x) \rightarrow 0$ as $s \rightarrow \infty$.

We set

$$\begin{aligned}
A_3 &= \left\{ \rho_s > 0 : \sup_k \left[M \left(\left\| \frac{\Delta_m^s x_k^s}{\rho_s}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \leq 1, \forall z_1, z_2, \dots, z_{n-1} \in X \right\}, \\
A_4 &= \left\{ \rho_{s'} > 0 : \sup_k \left[M \left(\left\| \frac{\Delta_m^s (x_k^s - x_k)}{\rho_{s'}}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \leq 1, \forall z_1, z_2, \dots, z_{n-1} \in X \right\}.
\end{aligned} \tag{3.12}$$

If $\rho_s \in A_3$ and $\rho_{s'} \in A_4$, we observe by the continuity of the Orlicz function M that

$$M \left(\left\| \frac{\Delta_m^s (\lambda^s x_k^s - \lambda x_k)}{(|\lambda^s - \lambda| \rho_s + |\lambda| \rho_{s'})}, z_1, z_2, \dots, z_{n-1} \right\| \right) \tag{3.13}$$

$$\begin{aligned}
&\leq M \left(\left\| \frac{\Delta_m^s (\lambda^s x_k^s - \lambda x_k^s)}{(|\lambda^s - \lambda| \rho_s + |\lambda| \rho_{s'})}, z_1, z_2, \dots, z_{n-1} \right\| \right) \\
&\quad + M \left(\left\| \frac{\Delta_m^s (\lambda x_k^s - \lambda x_k)}{(|\lambda^s - \lambda| \rho_s + |\lambda| \rho_{s'})}, z_1, z_2, \dots, z_{n-1} \right\| \right)
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&\leq \frac{|\lambda^s - \lambda| \rho_s}{|\lambda^s - \lambda| \rho_s + |\lambda| \rho_{s'}} M \left(\left\| \frac{\Delta_m^s (x_k^s)}{\rho_s}, z_1, z_2, \dots, z_{n-1} \right\| \right) \\
&\quad + \frac{|\lambda| \rho_{s'}}{|\lambda^s - \lambda| \rho_s + |\lambda| \rho_{s'}} M \left(\left\| \frac{\Delta_m^s (x_k^s - x_k)}{\rho_{s'}}, z_1, z_2, \dots, z_{n-1} \right\| \right).
\end{aligned} \tag{3.15}$$

From the above inequality, it follows that

$$\sup_k \left[M \left(\left\| \frac{\Delta_m^s (\lambda^s x_k^s - \lambda x_k)}{(|\lambda^s - \lambda| \rho_s + |\lambda| \rho_{s'})}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \leq 1, \tag{3.16}$$

and consequently

$$\begin{aligned}
& g_{\Delta}(\lambda^s x^s - \lambda x) \\
&= \sum_{k=1}^{ms} \left\| \lambda^s x_k^s - \lambda x_k, z_1, z_2, \dots, z_{n-1} \right\| + \inf \left\{ (|\lambda^s - \lambda| \rho_s + |\lambda| \rho_s')^{p_k/H} : \rho_s \in A_3, \rho_s' \in A_4 \right\} \\
&\leq |\lambda^s - \lambda| \sum_{k=1}^{ms} \left\| x_k^s, z_1, z_2, \dots, z_{n-1} \right\| + |\lambda^s - \lambda|^{p_k/H} \inf \left\{ (\rho_s)^{p_k/H} : \rho_s \in A_3 \right\} \\
&\quad + |\lambda| \sum_{k=1}^{ms} \left\| x_k^s - x_k, z_1, z_2, \dots, z_{n-1} \right\| + |\lambda|^{p_k/H} \inf \left\{ (\rho_s')^{p_k/H} : \rho_s' \in A_4 \right\} \\
&\leq \max \left\{ |\lambda^s - \lambda|, |\lambda^s - \lambda|^{p_k/H} \right\} g_{\Delta}(x^s) + \max \left\{ |\lambda|, |\lambda|^{p_k/H} \right\} g_{\Delta}(x^s - x).
\end{aligned} \tag{3.17}$$

Note that $g_{\Delta}(x^s) \leq g_{\Delta}(x) + g_{\Delta}(x^s - x)$, for all $s \in \mathbb{N}$.

