Research Article

Computation of the Different Errors in the Ballistic Missiles Range

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The ranges of the ballistic missile trajectories are very sensitive to any kind of errors. Most of the missile trajectory is a part of an elliptical orbit. In this work, the missile problem is stated. The variations in the orbital elements are derived using Lagrange planetary equations. Explicit expressions for the errors in the missile range due to the in-orbit plane changes are derived. Explicit expressions for the errors in the missile range due to the out-of-orbit plane changes are derived when the burnout point is assumed on the equator.

1. Introduction

The fundamental problem of astrodynamics is the orbit determination and orbit correction. For a spacecraft moving under the influence of gravitational field of Earth in free space (no air drag), the trajectory is an ellipse with the center of Earth lying at one of the foci of the ellipse. This constitutes a standard two-body-central-force problem, which has been treated, in detail, in many standard textbooks [1, 2]. The trick is to first reduce the problem to two dimensions by showing that the trajectory always lies in a plane perpendicular to the angular momentum vector. Then, the problem is set up in plane-polar coordinates. Angular momentum is conserved, and the problem, effectively, reduces to one-dimensional problem [3].

A ballistic missile is a missile that follows a suborbital ballistic flight path with the ballistic missile objective of delivering one or more warheads (often nuclear) to a predetermined target. The missile is only guided during the relatively brief initial powered phase of

Figure 1: (a) Geometry of a typical ICBM trajectory. (b) Some orbital parameters at the burnout point. where Γ, Ψ, Ω , and Λ are the powered flight, free flight, reentry, and total range angles, respectively, and R_b, R_{ff}, R_{re} , and R_t are their ground ranges.

flight, and its course is subsequently governed by the laws of orbital mechanics and ballistics. To date, ballistic missiles have been propelled during powered flight by chemical rocket engines of various types. Therefore, the ballistic missile trajectory consists of three parts; see Figure 1(a). (1) *The powered flight portion*, sometimes called boost phase, takes usually from 3 to 5 minutes (shorter for a solid rocket than for a liquid-propellant rocket); the altitude of the missile at the end of this phase is typically 150 to 400 km depending on the trajectory chosen, and the typical burnout speed is 7 km/s. (2) *The free-flight portion*, or the midcourse phase which constitutes most of the flight time, takes approximately 25 minutes. It is a part of an elliptic orbit with a vertical major axis; the apogee is at an altitude of approximately 1,200 km; the semimajor axis is between 3,186 km and 6,372 km; the projection of the orbit on the Earth's surface is close to a great circle, slightly displaced due to Earth rotation during the time of flight; the missile may release several independent warheads, and penetration aids such as metallic-coated balloons, aluminum chaff, and full-scale warhead decoys. (3) *The reentry phase* during which energy is dissipated as a result of friction with the atmosphere, starts at an ill-defined point at an altitude of 100 km. It takes about 2 minutes to impact at a speed of up to 4 km/s (for early ICBMs less than 1 km/s).

Ballistic missiles can be launched from fixed sites or mobile launchers, including vehicles (transporter erector launchers, TELs), aircraft, ships, and submarines. The powered flight portion can last from a few tens of seconds to several minutes and can consist of multiple rocket stages. When in space and no more thrust is provided, the missile enters free flight. In order to cover large distances, ballistic missiles are usually launched into a high suborbital spaceflight; for intercontinental missiles, the highest altitude (apogee) reached during free

flight is about 1200 km. The reentry stage begins at an altitude where atmospheric drag plays a significant role in missile trajectory and lasts until missile impact.

2. Types of Ballistic Missiles

Ballistic missiles are categorized according to their range, the maximum distance measured along the surface of the Earth's ellipsoid from the point of launch of a ballistic missile to the point of impact of the last element of its payload. Various schemes are used by different countries to categorize the ranges of ballistic missiles as follows: tactical ballistic missile: range between about 150 km and 300 km, battlefield range ballistic missile (BRBM), range less than 200 km, theatre ballistic missile (TBM): range between 300 km and 3500 km, short-range ballistic missile (SRBM): range 1000 km or less, medium-range ballistic missile (MRBM): range between 1000 km and 3500 km, intermediate-range ballistic missile (IRBM) or long-range ballistic missile (LRBM): range between 3500 km and 5500 km, intercontinental ballistic missile (ICBM): range greater than 5500 km, and submarine-launched ballistic missile (SLBM): launched from ballistic missile submarines (SSBNs), and all current designs have intercontinental range.

Short and medium-range missiles are often collectively referred to as theater or tactical ballistic missiles (TBMs). Long- and medium-range ballistic missiles are generally designed to deliver nuclear weapons, because their payload is too limited for conventional explosives to be efficient (though the U.S. may be evaluating the idea of a conventionally armed ICBM for near-instant global air strike capability despite the high costs). The flight phases are like those for ICBMs except with no exoatmospheric phase for missiles with ranges less than about 350 km.

Sometimes, the designers of the ballistic missiles need to perform maneuvers in flight or make unexpected changes in direction and range; this type is known as a quasi-ballistic missile or a semi ballistic missile. At a lower trajectory than a ballistic missile, a quasi-ballistic missile can maintain higher speed, thus allowing its target less time to react to the attack, at the cost of reduced range.

3. Literature Survey

McFarland [4] treated ballistic missile problem by modeling spherical Earth, Earth rotation, and addition of atmospheric drag using the state transition matrix. Forden [5] described the integration of the three degrees of freedom equations of motion, and approximations are made to the aerodynamic for simulating ballistic missiles. Bao and Murray [6] improved the ground range calculation of a ballistic missile trajectory on a nonrotating oblate Earth. Isaacson and Vaughan [7] described a method of estimating and predicting ballistic missile trajectories using a Kalman filter over a spherical, nonrotating Earth. They determined uncertainties in the missile launch point and missile position during flight. Harlin and Ciccini [8] developed a method for the determination of the trajectory of a ballistic missile over a rotating, spherical Earth given only the launch position and impact point. The iterative solution presented uses a state transition matrix to correct the initial conditions of the ballistic missile state vector based upon deviations from a desired set of final conditions. Akgül and Karasoy [9] developed a trajectory prediction program to predict the full trajectory of a tactical ballistic missile. Vinh et al. [10] obtained a minimum-fuel interception of a satellite, or a ballistic missile, in elliptic trajectory in a Newtonian central force field, via Lawden's

theory of primer vector. Kamal [11] developed an algorithm includes detection of cross-range error using Lambert scheme in free space in the absence of atmospheric drags. Bhowmik and Sadhukhan [12] investigated the advantages and performance of extended Kalman filter for the estimation of nonlinear system, where linearization takes place about a trajectory that was continually updated with the state estimates resulting from the measurement. They took tactile ballistic missile reentry problem as a nonlinear system model and extended Kalman filter technique is used to estimate the positions and velocities at the X and Y direction at different values of ballistic coefficients. Kamal [13] presented an innovative adaptive scheme which was called "the multistage Lambert scheme". Liu and Chen [14] presented a novel tracking algorithm by integrating input estimation and modified probabilistic data association filter to identify warhead among objects separation from the reentry vehicle in a clear environment.

