Research Article

Solution of the Lane-Emden Equation Using the Bernstein Operational Matrix of Integration

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Lane-Emden’s equation has fundamental importance in the recent analysis of many problems in relativity and astrophysics including some models of density profiles for dark matter halos. An efficient numerical method is presented for linear and nonlinear Lane-Emden-type equations using the Bernstein polynomial operational matrix of integration. The proposed approach is different from other numerical techniques as it is based on the Bernstein polynomial integration matrix. Some illustrative examples are given to demonstrate the efficiency and validity of the proposed algorithm.

1. Introduction

In recent years, the studies of singular initial value problems in some special second-order ordinary differential equations (ODEs) have attracted the attention of many mathematicians and physicists. One of the most intriguing equations is the Lane-Emden-type equations which models many phenomena in mathematical physics and astrophysics. It is a nonlinear ordinary differential equation which describes the equilibrium density distribution in self-gravitating sphere of polytrophic isothermal gas and has a singularity at the origin. This equation has fundamental importance in the field of radiative cooling and modeling of clusters of galaxies. It has also proven to be most versatile in the examination of a variety of situations, including the analysis of isothermal cores, convective stellar interiors, and fully degenerate stellar configurations. Moreover, it has been recently observed [1–3] that the density profiles of dark matter halos are often modeled by the isothermal Lane-Emden equation with suitable boundary conditions at the origin. Since the solution is often given by some numerical approximation, the chosen method would implies some consequences on the physical interpretation of the dark matter evolution. In the following we will give an efficient method for computing its numerical solution.

Lane-Emden’s equations [4, 5] are categorized as nonlinear ordinary differential equations with singular initial values. The more general Cauchy problem in this category is the following equation:

\[ y''(t) + \frac{\alpha}{t} y'(t) + f(t, y) = g(t), \quad \alpha, t \geq 0, \]  

with initial conditions (ICs)

\[ y(0) = a, \quad y'(0) = 0, \]  

where primes denote differentiation with respect to \( t \), \( \alpha \) is constant, and \( f(t, y) \) is a nonlinear function of \( t \) and \( y \).

It has been shown [6] that there exists an analytic solution of (1), (2) in the neighbourhood of the singular point \( t = 0 \).

In the special case, where \( \alpha = 2 \), \( f(t, y) = f(y) \), \( g(t) = 0 \) and IC (2) holds, we have one of the most studied cases

\[ y''(t) + \frac{2}{t} y'(t) + f(y) = 0, \quad t \geq 0. \]  

(1')
For instance, with \( f(y) = y^n \) and \( a = 1 \), we get
\[
y''(t) + \frac{2}{t} y'(t) + y^n = 0, \quad t \geq 0,
\]
which in the form
\[
\frac{1}{t^2} \frac{d}{dt} \left( t^2 \frac{dy}{dt} \right) + y^n = 0
\]
subject to IC
\[
y(0) = 1, \quad y'(0) = 0
\]
was originally given by Lane [4] and (later) Emden [5].

The parameter \( n \) has physical significance only in the range \( 0 \leq n \leq 5 \). The solution for a given index \( n \) is known as polytropic index \( n \). Equation (3) with IC (5) has well-known analytical solutions [7] for \( n = 0, 1, 5 \) while, for other values of \( n \), numerical solutions are still sought. The series solution can be found by perturbation techniques and Adomian decomposition methods (ADM). However, these solutions are often convergent in restricted regions. Thus, some techniques such as the Padé method are required to enlarge the convergent regions [8, 9].

Similarly, by choosing \( f(t, y) = e^y \) and \( a = 0 \) in (1') and (2), isothermal gas spheres equation are modeled by
\[
y''(t) + \frac{2}{t} y'(t) + e^{y(t)} = 0, \quad t \geq 0,
\]
with IC
\[
y(0) = 0, \quad y'(0) = 0.
\]
A number of methods have recently been proposed to solve (1'), (6). They are quasilinearization methods [10–12], a piecewise linearization technique [13], and the Lagrangian-based analytic solution [14]. The approximate solutions were also given by homotopy analysis method (HAM) [15, 16], variational iteration method [17], and variational approach method [18].

A numerical method based on conversion into integral equations solved by Legendre wavelets is given in [19]. Hybrid functions have also been used in [20] to find the numerical solutions of (1) for some particular nonlinear cases.

In [21] the transform \( t = e^{y} \) to (1’) is given to get
\[
\ddot{y}(x) + f(y(x)) = 0,
\]
such that the conditions
\[
\lim_{x \to -\infty} y(x) = a, \quad \lim_{x \to -\infty} e^{-y} \dot{y}(x) = 0,
\]
where dots denote differentiations with respect to \( x \). Then, an approximate solution of (8) is obtained in \([0, 1]\) by variational iteration method, for special cases when \( f(y) = y^n \) and \( n = 0, 1, 5 \).