Hence, by our assumption, the right hand side (3.17) tends to 0 as $s \rightarrow \infty$, and the result follows. This completes the proof of the theorem. $\square$

Theorem 3.3. Let M , M_1 , and M_2 be Orlicz functions. Then, the following hold:

- (a) $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|) \subseteq W_0^I(\Delta_m^s, M.M_1, p, \|\cdot, \dots, \cdot\|)$, provided (p_k) be such that $G_0 = \inf p_k > 0$,
- (b) $W_0^I(\Delta_m^s, M_1, p, \|\cdot, \dots, \cdot\|) \cap W_0^I(\Delta_m^s, M_2, p, \|\cdot, \dots, \cdot\|) \subseteq W_0^I(\Delta_m^s, M_1 + M_2, p, \|\cdot, \dots, \cdot\|)$.

Proof. (a) Let $\varepsilon > 0$ be given. Choose $\varepsilon_1 > 0$ such that $\max\{\varepsilon_1^G, \varepsilon_1^{G_0}\} < \varepsilon$.

Using the continuity of the Orlicz function M , choose $0 < \delta < 1$ such that $0 < t < \delta$ implies that $M(t) < \varepsilon_1$.

Let $x = (x_k)$ be any element in $W_0^I(\Delta_m^s, M_1, p, \|\cdot, \dots, \cdot\|)$. Put

$$A_{\delta} = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M_1 \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \delta^G \right\}. \tag{3.18}$$

Then, by definition of ideal, we have the set $A_{\delta} \in I$.

If $n \notin A_{\delta}$, then we have

$$\begin{aligned}
& \frac{1}{n} \sum_{k=1}^n \left[M_1 \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} < \delta^G \\
& \Rightarrow \sum_{k=1}^n \left[M_1 \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} < n\delta^G \\
& \Rightarrow \left[M_1 \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} < \delta^G, \quad \forall k = 1, 2, \dots, n \\
& \Rightarrow M_1 \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) < \delta, \quad \forall k = 1, 2, \dots, n.
\end{aligned} \tag{3.19}$$

Using the continuity of the Orlicz function M , then from the relation (3.19), we have

$$\sum_{k=1}^n M\left(M_1\left(\left\|\frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right) < \varepsilon_1, \quad \forall k = 1, 2, \dots, n. \quad (3.20)$$

Consequently, we get

$$\begin{aligned} \sum_{k=1}^n \left[M\left(M_1\left(\left\|\frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right) \right]^{p_k} &< n \max\{\varepsilon_1^G, \varepsilon_1^{G_0}\} < n\varepsilon \\ \Rightarrow \frac{1}{n} \sum_{k=1}^n \left[M\left(M_1\left(\left\|\frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right) \right]^{p_k} &< \varepsilon. \end{aligned} \quad (3.21)$$

This shows that

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M\left(M_1\left(\left\|\frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right) \right]^{p_k} \geq \varepsilon, \right\} \subseteq A_\delta \in I. \quad (3.22)$$

This proves the result.

(b) Let $x = (x_k) \in W_0^I(\Delta_m^s, M_1, p, \|\cdot, \dots, \cdot\|) \cap W_0^I(\Delta_m^s, M_2, p, \|\cdot, \dots, \cdot\|)$. Then, by the following inequality, the result follows:

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \left[(M_1 + M_2) \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ \leq D \frac{1}{n} \sum_{k=1}^n \left[M_1 \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ + D \frac{1}{n} \sum_{k=1}^n \left[M_2 \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k}. \end{aligned} \quad (3.23)$$

□

Theorem 3.4. *The inclusions $Z(\Delta_m^{s-1}, M, p, \|\cdot, \dots, \cdot\|) \subseteq Z(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ are strict for $s \geq 1$. In general, $Z(\Delta_m^j, M, p, \|\cdot, \dots, \cdot\|) \subseteq Z(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$, for $j = 0, 1, 2, \dots, s-1$, and the inclusion is strict, where $Z = W^I, W_0^I$, and W_∞^I .*

Proof. We will give the proof for $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ only. The others can be proved by similar arguments.

Let $x = (x_k)$ be any element in the space $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$. Let $\varepsilon > 0$ be given. Then, there exists $\rho > 0$ such that the set

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I. \quad (3.24)$$

Since M is nondecreasing and convex, it follows that

$$\begin{aligned}
& \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{2\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\
&= \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^{s-1} x_{k+1} - \Delta_m^{s-1} x_k}{2\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\
&\leq \frac{D}{n} \sum_{k=1}^n \left[\frac{1}{2} M \left(\left\| \frac{\Delta_m^{s-1} x_{k+1}}{\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\
&\quad + \frac{D}{n} \sum_{k=1}^n \left[\frac{1}{2} M \left(\left\| \frac{\Delta_m^{s-1} x_k}{\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\
&\leq \frac{DH}{n} \sum_{k=1}^n \left(\left[M \left(\left\| \frac{\Delta_m^{s-1} x_{k+1}}{\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} + \left[M \left(\left\| \frac{\Delta_m^{s-1} x_k}{\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right), \tag{3.25}
\end{aligned}$$

where $H = \max\{1, (1/2)^G\}$.