4. Statement of the Problem

Our ICBM problem concerns with the determination of the free-flight range angle taking into account the perturbation in the orbital elements. Let us define the dimensionless parameter as $Q = (v/v_c)^2 = rv^2/\mu$, where r is the magnitude of the position vector of the missile relative to the Earth, v is the missile speed at any point in its orbit, v_c is the corresponding missile circular speed at this point, and $\mu = G(m_1 + m_2) = \mu = Gm_1$, $m_2 \ll m_1$, where G is the gravitational constant, m_1 is the mass of the Earth, and m_2 is the missile mass. From the orbital mechanics of two body and the symmetry of the free-flight portion shown in Figures 1(a) and 1(b), we have [15]

$$\cos\left(\frac{\Psi}{2}\right) = -\cos f_{bo} = \frac{1 - Q_{bo}\cos^2\phi_{bo}}{\sqrt{1 + Q_{bo}(Q_{bo} - 2)\cos^2\phi_{bo}}}, \quad (4.1)$$

$$\sin\left(2\phi_{bo} + \frac{\Psi}{2}\right) = \frac{2 - Q_{bo}}{Q_{bo}} \sin\left(\frac{\Psi}{2}\right), \quad (4.2)$$

where ϕ_{bo} is the flight path angle, f_{bo} the true anomaly, and Q_{bo} the dimensionless parameter at the burnout point.

Equation (4.1) gives the free-flight range by which the free-flight range angle can be computed for any given combination of burnout conditions r_{bo} , v_{bo} , and ϕ_{bo} . The problem can now be specified as "given a particular launch point and target, it is required to calculate Λ , and knowing Γ , Ω , Ψ can be calculated".

Equation (4.2) gives the flight-path angle, which provides two trajectories to the missile. The trajectory corresponding to the larger value of ϕ_{bo} is called the high trajectory and to the smaller value is the low trajectory. The nature of the trajectory, high or low, depends primarily on the value of Q_{bo} . This is obvious from (4.2), where if

- (1) $Q_{bo} < 1$, Ψ is always less than 180° ; otherwise, the right side of (4.2) exceeds 1, and both high and low trajectories are possible,
- (2) $Q_{bo} = 1$, one trajectory is circular, and for $\Psi < 180^\circ$, both high and low trajectories are possible ($\phi_{bo} = 0$ for low), while high trajectory only is possible for $\Psi > 180^\circ$, and low trajectory skims Earth ($\phi_{bo} = 0$ for high),

- (3) $Q_{bo} > 1$, (4.2) yields one positive and one negative value for ϕ_{bo} regardless of range. The low trajectory, corresponding to the negative value, is not practical, since it would penetrate the Earth.

When the right hand side of (4.2) equals unity, we obtain a single trajectory called the maximum range trajectory, $\sin(\Psi/2) = Q_{bo}/(2 - Q_{bo})$, and the flight path angle is $\phi_{bo} = (1/4)(180^\circ - \Psi)$.

5. In-Orbit-Plane Changes

The variations in the parameters $\{\phi_{bo}, r_{bo}, v_{bo}\}$ due to changes in the orbital elements $\{a, e\}$ take place in the plane of the orbit. Therefore, in what follows, we will compute the errors in Ψ due to changes in the mentioned orbital elements. To do this, we need first the following partial derivatives:

$$\begin{aligned}\frac{\partial \Psi}{\partial \phi_{bo}} &= 2[\sin(\Psi + 2\phi_{bo})\csc 2\phi_{bo} - 1], \\ \frac{\partial \Psi}{\partial r_{bo}} &= \frac{4\mu}{r_{bo}^2 v_{bo}^2} \sin^2 \frac{\Psi}{2} \csc 2\phi_{bo}, \\ \frac{\partial \Psi}{\partial v_{bo}} &= \frac{8\mu}{r_{bo} v_{bo}^3} \sin^2 \frac{\Psi}{2} \csc 2\phi_{bo}.\end{aligned}\tag{5.1}$$

5.1. Error in Ψ due to the Change in the Semimajor Axis

We can write the change in the free flight range angle due to the change in the semimajor axis $\Delta_a \Psi$ as follows:

$$\Delta_a \Psi = \frac{\partial \Psi}{\partial r_{bo}} \frac{\partial r_{bo}}{\partial a} \Delta a + \frac{\partial \Psi}{\partial v_{bo}} \frac{\partial v_{bo}}{\partial a} \Delta a.\tag{5.2}$$

The required derivatives are given by

$$\begin{aligned}\frac{\partial r_{bo}}{\partial a} &= \sum_{n=0}^{\infty} (-1)^n (e^n - e^{n+2}) \cos^n f, \\ \frac{\partial v_{bo}}{\partial a} &= -\frac{1}{2} \sqrt{\frac{\mu}{a^3}} (1 + e^2 + 2e \cos f)^{1/2} (1 - e^2)^{-1/2}.\end{aligned}\tag{5.3}$$

The involved products are

$$\begin{aligned}\frac{\partial \Psi}{\partial r_{bo}} \frac{\partial r_{bo}}{\partial a} &= \frac{4\mu}{r_{bo}^2 v_{bo}^2} \sin^2 \frac{\Psi}{2} \csc 2\phi_{bo} \sum_{n=0}^{\infty} (-1)^n (e^n - e^{n+2}) \cos^n f, \\ \frac{\partial \Psi}{\partial v_{bo}} \frac{\partial v_{bo}}{\partial a} &= -\frac{1}{2} \sqrt{\frac{\mu^3}{a^3}} \frac{8}{r_{bo} v_{bo}^3} \sin^2 \frac{\Psi}{2} \csc 2\phi_{bo} (1 + e^2 + 2e \cos f)^{1/2} (1 - e^2)^{-1/2}.\end{aligned}\tag{5.4}$$

Using the Lagrange planetary equations, we computed Δa taking into account the oblate model of the Earth (retaining the zonal harmonics up to J_4). The integration is performed between (f_{bo}) and $(2\pi - f_{bo})$. Substitution of the obtained expression into (5.2) yields

$$\begin{aligned} \Delta_a \Psi = & \left\{ \frac{4\mu}{r_{bo}^2 v_{bo}^2} \sin^2\left(\frac{\Psi}{2}\right) \csc 2\phi_{bo} \sum_{n=0}^{\infty} (-1)^n (e^n - e^{n+2}) \cos^n f_{bo} \right. \\ & \left. - \sqrt{\frac{\mu^3}{a^3}} \frac{8}{r_{bo} v_{bo}^3} \sin^2\left(\frac{\Psi}{2}\right) \csc 2\phi_{bo} (1 + e^2 + 2e \cos f_{bo})^{1/2} (1 - e^2)^{-1/2} \right\} \\ & \times J_2 \sum_{i=1}^5 \sum_{j=-2}^2 \alpha''_{ij} \sin(if_{bo}) \sin(j\omega) - J_3 \sum_{k=1}^7 \sum_{l=-3}^3 \beta''_{kl} \sin(kf_{bo}) \cos(l\omega) \\ & + J_4 \sum_{m=1}^9 \sum_{n=-4}^4 \gamma''_{mn} \sin(mf_{bo}) \sin(n\omega), \end{aligned} \quad (5.5)$$