Legendre’s spectral method for solving only singular IVPs is given in [22]. In [23], modified homotopy analysis methods (MHAMs) enable to obtain approximate solution and to show that MHAM solution contains the previous solutions obtained by ADM and HPM.

A collocation method based on Chebyshev’s polynomials is proposed in [24]. In [25–27], three different methods are presented, to solve (1), based on the Hermite function collocation method, the Lagrangian method, and radial basis function approximation, respectively. The Jacobi-Gauss collocation method is given in [28]. In [29] the optimal homotopy asymptotic method is applied to obtain the analytic solution of singular Lane-Emden-type equation. The perturbation technique and delta-expansion method are presented in [30, 31], respectively.

The aim of the present paper is to apply the Bernstein polynomial operational matrix of integration for the first time, to propose a reliable numerical technique for solving linear and nonlinear Lane-Emden’s equations. Some special cases of the problem are solved to show its validity and efficiency in comparison with other existing numerical methods. The approximate solution, obtained by the proposed method, shows its superiority on the other existing numerical solution.

### 2. Bernstein Polynomials

A Bernstein polynomial [32] is a polynomial in the Bernstein form that is a linear combination of the Bernstein basis polynomials. The Bernstein basis polynomials of degree \( n \) are defined by
\[
B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad \text{for} \quad i = 0, 1, 2, \ldots, n. \quad (10)
\]
There are \((n + 1)\)-th degree Bernstein basis polynomials forming a basis for the linear space \( V_n \) consisting of all polynomials of degree less than or equal to \( n \) in \( \mathbb{R} \). The ring of polynomials over the field \( \mathbb{R} \). For mathematical convenience, we usually set \( B_{i,n} = 0 \) if \( i < 0 \) or \( i > n \). Any polynomial \( B(t) \) in \( V_n \) may be written as
\[
B(t) = \sum_{i=0}^{n} \beta_i B_{i,n}(t). \quad (11)
\]
Then \( B(t) \) is called a polynomial in the Bernstein form or the Bernstein polynomial of degree \( n \). The coefficients \( \beta_i \) are called Bernstein’s or Bezier’s coefficients. Often, the Bernstein basis polynomials \( B_{i,n}(t) \) are called the Bernstein polynomials. We will follow this convention as well. A function \( f \in L^2[0,1] \) may be written as
\[
f(t) = \lim_{n \to \infty} \sum_{i=0}^{n} c_i B_{i,n}(t), \quad (12)
\]
where \( c_i = \langle \psi, B_{i,n} \rangle \) and \( \langle \cdot, \cdot \rangle \) is the standard inner product on \( L^2[0,1] \).

If (3) is truncated at \( n = m' \), then we have
\[
f \equiv \sum_{i=0}^{m} c_i B_{i,m} = C^T \Psi(t), \quad (13)
\]
where $C$ and $\Psi(t)$ are $(m'+1) \times 1$ matrices given by

$$\begin{align*}
C &= \begin{bmatrix} c_{0,m'} & c_{1,m'} & \cdots & c_{m',m'} \end{bmatrix}^T, \\
\Psi(t) &= \begin{bmatrix} B_{0,m'}(t) & B_{1,m'}(t) & \cdots & B_{m',m'}(t) \end{bmatrix}^T.
\end{align*}$$

For taking the collocation points, let $t_{0}$ be any point near to zero and other point as follows:

$$t_{i} = t_{0} + \frac{i}{m'+1}, \quad i > 0.$$  \hfill (15)

Let us use the notation $m = m'+1$, for defining the Bernstein operational matrix $\Phi_{m \times m}$ as follows:

$$\Phi_{m \times m} = \begin{bmatrix} B_{0,m'}(t_{0}) & B_{1,m'}(t_{1}) & \cdots & B_{m',m'}(t_{m'}) \end{bmatrix}^T.$$  \hfill (16)

For example, when $m = 6$, the Bernstein operational matrix is expressed as

$$\begin{align*}
\Phi_{6 \times 6} &= \begin{bmatrix} 0.0060 & 0.4019 & 0.2622 & 0.0930 & 0.0162 & 0.0006 \\
0.0000 & 0.2024 & 0.3292 & 0.2334 & 0.0816 & 0.0079 \\
0.0000 & 0.0544 & 0.2205 & 0.3125 & 0.2185 & 0.0528 \\
0.0000 & 0.0082 & 0.0830 & 0.2353 & 0.3292 & 0.1995 \\
0.0000 & 0.0007 & 0.0167 & 0.0945 & 0.2646 & 0.4019 \\
0.0000 & 0.0000 & 0.0014 & 0.0158 & 0.0886 & 0.3373
\end{bmatrix}.
\end{align*}$$  \hfill (17)