Thus, we have

$$\begin{aligned}
& \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{2\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \\
&\subseteq \left\{ n \in \mathbb{N} : \frac{DH}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^{s-1} x_{k+1}}{\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \tag{3.26} \\
&\cup \left\{ n \in \mathbb{N} : \frac{DH}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^{s-1} x_k}{\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\}.
\end{aligned}$$

Since both the sets in the right side of the relation (3.26) belongs to I , we get the set

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{2\rho}, z_1 z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I. \tag{3.27}$$

Let $M(x) = x^2$, for all $x \in [0, \infty)$, $p_k = 1$, for all $k \in \mathbb{N}$. Consider a sequence $x = (x_k) = ((k+1)^r)$. Then, x belongs to $W_0^I(\Delta_m^{s-1}, M, p, \|\cdot, \dots, \cdot\|)$ but does not belong to $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$. This shows that the inclusion $W_0^I(\Delta_m^{s-1}, M, p, \|\cdot, \dots, \cdot\|) \subset W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ is strict. $\square$

Theorem 3.5. Let $0 < p_k \leq q_k$ and (q_k/p_k) be bounded, then

$$W^I(\Delta_m^s, M, q, \|\cdot, \dots, \cdot\|) \subseteq W^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|). \tag{3.28}$$

Proof. Let $x = (x_k) \in W^I(\Delta_m^s, M, q, \|\cdot, \dots, \cdot\|)$. We put

$$y_k = \left(M \left(\left\| \frac{\Delta_m^s x_k - l}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right)^{q_k} \text{ and } \beta_k = \frac{p_k}{q_k}, \quad (3.29)$$

$\forall k \in \mathbb{N}$. Then, $0 < \beta_k \leq 1$.

For all $k \in \mathbb{N}$. Let β be such that $0 < \beta < \beta_k$, for all $k \in \mathbb{N}$.

Define the sequences (a_k) and (b_k) as follows: for $y_k \geq 1$, let $a_k = y_k$ and $b_k = 0$; for $y_k < 1$, let $a_k = 0$ and $b_k = y_k$.

Then, clearly, for all $k \in \mathbb{N}$, we have $y_k = a_k + b_k$, $y_k^{\beta_k} = a_k^{\beta_k} + b_k^{\beta_k}$; $a_k^{\beta_k} \leq a_k \leq y_k$ and $b_k^{\beta_k} \leq b_k$. Therefore, we have

$$\frac{1}{n} \sum_{k=1}^n y_k^{\beta_k} \leq \frac{1}{n} \sum_{k=1}^n y_k \leq \left[\frac{1}{n} \sum_{k=1}^n b_k \right]^{\beta}. \quad (3.30)$$

Hence, $(x_k) \in W^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$. □

Theorem 3.6. For any two sequences $p = (p_k)$ and $q = (q_k)$ of positive real numbers and for any two n -norms $\|\cdot, \dots, \cdot\|_1$ and $\|\cdot, \dots, \cdot\|_2$ on X , the following holds:

$$Z(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|_1) \cap Z(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|_2) \neq \emptyset, \quad (3.31)$$

where $Z = W^I, W_0^I, W_\infty^I$, and W_∞ .

Proof. Proof of the theorem is obvious, because the zero element belongs to each of the sequence spaces involved in the intersection. □

Theorem 3.7. The sequence spaces $Z(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ are normal as well as monotone, where $Z = W_0^I, W_\infty^I$.

Proof. We will give the proof for $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ only. Let $x = (x_k)$ be any element in $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$, and let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$. Then, we have

$$\begin{aligned} & \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s (\alpha_k x_k)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \\ & \subseteq \left\{ n \in \mathbb{N} : \frac{E}{n} \sum_{k=1}^n \left[M \left(\left\| \frac{\Delta_m^s x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\}, \end{aligned} \quad (3.32)$$

where $E = \max\{1, |\alpha_k|^G\}$.

Hence, $(\alpha_k x_k) \in W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$, for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$, whenever $(x_k) \in W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$.