where nonvanishing coefficients are given by

$$\begin{aligned} \alpha'_{ij} &= \frac{R^2}{2a(1-e^2)} \alpha_{ij}, & \beta'_{kl} &= \frac{R^3}{4a^2(1-e^2)} \beta_{kl}, & \gamma'_{mn} &= \frac{R^4}{32a^3(1-e^2)} \gamma_{mn}, \\ \alpha_{1,0} &= -\frac{3}{2}(-4e + 6es^2), & \alpha_{1,2} &= -\frac{3}{2}es^2, & \alpha_{2,2} &= -6s^2, \\ \alpha_{3,2} &= \frac{15}{2}es^2, & \beta_{0,1} &= \frac{1}{2}(36es + 45es^3), & \beta_{2,3} &= \frac{5}{2}es^3, \\ \beta_{2,1} &= \frac{1}{2}(-60es + 75es^3), & \beta_{1,1} &= \frac{1}{2}(-24s + 30s^3), & \beta_{3,3} &= -15s^3, \\ \beta_{4,3} &= -\frac{35}{2}es^3, & \gamma_{1,0} &= -\frac{5}{2}(48e - 240es^2 + 210es^4), & \gamma_{5,4} &= -\frac{315}{2}es^4, \\ \gamma_{2,2} &= -\frac{5}{2}(96s^2 - 112s^4), & \gamma_{3,2} &= -\frac{5}{2}(168es^2 - 196es^4), & \gamma_{3,4} &= \frac{35}{2}es^4, \\ \gamma_{4,4} &= -140s^4, & \gamma_{1,2} &= -\frac{5}{2}(-72es^2 - 84es^4), & \beta''_{1,-3} &= \eta^6 \frac{e^3}{8} \beta'_{2,3}, \\ \alpha''_{1,-2} &= \eta^4 \frac{e^2}{4} \alpha'_{1,2}, & \alpha''_{1,0} &= -\eta^4 \left(1 + \frac{e^2}{4}\right) \alpha'_{1,0}, & \alpha''_{2,0} &= -\eta^4 \frac{e}{2} \alpha'_{1,0}, \\ \alpha''_{1,2} &= \eta^4 \left[-\left(1 + \frac{e^2}{2}\right) \alpha'_{1,2} - e \alpha'_{2,2} - \frac{e^2}{4} \alpha'_{3,2} \right], & \alpha''_{3,0} &= -\eta^4 \frac{e^2}{12} \alpha'_{1,0}, \\ \alpha''_{2,2} &= -\eta^4 \left[\frac{e}{2} \alpha'_{1,2} + \frac{1}{2} \left(1 + \frac{e^2}{2}\right) \alpha'_{2,2} + \frac{e}{2} \alpha'_{3,2} \right], & \alpha''_{5,2} &= -\eta^4 \frac{e^2}{20} \alpha'_{3,2}, \\ \alpha''_{3,2} &= -\eta^4 \left[\frac{e^2}{12} \alpha'_{1,2} + \frac{e}{3} \alpha'_{2,2} + \frac{1}{3} \left(1 + \frac{e^2}{2}\right) \alpha'_{3,2} \right], & \beta''_{3,-1} &= \eta^6 \frac{e^3}{24} \beta'_{0,1}, \end{aligned}$$

$$\begin{aligned}
\alpha''_{4,2} &= -\eta^4 \left[\frac{e^2}{16} \alpha'_{2,2} + \frac{e}{4} \alpha'_{3,2} \right], & \beta''_{5,1} &= \eta^6 \frac{e^3}{40} \beta'_{2,1}, \\
\beta''_{1,-1} &= \eta^6 \left[\left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{0,1} + \frac{3e^2}{4} \beta'_{1,1} + \frac{e^3}{8} \beta'_{2,1} \right], & \beta''_{7,3} &= \eta^6 \frac{e^3}{56} \beta'_{4,3}, \\
\beta''_{1,1} &= \eta^6 \left[\left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{0,1} + \left(1 + \frac{3e^2}{2} \right) \beta'_{1,1} + \left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{2,1} \right], & \gamma''_{1,-4} &= \eta^8 \frac{e^4}{16} \gamma'_{3,4}, \\
\beta''_{1,3} &= \eta^6 \left[\left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{2,3} + \frac{3e^2}{4} \beta'_{3,3} + \frac{e^3}{8} \beta'_{4,3} \right], & \gamma''_{3,-2} &= \eta^8 \frac{e^4}{48} \gamma'_{1,2}, \\
\beta''_{2,-1} &= \eta^6 \left[\frac{3e^2}{8} \beta'_{0,1} + \frac{e^3}{16} \beta'_{1,1} \right], & \gamma''_{4,0} &= -\eta^8 \frac{e^3}{8} \gamma'_{1,0}, \\
\beta''_{2,1} &= \eta^6 \left[\frac{3e^2}{8} \beta'_{0,1} + \frac{1}{2} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{1,1} + \frac{1}{2} \left(1 + \frac{3e^2}{2} \right) \beta'_{2,1} \right], & \gamma''_{5,0} &= -\eta^8 \frac{e^4}{80} \gamma'_{1,0}, \\
\beta''_{2,3} &= \eta^6 \left[\frac{1}{2} \left(1 + \frac{3e^2}{2} \right) \beta'_{2,3} + \frac{1}{2} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{3,3} + \frac{3e^2}{8} \beta'_{4,3} \right], & \gamma''_{7,2} &= -\eta^8 \frac{e^4}{112} \gamma'_{3,2}, \\
\beta''_{3,1} &= \eta^6 \left[\frac{e^3}{24} \beta'_{0,1} + \frac{3e^2}{12} \beta'_{1,1} + \frac{1}{3} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{2,1} \right], & \gamma''_{9,4} &= -\eta^8 \frac{e^4}{144} \gamma'_{5,4}, \\
\beta''_{3,3} &= \eta^6 \left[\frac{1}{3} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{2,3} + \frac{1}{3} \left(1 + \frac{3e^2}{2} \right) \beta'_{3,3} + \frac{1}{3} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{4,3} \right], \\
\beta''_{4,1} &= \eta^6 \left[\frac{e^3}{32} \beta'_{1,1} + \frac{3e^2}{16} \beta'_{2,1} \right], \\
\beta''_{4,3} &= \eta^6 \left[\frac{3e^2}{16} \beta'_{2,3} + \frac{1}{4} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{3,3} + \frac{1}{4} \left(1 + \frac{3e^2}{2} \right) \beta'_{4,3} \right], \\
\beta''_{6,3} &= \eta^6 \left[\frac{e^3}{48} \beta'_{3,3} + \frac{3e^2}{24} \beta'_{4,3} \right], \\
\beta''_{5,3} &= \eta^6 \left[\frac{e^3}{40} \beta'_{2,3} + \frac{3e^2}{20} \beta'_{3,3} + \frac{1}{5} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) \beta'_{4,3} \right], \\
\gamma''_{1,-2} &= \eta^8 \left[\left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \gamma'_{1,2} + \frac{e^3}{2} \gamma'_{2,2} + \frac{e^4}{16} \gamma'_{3,2} \right], \\
\gamma''_{1,0} &= \eta^8 \left[\left(\frac{3e^2}{2} + \frac{e^4}{4} \right) - \left(1 + 3e^2 + \frac{3e^4}{8} \right) \right] \gamma'_{1,0}, \\
\gamma''_{1,2} &= -\eta^8 \left[\left(1 + 3e^2 + \frac{3e^4}{8} \right) \gamma'_{1,2} + \left(2e + \frac{3e^3}{2} \right) \gamma'_{2,2} + \left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \gamma'_{3,2} \right], \\
\gamma''_{2,-2} &= \eta^8 \left[\frac{e^3}{4} \gamma'_{1,2} + \frac{e^4}{32} \gamma'_{2,2} \right],
\end{aligned}$$