### 3. Block Pulse Function and Operational Matrix of Integration

A set of block pulse functions (BPFs) is defined on $[0,1)$ as

$$b_{i}(t) = \begin{cases} 1, & \frac{i}{m} \leq t < \frac{i+1}{m}, \\
0, & \text{otherwise,}
\end{cases} \quad \text{where } i = 0, 1, \ldots, m - 1.$$  \hfill (18)

The functions $b_{i}(t)$ are disjoint and orthogonal, that is,

$$b_{i}(t)b_{j}(t) = \begin{cases} 0, & i \neq j, \\
1, & i = j,
\end{cases} \quad \text{and} \quad \int_{0}^{1} b_{i}(t)b_{j}(t) dt = \begin{cases} 0, & i \neq j, \\
\frac{1}{m}, & i = j.
\end{cases}$$  \hfill (19)

The block pulse operational matrix of the integration $F^a$ is defined [33] as following:

$$(I^a B_m)(t) \approx F^a B_m(t),$$  \hfill (20)

where $I^a = \begin{bmatrix} I^a \end{bmatrix}$.

Let us define block pulse functions (BPFs) as:

$$B_m(t) = \begin{bmatrix} b_0(t) & b_1(t) & \cdots & b_{m-1}(t) \end{bmatrix},$$

where $F^a = \begin{bmatrix} 1 & \varepsilon_1 & \cdots & \varepsilon_{m-1} \\
0 & 1 & \varepsilon_1 & \cdots & \varepsilon_{m-2} \\
0 & 0 & 1 & \cdots & \varepsilon_{m-3} \\
0 & 0 & 0 & \cdots & \varepsilon_{m-4} \\
0 & 0 & 0 & \cdots & 1 \end{bmatrix}$,  \hfill (21)

with

$$\varepsilon_k = (k+1)^{a+1}2^{a+1} + (k-1)^{a+1}.$$  \hfill (22)

In general the operational matrix of integration of the vector $\psi_m(t)$ can be obtained as

$$\int_{0}^{1} \psi_m(t) dt \approx P_{m \times m} \psi_m(t),$$  \hfill (23)

where $P$ is the $m \times m$ operational matrix for integration.

The Bernstein polynomial can also be expanded and approximated into an $m$-term block pulse function (BPF) as

$$\psi_m(t) = \Phi_{m \times m} B_m(t).$$  \hfill (24)

Let us consider that the matrix $P_{m \times m}$ is the Bernstein polynomial operational matrix of the integration, then we have

$$(I^a \psi_m)(t) \approx P_{m \times m}^a \psi_m(t).$$  \hfill (25)

Now, we have

$$(I^a \psi_m)(t) \approx (I^a \Phi_{m \times m} B_m)(t) = \Phi_{m \times m} (I^a B_m)(t) \approx \Phi_{m \times m} F^a B_m(t).$$  \hfill (26)

From (25) and (18) we get

$$P_{m \times m}^a \psi_m(t) \approx \Phi_{m \times m} F^a \Phi_{m \times m}^{-1} \psi_m(t).$$  \hfill (27)

Then, the Bernstein polynomial operational matrix of the integration $P_{m \times m}$ is given by

$$P_{m \times m}^a = \Phi_{m \times m} F^a \Phi_{m \times m}^{-1}.$$  \hfill (28)

### 4. Outline of the Method

In this section, the method presented in Section 3 is applied to solve the linear and nonlinear Lane-Emden equations. Letting

$$F(t, y) = f(y) = y^a(t) \quad \text{with } a = 1,$$  \hfill (29)
In this section some special cases of (1) are considered.

5. Numerical Results and Discussions

5.1. Standard Lane-Emden Equation

Consider the standard Lane-Emden equation:

\[ y''(t) + \frac{2}{t} y'(t) + y^n(t) = 0, \tag{36} \]

with initial condition \( y(0) = 1 \) and \( y'(0) = 0 \), which has the exact solution for the case \( n = 0, 1, \) and 5 as follows:

\[ y(t) = (\frac{M}{\pi t})^{\frac{1}{n}}. \]

Table 1: Comparison of the numerical solution and error obtained by present method for \( n = 3 \) in (36) with series solution [25].