By Lemma 2.18, we have the space $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ is monotone. □

Remark 3.8. If we replace the difference operator Δ_m^s by $\Delta_m^{(s)}$, then for each $\varepsilon > 0$, we get the following sequence spaces:

$$\begin{aligned} & W^I\left(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|\right) \\ &= \left\{ (x_k) \in w(n-X) : \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M\left(\left\| \frac{\Delta_m^{(s)} x_k - l}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right) \right]^{p_k} \geq \varepsilon \right\} \in I, \right. \\ & \quad \left. \text{for some } \rho > 0, l \in X \text{ and } \forall z_1, z_2, \dots, z_{n-1} \in X \right\}, \end{aligned} \quad (3.33)$$

$$\begin{aligned} & W_0^I\left(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|\right) \\ &= \left\{ (x_k) \in w(n-X) : \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M\left(\left\| \frac{\Delta_m^{(s)} x_k}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right) \right]^{p_k} \geq \varepsilon \right\} \in I, \right. \\ & \quad \left. \text{for some } \rho > 0, l \in X \text{ and } \forall z_1, z_2, \dots, z_{n-1} \in X \right\}, \end{aligned} \quad (3.34)$$

$$\begin{aligned} & W_\infty\left(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|\right) \\ &= \left\{ (x_k) \in w(n-X) : \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^n \left[M\left(\left\| \frac{\Delta_m^{(s)} x_k}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right) \right]^{p_k} < \infty, \right. \\ & \quad \left. \text{for some } \rho > 0, l \in X \text{ and } \forall z_1, z_2, \dots, z_{n-1} \in X \right\}, \end{aligned} \quad (3.35)$$

$$\begin{aligned} & W_\infty^I\left(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|\right) \\ &= \left\{ (x_k) \in w(n-X) : \exists K > 0 \text{ s.t. } \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left[M\left(\left\| \frac{\Delta_m^{(s)} x_k}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right) \right]^{p_k} \geq K \right\} \in I, \right. \\ & \quad \left. \text{for some } \rho > 0, l \in X \text{ and } \forall z_1, z_2, \dots, z_{n-1} \in X \right\}, \end{aligned} \quad (3.36)$$

Note 2. For $s = 0$, we write the above spaces as $Z(M, p, \|\cdot, \dots, \cdot\|)$, where $Z = W^I, W_0^I, W_\infty$, and W_∞^I .

It is clear from definitions that

$$W_0^I\left(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|\right) \subset W^I\left(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|\right) \subset W_\infty\left(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|\right). \quad (3.37)$$

Corollary 3.9. *The sequence spaces $Z(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|)$ are paranormed spaces (not totally paranormed) with respect to the paranorm h_Δ defined by*

$$h_\Delta(x) = \inf \left\{ \rho^{p_k/H} : \sup_k M \left(\left\| \frac{\Delta_m^{(s)} x_k}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \leq 1, \right. \\ \left. \text{for some } \rho > 0 \text{ and } \forall z_1, z_2, \dots, z_{n-1} \in X \right\}, \quad (3.38)$$

where $H = \max\{1, \sup p_k\}$ and $Z = W^I, W_0^I, W_\infty$, and W_∞^I .

Remark 3.10. It is obvious that $(x_k) \in Z(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|)$ if and only if $Z(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|)$, for $Z = W^I, W_0^I, W_\infty^I$ and W_∞ .

Also it is clear that the paranorm g_Δ and h_Δ are equivalent.

We state the following Theorem in view of the Lemma 2.17.

Theorem 3.11. *Let X be a standard n -normed space and $\{e_1, e_2, \dots, e_n\}$ an orthogonal set in X . Then, the following hold:*

- (a) $W^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|_\infty) = W^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|_{(n-1)})$,
- (b) $W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|_\infty) = W_0^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|_{(n-1)})$,
- (c) $W_\infty(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|_\infty) = W_\infty(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|_{(n-1)})$,
- (d) $W_\infty^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|_\infty) = W_\infty^I(\Delta_m^s, M, p, \|\cdot, \dots, \cdot\|_{(n-1)})$, where $\|\cdot, \dots, \cdot\|_\infty$ is the derived $(n-1)$ -norm defined with respect to the set $\{e_1, e_2, \dots, e_n\}$ and $\|\cdot, \dots, \cdot\|_{(n-1)}$ is the standard $(n-1)$ -norm on X .

Remark 3.12. Theorem 3.11 also holds if we replace the difference operator Δ_m^s by the difference operator $\Delta_m^{(s)}$.

Theorem 3.13. *The spaces $Z(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|)$ and $Z(M, p, \|\cdot, \dots, \cdot\|)$ are equivalent as topological spaces, where $Z = W^I, W_0^I, W_\infty^I$, and W_∞ .*

Proof. Consider the mapping $T : Z(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|) \rightarrow Z(M, p, \|\cdot, \dots, \cdot\|)$ defined by

$$Tx = (\Delta_m^{(s)} x_k), \quad \text{for each } x = (x_k) \in Z(\Delta_m^{(s)}, M, p, \|\cdot, \dots, \cdot\|). \quad (3.39)$$

Then, clearly, T is a linear homeomorphism and the proof follows. $\square$

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