$$\begin{aligned}
\gamma''_{1,4} &= -\eta^8 \left[\left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \gamma'_{3,4} + \frac{e^3}{2} \gamma'_{4,4} + \frac{e^4}{16} \gamma'_{5,4} \right], & \gamma''_{2,0} &= \eta^8 \left[\frac{e^3}{4} - \frac{1}{2} \left(2e + \frac{3e^3}{2} \right) \right] \gamma'_{1,0}, \\
\gamma''_{2,2} &= -\eta^8 \left[\frac{1}{2} \left(2e + \frac{3e^3}{2} \right) \gamma'_{1,2} + \frac{1}{2} \left(1 + 3e^2 + \frac{3e^4}{8} \right) \gamma'_{2,2} + \frac{1}{2} \left(2e + \frac{3e^3}{2} \right) \gamma'_{3,2} \right], \\
\gamma''_{2,4} &= -\eta^8 \left[\frac{1}{2} \left(2e + \frac{3e^3}{2} \right) \gamma'_{3,4} + \frac{1}{2} \left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \gamma'_{4,4} + \frac{e^3}{4} \gamma'_{5,4} \right], \\
\gamma''_{3,0} &= \eta^8 \left[\frac{e^4}{48} - \frac{1}{3} \left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \right] \gamma'_{1,0}, \\
\gamma''_{3,2} &= -\eta^8 \left[\frac{1}{3} \left(\frac{3}{2} e^2 + \frac{e^4}{4} \right) \gamma'_{1,2} + \frac{1}{3} \left(2e + \frac{3e^3}{2} \right) \gamma'_{2,2} + \frac{1}{3} \left(1 + 3e^2 + \frac{3e^4}{8} \right) \gamma'_{3,2} \right], \\
\gamma''_{3,4} &= -\eta^8 \left[\frac{1}{3} \left(1 + 3e^2 + \frac{3e^4}{8} \right) \gamma'_{3,4} + \frac{1}{3} \left(2e + \frac{3e^3}{2} \right) \gamma'_{4,4} + \frac{1}{3} \left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \gamma'_{5,4} \right], \\
\gamma''_{4,2} &= -\eta^8 \left[\frac{e^3}{8} \gamma'_{1,2} + \frac{1}{4} \left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \gamma'_{2,2} + \frac{1}{4} \left(2e + \frac{3e^3}{2} \right) \gamma'_{3,2} \right], \\
\gamma''_{4,4} &= -\eta^8 \left[\frac{1}{4} \left(2e + \frac{3e^3}{2} \right) \gamma'_{3,4} + \frac{1}{4} \left(1 + 3e^2 + \frac{3e^4}{8} \right) \gamma'_{4,4} + \frac{1}{4} \left(2e + \frac{3e^3}{2} \right) \gamma'_{5,4} \right], \\
\gamma''_{5,2} &= -\eta^8 \left[\frac{e^4}{80} \gamma'_{1,2} + \frac{e^3}{10} \gamma'_{2,2} + \frac{1}{5} \left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \gamma'_{3,2} \right], \\
\gamma''_{5,4} &= -\eta^8 \left[\frac{1}{5} \left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \gamma'_{3,4} + \frac{1}{5} \left(2e + \frac{3e^3}{2} \right) \gamma'_{4,4} + \frac{1}{5} \left(1 + 3e^2 + \frac{3e^4}{8} \right) \gamma'_{5,4} \right], \\
\gamma''_{6,4} &= -\eta^8 \left[\frac{e^3}{12} \gamma'_{3,4} + \frac{1}{6} \left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \gamma'_{4,4} + \frac{1}{6} \left(2e + \frac{3e^3}{2} \right) \gamma'_{5,4} \right], \\
\gamma''_{6,2} &= -\eta^8 \left[\frac{e^4}{96} \gamma'_{2,2} + \frac{e^3}{12} \gamma'_{3,2} \right], \\
\gamma''_{7,4} &= -\eta^8 \left[\frac{e^4}{112} \gamma'_{3,4} + \frac{e^3}{14} \gamma'_{4,4} + \frac{1}{7} \left(\frac{3e^2}{2} + \frac{e^4}{4} \right) \gamma'_{5,4} \right] & \gamma''_{8,4} &= -\eta^8 \left[\frac{e^4}{128} \gamma'_{4,4} + \frac{e^3}{16} \gamma'_{5,4} \right].
\end{aligned} \tag{5.6}$$

5.2. Error in Ψ due to the Change in the Eccentricity

We can write the change in the free flight range angle due to the change in the eccentricity $\Delta_e \Psi$ as follows:

$$\Delta_e \Psi = \frac{\partial \Psi}{\partial \phi_{bo}} \frac{\partial \phi_{bo}}{\partial e} \Delta e + \frac{\partial \Psi}{\partial r_{bo}} \frac{\partial r_{bo}}{\partial e} \Delta e + \frac{\partial \Psi}{\partial v_{bo}} \frac{\partial v_{bo}}{\partial e} \Delta e. \tag{5.7}$$

The required derivatives are given by

$$\begin{aligned}\frac{\partial \phi_{\text{bo}}}{\partial e} &= \sin f \left[1 + e^2 + 2e \cos f \right]^{-1}, \\ \frac{\partial r_{\text{bo}}}{\partial e} &= a \sum_{n=0}^{\infty} (-1)^n \left(n e^{n-1} - (n+2) e^{n+1} \right) \cos^n f, \\ \frac{\partial v_{\text{bo}}}{\partial e} &= \sqrt{\frac{\mu}{a}} \left(1 + e^2 + 2e \cos f \right)^{1/2} \left(1 - e^2 \right)^{-1/2}.\end{aligned}\tag{5.8}$$

The bracket $[1 + e^2 + 2e \cos f]$ vanishes when $\{e = \cos f \pm i \sin f, i = \sqrt{-1}\}$, which is impossible due to the fact that the eccentricity is a real value.

The involved products are

$$\begin{aligned}\frac{\partial \Psi}{\partial \phi_{\text{bo}}} \frac{\partial \phi_{\text{bo}}}{\partial e} &= 2 \sin f \left[1 + e^2 + 2e \cos f \right]^{-1} [\sin(\Psi + 2\phi_{\text{bo}}) \csc 2\phi_{\text{bo}} - 1], \\ \frac{\partial \Psi}{\partial r_{\text{bo}}} \frac{\partial r_{\text{bo}}}{\partial e} &= \frac{4a\mu}{r_{\text{bo}}^2 v_{\text{bo}}^2} \sin^2 \frac{\Psi}{2} \csc 2\phi_{\text{bo}} \sum_{n=0}^{\infty} (-1)^n \left(n e^{n-1} - (n+2) e^{n+1} \right) \cos^n f, \\ \frac{\partial \Psi}{\partial v_{\text{bo}}} \frac{\partial v_{\text{bo}}}{\partial e} &= \frac{8\mu^{3/2}}{r_{\text{bo}} v_{\text{bo}}^3 a^{1/2}} \left(1 + e^2 + 2e \cos f \right)^{1/2} \left(1 - e^2 \right)^{-1/2} \sin^2 \frac{\Psi}{2} \csc 2\phi_{\text{bo}}.\end{aligned}\tag{5.9}$$