<table>
<thead>
<tr>
<th>( t )</th>
<th>( m = 32 )</th>
<th>Our method ( m = 128 )</th>
<th>Our method ( m = 256 )</th>
<th>Series method [25]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00E + 00</td>
<td>0.00E + 00</td>
<td>0.00E + 00</td>
<td>0.00E + 00</td>
</tr>
<tr>
<td>0.1</td>
<td>4.02E - 05</td>
<td>2.51E - 06</td>
<td>6.28E - 07</td>
<td>1.40E - 06</td>
</tr>
<tr>
<td>0.5</td>
<td>2.89E - 05</td>
<td>1.85E - 06</td>
<td>4.96E - 07</td>
<td>2.99E - 06</td>
</tr>
<tr>
<td>1.0</td>
<td>4.39E - 05</td>
<td>3.79E - 05</td>
<td>3.76E - 05</td>
<td>1.99E - 06</td>
</tr>
</tbody>
</table>

(1') can be written as

\[ y''(t) + \frac{2}{t} y'(t) + y^n(t) = 0, \quad \text{with IC } y(0) = 1, \ y'(0) = 0, \tag{30} \]

where \( t, n \geq 0 \).

Since exact solutions for the case \( n = 0, 1, 5 \) are known, we solve them first by the proposed algorithm developed in Section 3.

Let

\[ D^2 y(t) = k_T^m \psi_m(t), \tag{31} \]

where \( k_T^m \) is unknown,

\[ \Rightarrow D y(t) = k_T^m p_1^m x_m \psi_m(t), \]

\[ \Rightarrow y(t) = k_T^m p_2^m x_m \psi_m(t). \tag{32} \]

We have

\[ y(t) = k_T^m p_2^m x_m \psi_m B_m(t) + 1. \tag{33} \]

Assume

\[ k_T^m p_2^m x_m \psi_m B_m(t) + 1 = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix}, \]

\[ [y(t)]^n = \begin{bmatrix} a_1^n & a_2^n & \cdots & a_m^n \end{bmatrix} B_m(t). \tag{34} \]

Substituting (31)–(34) in (30), we get

\[ k_T^m \psi_m B_m(t) \]

\[ + \begin{bmatrix} a_1^n & a_2^n & \cdots & a_m^n \end{bmatrix} B_m(t) = 0. \tag{35} \]

Solution of (35) yields the value of \( k_T^m \).

5.2. Isothermal Gas Spheres Equation

Letting \( f(t, y) = e^y \) with \( a = 0 \), (1) can be written as

\[ y''(t) + \frac{2}{t} y'(t) + e^y = 0, \tag{37} \]

with IC \( y(0) = 0 \), \( y'(0) = 0 \). The isothermal gas spheres are modeled in [6]. Equation (38) is solved with the presented method, and the obtained solution is compared with the existing solutions. The plot of proposed and exact solution is given in Figure 2 and comparison has been done in Table 2.
5.3. Nonlinear Homogeneous Lane-Emden Equation. Let $f(t, y) = 4(2e^y + e^{y^2})$ with $a = 0$, (1) be written as

$$y''(t) + \frac{2}{t}y'(t) + 4\left(2e^y + e^{y^2}\right) = 0, \quad t \geq 0 \quad (38)$$

with IC $y(0) = 0$ and $y'(0) = 0$, where the exact solution is $y(t) = -\ln(1 + t^2)$. We solve the above problem, by applying the technique described in Section 4 with $m = 32, 128$ and $256$ and plotted in Figures 3 and 4, and comparison has been done in Table 3.

6. Conclusions

The Bernstein polynomial operational matrix of integrations has been applied for solving one of the most popular and intriguing differential equations, that is, the Lane-Emden equations. These results are useful in a few respects and deal with some actual state equation for stars. Though these two solutions for $n = 1$ and $n = 0$ share many characteristics, the solution for the polytrope of index $n = 5$ contains some radically different and unexpected characteristics. In this case the behavior of the function is markedly different than that of its predecessors. Here the density of the star $\rho = \lambda y^n$, where $\lambda$ represents the central density of the star and $y$ that of a related dimensionless quantity, initially decreases rapidly as radius increases but slows rapidly once a $t$-value of around
three is reached. At this point the decrease slows continually. Though it may not be apparent on the graphic provided, the function never reaches 0. It is, therefore, evident that a polytropic star of index $n = 5$ has an infinite radius and in reality cannot exist. Despite this fact, such a model provides important theoretical perspective concerning the theory, as one may view this as the border between polytropic one that are physically feasible. It is also of interest to note that such a stellar model has, in spite of the infinite radius, a finite mass. Additionally, other stellar models, which are created in a “layered” fashion where each layer consists of a polytrope of a different index, may also utilize this function for a portion of the star, in which case a finite radius would be possible. In addition to these relations, there are also a number of other conclusions that one can draw from the polytropic model of stars. For relations of this type, there exists a relation between the polytropic index, mass of a star, and the radius. It is perhaps evident in the discussion of the analytic solutions of the polytropic index that one could possibly infer a relation between the polytropic index of the star and the radius that one would calculate from that star. In the attempt to find a relation, the most immediate result is obtained from the simple equations of stellar state.

References