Using the Lagrange planetary equations, we computed Δe (retaining the zonal harmonics up to J_4). The integration is performed between (f_{bo}) and $(2\pi - f_{\text{bo}})$. Substitution of the obtained expression into (5.7) yields

$$\begin{aligned}\Delta_e \Psi &= \left\{ 2 \sin f \left[1 + e^2 + 2e \cos f_{\text{bo}} \right]^{-1} [\sin(\Psi + 2\phi_{\text{bo}}) \csc 2\phi_{\text{bo}} - 1] \right. \\ &\quad + \frac{4a\mu}{r_{\text{bo}}^2 v_{\text{bo}}^2} \sin^2 \left(\frac{\Psi}{2} \right) \csc 2\phi_{\text{bo}} \sum_{n=0}^{\infty} (-1)^n \left(n e^{n-1} - (n+2) e^{n+1} \right) \cos^n f_{\text{bo}} \\ &\quad \left. + \frac{16}{r_{\text{bo}} v_{\text{bo}}^3} \sqrt{\frac{\mu^3}{a}} \left(1 + e^2 + 2e \cos f_{\text{bo}} \right)^{1/2} \left(1 - e^2 \right)^{-1/2} \sin^2 \frac{\Psi}{2} \csc 2\phi_{\text{bo}} \right\} \\ &\quad \times \left\{ J_2 \sum_{i=1}^5 \sum_{j=-2}^2 \bar{\delta}_{ij} \cos(i f_{\text{bo}}) \sin(j \omega) \right. \\ &\quad - J_3 \left[2\pi X_1 + \sum_{m=1}^7 \sum_{n=-3}^3 \bar{\sigma}_{mn} \sin(m f_{\text{bo}}) \cos(n \omega) \right] \\ &\quad \left. + J_4 \left[2\pi X_2 + \sum_{k=1}^9 \sum_{l=-4}^4 2\bar{Q}_{kl} \sin(k f_{\text{bo}}) \sin(l \omega) \right] \right\},\end{aligned}\tag{5.10}$$

where

$$X_1 = \eta^4 (48s - 60s^3) \left(1 + \frac{e^2}{2}\right) \cos \omega, \quad X_2 = \eta^4 (720s^2 - 840s^4) e \sin 2\omega, \quad (5.11)$$

where nonvanishing coefficients are given by

$$\begin{aligned} \delta_{1,0} &= -\frac{3R^2(-8+12s^2)}{16a^2(1-e^2)}, & \delta_{2,0} &= -\frac{3R^2e(-4+6s^2)}{16a^2(1-e^2)}, & \delta_{0,2} &= -\frac{3R^2es^2}{16a^2(1-e^2)}, \\ \delta_{1,2} &= \frac{3R^2s^2}{8a^2(1-e^2)}, & \delta_{2,2} &= \frac{9R^2es^2}{4a^2(1-e^2)}, & \delta_{3,2} &= \frac{21R^2s^2}{8a^2(1-e^2)}, \\ \delta_{4,2} &= \frac{15R^2es^2}{16a^2(1-e^2)}, & \sigma_{1,-1} &= \frac{R^3e(36s-45s^3)}{32a^3(1-e^2)}, & \sigma_{0,1} &= \frac{R^3(48s-60s^3)}{32a^3(1-e^2)}, \\ \sigma_{1,1} &= \frac{R^3e(-72s+90s^3)}{32a^3(1-e^2)}, & \sigma_{2,1} &= \frac{R^3(-144s+180s^3)}{32a^3(1-e^2)}, & \sigma_{3,1} &= \frac{R^3e(-60s+75s^3)}{32a^3(1-e^2)}, \\ \sigma_{1,3} &= \frac{5R^3es^3}{32a^3(1-e^2)}, & \sigma_{2,3} &= -\frac{5R^3s^3}{8a^3(1-e^2)}, & \sigma_{3,3} &= -\frac{45R^3es^3}{16a^3(1-e^2)}, \\ \sigma_{4,3} &= -\frac{25R^3s^3}{8a^3(1-e^2)}, & \sigma_{5,3} &= -\frac{35R^3es^3}{32a^3(1-e^2)}, \\ Q_{1,0} &= \frac{2}{e}Q_{2,0} = -\frac{5R^4(96-480s^2+420s^4)}{256a^4(1-e^2)}, & Q_{0,2} &= -\frac{5R^4e(-72s^2+84s^4)}{256a^4(1-e^2)}, \\ Q_{1,2} &= -\frac{5R^4(-48s^2+56s^4)}{256a^4(1-e^2)}, & Q_{2,2} &= -\frac{5R^4e(288s^2-336s^4)}{256a^4(1-e^2)}, & Q_{3,2} &= -\frac{5R^4(432s^2-504s^4)}{256a^4(1-e^2)}, \\ Q_{4,2} &= -\frac{5R^4e(168s^2-196s^4)}{256a^4(1-e^2)}, & Q_{2,4} &= \frac{35R^4es^4}{256a^4(1-e^2)}, & Q_{3,4} &= -\frac{105R^4s^4}{128a^4(1-e^2)}, \\ Q_{4,4} &= -\frac{210R^4es^4}{64a^4(1-e^2)}, & Q_{5,4} &= -\frac{445R^4s^4}{128a^4(1-e^2)}, & Q_{6,4} &= -\frac{315R^4es^4}{256a^4(1-e^2)}, \\ \bar{\delta}_{1,-2} &= \eta^2 \frac{e}{2} \delta_{0,2}, & \bar{\delta}_{1,0} &= \eta^2 \left[-\delta_{1,0} - \frac{e}{2} \delta_{2,0} \right], \\ \bar{\delta}_{1,2} &= \eta^2 \left[-\delta_{1,2} - \frac{e}{2} \delta_{2,2} - \frac{e}{2} \delta_{0,2} \right], & \bar{\delta}_{2,0} &= \eta^2 \left[-\frac{e}{4} \delta_{1,0} - \frac{1}{2} \delta_{2,0} \right], \\ \bar{\delta}_{2,2} &= \eta^2 \left[-\frac{e}{4} \delta_{1,2} - \frac{1}{2} \delta_{2,2} - \frac{e}{4} \delta_{3,2} \right], & \bar{\delta}_{3,0} &= -\eta^2 \frac{e}{6} \delta_{2,0}, & \bar{\delta}_{5,2} &= -\eta^2 \frac{e}{10} \delta_{4,2}, \\ \bar{\delta}_{3,2} &= \eta^2 \left[-\frac{e}{6} \delta_{2,2} - \frac{1}{3} \delta_{3,2} - \frac{e}{6} \delta_{4,2} \right], & \bar{\delta}_{4,2} &= \eta^2 \left[-\frac{e}{8} \delta_{3,2} - \frac{1}{4} \delta_{4,2} \right], \\ \bar{\sigma}_{1,-1} &= \eta^4 \left[e\sigma_{0,1} + \left(1 + \frac{e^2}{2}\right) \sigma_{1,-1} + \frac{e^2}{4} \sigma_{1,1} \right], & \bar{\sigma}_{1,-3} &= \eta^4 \frac{e^2}{4} \sigma_{1,3}, \end{aligned}$$

$$\begin{aligned}
\bar{\sigma}_{1,1} &= \eta^4 \left[e\sigma_{0,1} + \frac{e^2}{4}\sigma_{1,-1} + \left(1 + \frac{e^2}{2}\right)\sigma_{1,1} + e\sigma_{2,1} + \frac{e^2}{4}\sigma_{3,1} \right], \\
\bar{\sigma}_{2,1} &= \eta^4 \left[\frac{e^2}{8}\sigma_{0,1} + \frac{e}{2}\sigma_{1,1} + \frac{1}{2}\left(1 + \frac{e^2}{2}\right)\sigma_{2,1} + \frac{e}{2}\sigma_{3,1} \right], \quad \bar{\sigma}_{2,-1} = \eta^4 \left[\frac{e^2}{8}\sigma_{0,1} + \frac{e}{2}\sigma_{1,-1} \right], \\
\bar{\sigma}_{2,3} &= \eta^4 \left[\frac{e}{2}\sigma_{1,3} + \frac{1}{2}\left(1 + \frac{e^2}{2}\right)\sigma_{2,3} + \frac{e}{2}\sigma_{3,3} + \frac{e^2}{8}\sigma_{4,3} \right], \quad \bar{\sigma}_{3,-1} = \eta^4 \frac{e^2}{12}\sigma_{1,-1}, \\
\bar{\sigma}_{3,1} &= \eta^4 \left[\frac{e^2}{12}\sigma_{1,1} + \frac{e}{3}\sigma_{2,1} + \frac{1}{3}\left(1 + \frac{e^2}{2}\right)\sigma_{3,1} \right], \quad \bar{\sigma}_{4,1} = \eta^4 \left[\frac{e^2}{16}\sigma_{2,1} + \frac{e}{4}\sigma_{3,1} \right], \\
\bar{\sigma}_{4,3} &= \eta^4 \left[\frac{e^2}{16}\sigma_{2,3} + \frac{e}{4}\sigma_{3,3} + \frac{1}{4}\left(1 + \frac{e^2}{2}\right)\sigma_{4,3} + \frac{e}{4}\sigma_{5,3} \right], \\
\bar{\sigma}_{1,3} &= \eta^4 \left[\left(1 + \frac{e^2}{2}\right)\sigma_{1,3} + e\sigma_{2,3} + \frac{e^2}{4}\sigma_{3,3} \right], \\
\bar{\sigma}_{3,3} &= \eta^4 \left[\frac{e^2}{12}\sigma_{1,3} + \frac{e}{3}\sigma_{2,3} + \frac{1}{3}\left(1 + \frac{e^2}{2}\right)\sigma_{3,3} + \frac{e}{3}\sigma_{4,3} + \frac{e^2}{12}\sigma_{5,3} \right], \\
\bar{\sigma}_{5,3} &= \eta^4 \left[\frac{e^2}{20}\sigma_{3,3} + \frac{e}{5}\sigma_{4,3} + \frac{1}{5}\left(1 + \frac{e^2}{2}\right)\sigma_{5,3} \right], \quad \bar{\sigma}_{5,1} = \eta^4 \frac{e^2}{20}\sigma_{3,1}, \\
\bar{Q}_{1,-2} &= \eta^6 \left[\left(\frac{3e}{2} + \frac{3e^3}{8}\right)Q_{0,2} + \frac{3e^2}{4}Q_{1,2} + \frac{e^2}{8}Q_{2,2} \right], \quad \bar{\sigma}_{6,3} = \eta^4 \left[\frac{e^2}{24}\sigma_{4,3} + \frac{e}{6}\sigma_{5,3} \right], \\
\bar{Q}_{1,0} &= \eta^6 \left[-\left(1 + \frac{3e^2}{4}\right)Q_{1,0} - \left(\frac{3e}{2} + \frac{3e^3}{8}\right)Q_{2,0} - \frac{e^3}{8}Q_{2,0} \right], \quad \bar{\sigma}_{7,3} = \eta^4 \frac{e^2}{28}\sigma_{5,3}, \\
\bar{Q}_{1,2} &= \eta^6 \left[-\left(\frac{3e}{2} + \frac{3e^3}{8}\right)Q_{0,2} - \left(1 + \frac{3e^2}{2}\right)Q_{1,2} - \left(\frac{3e}{2} + \frac{3e^3}{8}\right)Q_{2,2} - \frac{3e^2}{4}Q_{3,2} - \frac{e^3}{8}Q_{4,2} \right], \\
\bar{Q}_{2,0} &= \eta^6 \left[-\left(\frac{3e}{4} + \frac{e^3}{8}\right)Q_{1,0} - \frac{1}{2}\left(1 + \frac{3e^2}{2}\right)Q_{2,0} \right], \quad \bar{Q}_{2,-2} = \eta^6 \left[\frac{3e^2}{8}Q_{0,2} + \frac{e^3}{16}Q_{1,2} \right], \\
\bar{Q}_{1,4} &= \eta^6 \left[-\left(\frac{3e}{2} + \frac{3e^3}{8}\right)Q_{2,4} - \frac{3e^2}{4}Q_{3,4} - \frac{e^3}{8}Q_{4,4} \right], \quad \bar{Q}_{7,2} = -\eta^6 \frac{e^3}{56}Q_{4,2}, \\
\bar{Q}_{2,2} &= \eta^6 \left[-\frac{3e^2}{8}Q_{0,2} - \frac{1}{2}\left(\frac{3e}{2} + \frac{3e^3}{8}\right)(Q_{1,2} + Q_{3,2}) - \frac{1}{2}\left(1 + \frac{3e^2}{2}\right)Q_{2,2} - \frac{3e^2}{8}Q_{4,2} \right], \\
\bar{Q}_{2,4} &= \eta^6 \left[-\frac{1}{2}\left(1 + \frac{3e^2}{4}\right)Q_{2,4} - \frac{1}{2}\left(\frac{3e}{2} + \frac{3e^3}{8}\right)Q_{3,4} - \frac{3e^2}{8}Q_{4,4} - \frac{e^3}{16}Q_{5,4} \right], \\
\bar{Q}_{3,2} &= \eta^6 \left[-\frac{e^3}{24}Q_{0,2} - \frac{3e^2}{12}Q_{1,2} - \frac{1}{3}\left(\frac{3e}{2} + \frac{3e^3}{8}\right)Q_{2,2} - \frac{1}{3}\left(1 + \frac{3e^2}{2}\right)Q_{3,2} - \frac{1}{3}\left(\frac{3e}{2} + \frac{3e^3}{8}\right)Q_{4,2} \right],
\end{aligned}$$

$$\begin{aligned}
\bar{Q}_{3,0} &= \eta^6 \left[-\frac{3e^2}{12} Q_{1,0} - \frac{1}{3} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) Q_{2,0} \right], & \bar{Q}_{3,-2} &= \eta^6 \frac{e^3}{24} Q_{0,2}, \\
\bar{Q}_{4,0} &= \eta^6 \left[-\frac{e^3}{32} Q_{1,0} - \frac{3e^2}{16} Q_{2,0} \right], & \bar{Q}_{5,0} &= -\eta^6 \frac{e^3}{40} Q_{2,0}, \\
\bar{Q}_{6,2} &= \eta^6 \left[-\frac{e^3}{48} Q_{3,2} - \frac{3e^2}{24} Q_{4,2} \right], & \bar{Q}_{1,-4} &= \eta^6 \frac{e^3}{8} Q_{2,4}, \\
\bar{Q}_{4,2} &= \eta^6 \left[-\frac{e^3}{32} Q_{1,2} - \frac{3e^2}{16} Q_{2,2} - \frac{1}{4} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) Q_{3,2} - \frac{1}{4} \left(1 + \frac{3e^2}{2} \right) Q_{4,2} \right], \\
\bar{Q}_{4,4} &= \eta^6 \left[-\frac{3e^2}{16} Q_{2,4} - \frac{1}{4} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) (Q_{3,4} + Q_{5,4}) - \frac{1}{4} \left(1 + \frac{3e^2}{2} \right) Q_{4,4} - \frac{3e^2}{16} Q_{5,4} \right], \\
\bar{Q}_{5,4} &= \eta^6 \left[-\frac{e^3}{40} Q_{2,4} - \frac{3e^2}{20} Q_{3,4} - \frac{1}{5} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) (Q_{4,4} + Q_{6,4}) - \frac{1}{5} \left(1 + \frac{3e^2}{2} \right) Q_{5,4} \right], \\
\bar{Q}_{6,4} &= \eta^6 \left[-\frac{e^3}{48} Q_{3,4} - \frac{3e^2}{24} Q_{4,4} - \frac{1}{6} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) Q_{5,4} - \frac{1}{6} \left(1 + \frac{3e^2}{2} \right) Q_{6,4} \right], \\
\bar{Q}_{7,4} &= \eta^6 \left[-\frac{e^3}{56} Q_{4,4} - \frac{3e^2}{28} Q_{5,4} - \frac{1}{7} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) Q_{6,4} \right], & \bar{Q}_{9,4} &= -\eta^6 \frac{e^3}{72} Q_{6,4}, \\
\bar{Q}_{5,2} &= \eta^6 \left[-\frac{e^3}{40} Q_{2,2} - \frac{3e^2}{20} Q_{3,2} - \frac{1}{5} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) Q_{4,2} \right], & \bar{Q}_{8,4} &= \eta^6 \left[-\frac{e^3}{64} Q_{5,4} - \frac{3e^2}{32} Q_{6,4} \right].
\end{aligned} \tag{5.12}$$

6. Out-of-Orbit Plane Changes

All out-of-orbit plane changes, for example, $\Delta\Omega$, Δi will cause a cross-range errors $\Delta\psi_{\times}$.

6.1. Error in Ψ due to the Change in the Ascending Node

For the sake of the simplicity, let us take the burnout point on the equator. For some reason, it was displaced by an amount, Δx . This displacement could be interpreted as a change in the longitude of the ascending node $\Delta\Omega$ if the rest of the orbital elements were kept fixed. Due to this change, a cross-range error, $\Delta_{\Omega}\psi_{\times}$, at impact occurs, the value of which is obtained, in a similar manner as the previous subsection, by applying the law of cosines. Hence,

$$\cos \Delta_{\Omega}\psi_{\times} = \sin^2 \Psi + \cos^2 \Psi \cos \Delta\Omega. \tag{6.1}$$

Since $\Delta_{\Omega}\psi_{\times}$, $\Delta\Omega$ are small angles, then we have

$$\Delta_{\Omega}\psi_{\times} \approx \Delta\Omega \cos \Psi. \tag{6.2}$$

Using the Lagrange planetary equations, we computed $\Delta\Omega$ (retaining the zonal harmonics up to J_4). The integration is performed between (f_{bo}) and $(2\pi - f_{\text{bo}})$. Substitution of the obtained expression into (6.2) yields

$$\begin{aligned} \Delta\Omega\psi_{\times} \approx & \left\{ J_2 \left\{ 2\pi X_8 + \sum_{b=1}^3 \sum_{c=0}^2 -\bar{\rho}_{bc} [(-1)^b + 1] \sin[b f_{\text{bo}} + c\omega] \right\} \right. \\ & + J_3 \left\{ 2\pi X_9 + \sum_{\alpha=1}^5 \sum_{\beta=-1}^3 \bar{q}_{\alpha\beta} [(-1)^\alpha - 1] \cos[\alpha f_{\text{bo}} + \beta\omega] \right\} \\ & \left. + J_4 \left\{ 2\pi X_{10} + \sum_{i=1}^7 \sum_{j=-2}^4 -\bar{p}_{ij} [(-1)^\alpha + 1] \sin[i f_{\text{bo}} + j\omega] \right\} \right\} \cos \Psi, \end{aligned} \quad (6.3)$$

where

$$X_8 = \eta^2 \rho'_{0,0}, \quad X_9 = \eta^4 e q'_{1,1} \sin \omega, \quad X_{10} = \eta^6 \left\{ \left(1 + \frac{3e^2}{2} \right) P'_{0,0} + \frac{3e^2}{4} P'_{2,2} \cos \omega \right\}, \quad (6.4)$$

where nonvanishing coefficients are given by

$$\begin{aligned} \rho'_{bc} &= \frac{R^2}{4a^2 s(1-e^2)} \rho_{bc}, & q'_{\alpha\beta} &= \frac{R^3}{8a^3 s(1-e^2)} q_{\alpha\beta}, & P'_{ij} &= \frac{R^4}{64a^4 s(1-e^2)} P_{ij}, \\ \rho_{0,0} &= -\rho_{2,2} = 6s\sqrt{1-s^2}, & P_{4,4} &= 140s^3\sqrt{1-s^2}, & q_{1,1} &= \sqrt{1-s^2}(45s^2 - 12), \\ q_{3,3} &= -15s^2\sqrt{1-s^2}, & P_{0,0} &= s\sqrt{1-s^2}(-240 + 420s^2), & P_{2,2} &= s\sqrt{1-s^2}(240 - 560s^2), \\ \bar{\rho}_{1,0} &= \eta^2 \left\{ e \rho'_{0,0} \right\}, & \bar{\rho}_{1,2} &= \eta^2 \left\{ \frac{e}{2} \rho'_{2,2} \right\}, & \bar{\rho}_{2,2} &= \eta^2 \left\{ \frac{1}{2} \rho'_{2,2} \right\}, \\ \bar{\rho}_{3,2} &= \eta^2 \left\{ \frac{e}{6} \rho'_{2,2} \right\}, & \bar{q}_{1,-1} &= \eta^4 \left\{ \frac{e^2}{4} q'_{1,1} \right\}, & \bar{q}_{1,1} &= \eta^4 \left\{ -\left(1 + \frac{e^2}{2} \right) q'_{1,1} \right\}, \\ \bar{q}_{1,3} &= \eta^4 \left\{ -\frac{e^2}{4} q'_{3,3} \right\}, & \bar{q}_{2,1} &= \eta^4 \left\{ -\frac{e}{2} q'_{1,1} \right\}, & \bar{q}_{2,3} &= \eta^4 \left\{ -\frac{e}{2} q'_{3,3} \right\}, \\ \bar{q}_{3,1} &= \eta^4 \left\{ -\frac{e^2}{12} q'_{1,1} \right\}, & \bar{q}_{3,3} &= \eta^4 \left\{ -\frac{1}{3} \left(1 + \frac{e^2}{2} \right) q'_{3,3} \right\}, & \bar{q}_{4,3} &= \eta^4 \left\{ -\frac{e}{4} q'_{3,3} \right\}, \\ \bar{q}_{5,3} &= \eta^4 \left\{ -\frac{e^2}{20} q'_{3,3} \right\}, & \bar{P}_{1,-2} &= \eta^6 \left\{ \frac{e^3}{8} P'_{2,2} \right\}, & \bar{P}_{1,0} &= \eta^6 \left\{ 2 \left(\frac{3e}{2} + \frac{3e^3}{8} \right) P'_{0,0} \right\}, \\ \bar{P}_{1,2} &= \eta^6 \left\{ \left(\frac{3e}{2} + \frac{3e^3}{8} \right) P'_{2,2} \right\}, & \bar{P}_{1,4} &= \eta^6 \left\{ \frac{e^3}{8} P'_{4,4} \right\}, & \bar{P}_{2,0} &= \eta^6 \left\{ \frac{6e^2}{8} P'_{0,0} \right\}, \end{aligned}$$

$$\begin{aligned}
\bar{P}_{2,2} &= \eta^6 \left\{ \frac{1}{2} \left(1 + \frac{3e^2}{2} \right) P'_{2,2} \right\}, & \bar{P}_{2,4} &= \eta^6 \left\{ \frac{3e^2}{8} P'_{4,4} \right\}, & \bar{P}_{3,0} &= \eta^6 \left\{ \frac{e^3}{12} P'_{0,0} \right\}, \\
\bar{P}_{3,2} &= \eta^6 \left\{ \frac{1}{3} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) P'_{2,2} \right\}, & \bar{P}_{3,4} &= \eta^6 \left\{ \frac{1}{3} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) P'_{4,4} \right\}, & \bar{P}_{4,2} &= \eta^6 \left\{ \frac{3e^2}{16} P'_{2,2} \right\}, \\
\bar{P}_{4,4} &= \eta^6 \left\{ \frac{1}{4} \left(1 + \frac{3e^2}{2} \right) P'_{4,4} \right\}, & \bar{P}_{5,2} &= \eta^6 \left\{ \frac{e^3}{40} P'_{2,2} \right\}, & \bar{P}_{5,4} &= \eta^6 \left\{ \frac{1}{5} \left(\frac{3e}{2} + \frac{3e^3}{8} \right) P'_{4,4} \right\}, \\
\bar{P}_{6,4} &= \eta^6 \left\{ \frac{3e^2}{24} P'_{4,4} \right\}, & \bar{P}_{7,4} &= \eta^6 \left\{ \frac{e^3}{56} P'_{4,4} \right\}.
\end{aligned} \tag{6.5}$$

6.2. Error in Ψ due to the Change in the Inclination

Again, assume that the burnout point is on the equator and that the actual launch azimuth differs from the intended value by an amount $\Delta\beta$. This amount could be interpreted as a change in the orbital inclination if all other orbital elements were kept fixed. Due to this change, a cross-range error, $\Delta_i\psi_\times$, at impact occurs, the value of which is obtained, in a similar manner as the previous subsection, by applying the law of cosines. Hence,

$$\cos \Delta_i\psi_\times = \cos^2\Psi + \sin^2\Psi \cos \Delta i. \tag{6.6}$$

Since $\Delta_i\psi_\times, \Delta i$ are small angles, then we have

$$\Delta_i\psi_\times \approx \Delta i \sin \Psi. \tag{6.7}$$

Using the Lagrange planetary equations, we computed Δi (retaining the zonal harmonics up to J_4). The integration is performed between (f_{bo}) and $(2\pi - f_{\text{bo}})$. Substitution of the obtained expression into (6.7) yields

$$\begin{aligned}
\Delta_i\psi_\times \approx & \left\{ J_2 \sum_{i=1}^3 Z_i [(-1)^i - 1] \cos[i f_{\text{bo}} + 2\omega] \right. \\
& + J_3 \left\{ 2\pi X_3 + \sum_{k=1}^5 \sum_{l=-1}^3 -\bar{\bar{C}}_{kl} [(-1)^k + 1] \sin[k f_{\text{bo}} + l\omega] \right. \\
& \left. \left. + 2\pi X_9 + \sum_{\alpha=1}^5 \sum_{\beta=-1}^3 \bar{\bar{q}}_{\alpha\beta} [(-1)^\alpha - 1] \cos[\alpha f_{\text{bo}} + \beta\omega] \right\} \right. \\
& \left. + J_4 \left\{ 2\pi X_4 + \sum_{p=1}^7 \sum_{q=-2}^4 \bar{\bar{S}}_{pq} [(-1)^p - 1] \cos[p f_{\text{bo}} + q\omega] \right\} \right\} \sin \Psi,
\end{aligned} \tag{6.8}$$

where

$$X_3 = \eta^4 e \bar{C}_{11}' \cos \omega, \quad X_4 = \eta^6 \frac{3e^2}{4} \bar{S}_{22}' \sin 2\omega, \quad (6.9)$$

where nonvanishing coefficients are given by

$$\begin{aligned} \bar{C}_{1,1} &= 15s^3 - 12s, & \bar{C}_{3,3} &= -15s^3, & \bar{S}_{2,2} &= -240s^2 + 280s^4, \\ \bar{S}_{4,4} &= -140s^4, \\ z &= \frac{R^2}{4a^2(1-e^2)} 6s\sqrt{1-s^2}, & \bar{C}_{kl}' &= \frac{R^3\sqrt{1-s^2}}{8a^3s(1-e^2)} \bar{C}_{kl}, & \bar{S}_{pq}' &= \frac{R^4\sqrt{1-s^2}}{64a^4s(1-e^2)} \bar{S}_{pq}, \\ Z_1 &= -\frac{e}{2}\eta^2 z, & Z_2 &= -\frac{1}{2}\eta^2 z, & Z_3 &= -\frac{e}{6}\eta^2 z, \\ \bar{\bar{C}}_{1,-1} &= \frac{e^2}{4}\eta^4 \bar{C}_{1,1}', & \bar{\bar{C}}_{1,1} &= \left(1 + \frac{e^2}{2}\right)\eta^4 \bar{C}_{1,1}', & \bar{\bar{C}}_{1,3} &= \frac{e^2}{4}\eta^4 \bar{C}_{3,3}', \\ \bar{\bar{C}}_{2,1} &= \frac{e}{2}\eta^4 \bar{C}_{1,1}', & \bar{\bar{C}}_{2,3} &= \frac{e}{2}\eta^4 \bar{C}_{3,3}', & \bar{\bar{C}}_{3,1} &= \frac{e^2}{12}\eta^4 \bar{C}_{1,1}', \\ \bar{\bar{C}}_{3,3} &= \frac{1}{3}\left(1 + \frac{e^2}{2}\right)\eta^4 \bar{C}_{3,3}', & \bar{\bar{C}}_{4,3} &= \frac{e}{4}\eta^4 \bar{C}_{3,3}', & \bar{\bar{C}}_{5,3} &= \frac{e^2}{20}\eta^4 \bar{C}_{3,3}', \\ \bar{\bar{S}}_{1,-2} &= \frac{e^3}{8}\eta^6 \bar{S}_{2,2}', & \bar{\bar{S}}_{1,2} &= -\left(\frac{3e}{2} + \frac{3e^3}{8}\right)\eta^6 \bar{S}_{2,2}', & \bar{\bar{S}}_{1,4} &= -\frac{e^3}{8}\eta^6 \bar{S}_{4,4}', \\ \bar{\bar{S}}_{2,2} &= -\frac{1}{2}\left(1 + \frac{3e^2}{2}\right)\eta^6 \bar{S}_{2,2}', & \bar{\bar{S}}_{2,4} &= -\frac{3e^2}{8}\eta^6 \bar{S}_{4,4}', & \bar{\bar{S}}_{3,2} &= -\frac{1}{3}\left(\frac{3e}{2} + \frac{3e^3}{8}\right)\eta^6 \bar{S}_{2,2}', \\ \bar{\bar{S}}_{3,4} &= -\frac{1}{3}\left(\frac{3e}{2} + \frac{3e^3}{8}\right)\eta^6 \bar{S}_{4,4}', & \bar{\bar{S}}_{4,2} &= -\frac{3e^2}{16}\eta^6 \bar{S}_{2,2}', & \bar{\bar{S}}_{4,4} &= -\frac{1}{4}\left(1 + \frac{3e^2}{2}\right)\eta^6 \bar{S}_{4,4}', \\ \bar{\bar{S}}_{5,2} &= -\frac{e^3}{40}\eta^6 \bar{S}_{2,2}', & \bar{\bar{S}}_{5,4} &= -\frac{1}{5}\left(\frac{3e}{2} + \frac{3e^3}{8}\right)\eta^6 \bar{S}_{4,4}', & \bar{\bar{S}}_{6,4} &= -\frac{3e^2}{24}\eta^6 \bar{S}_{4,4}', \\ \bar{\bar{S}}_{7,4} &= -\frac{e^3}{56}\eta^6 \bar{S}_{4,4}', \end{aligned} \quad (6.10)$$

7. Conclusions and Future Work

Due to the high sensitivity of the ballistic missile range to the different kinds of errors, we computed the explicit expressions for the errors in the missile range due to the in-orbit plane

changes. We derived explicit expressions for the errors in the missile range due to the out-of-orbit plane changes when the burnout point is assumed on the equator. In a forthcoming work, we aim to generalize this situation. Also, we aim to do the corresponding algorithms and give numerical examples.

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