Research Article

Bayes Estimation of Two-Phase Linear Regression Model

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Let the regression model be $Y_i = \beta_1 X_i + \varepsilon_i$, where ε_i are i. i. d. $N(0, \sigma^2)$ random errors with variance $\sigma^2 > 0$ but later it was found that there was a change in the system at some point of time m and it is reflected in the sequence after X_m by change in slope, regression parameter β_2 . The problem of study is when and where this change has started occurring. This is called change point inference problem. The estimators of m, β_1, β_2 are derived under asymmetric loss functions, namely, Linex loss & General Entropy loss functions. The effects of correct and wrong prior information on the Bayes estimates are studied.

1. Introduction

Regression analysis is an important statistical technique to analyze data in social, medical, and engineering sciences. Quite often, in practice, the regression coefficients are assumed constant. In many real-life problems, however, theoretical or empirical deliberations suggest models with occasionally changing one or more of its parameters. The main parameter of interest in such regression analyses is the shift point parameter, which indexes when or where the unknown change occurred.

A variety of problems, such as switching straight lines [1], shifts of level or slope in linear time series models [2], detection of ovulation time in women [3], and many others, have been studied during the last two decades. Holbert [4], while reviewing Bayesian developments in structural change from 1968 onward, gives a variety of interesting examples from economics and biology. The monograph by Broemeling and Tsurumi [5] provides a complete study of structural change in the linear model from the Bayesian viewpoint.

Bayesian inference of the shift point parameter assumes availability of the prior distribution of the changing model parameters. Bansal and Chakravarty [6] had proposed to study the effect of an ESD prior for the changed slope parameter of the two-phase linear regression (TPLR) model on the Bayes estimates of the shift point and also on the posterior odds ratio (POR) to detect a change in the simple regression model.

In this paper, we studied a TPLR model. In Section 2, we have given a change point model TPLR. In Sections 3.1 and 3.2, we obtained posterior densities of m considering σ^2 unknown and of β_1, β_2 and m considering σ^2 known, respectively. We derive Bayes estimators of β_1, β_2 , and m under symmetric loss functions in Section 4 and asymmetric loss functions in Section 5. We have studied the sensitivity of the Bayes estimators of m when prior specifications deviate from the true values in Section 6. In Section 7, we have presented a numerical study to illustrate the above technique on generated observations. In this study, we have generated observations from the proposed model and have computed the Bayes estimates of m and of other parameters. Section 8 concludes the research paper.

2. Two-Phase Linear Regression Model

The TPLR model is one of the many models, which exhibits structural change. Holbert [4] used a Bayesian approach, based on TPLR model, to reexamine the McGee and Kotz [7] data for stock market sales volume and reached the same conclusion that the abolition of splitups did hurt the regional exchanges.

The TPLR model is defined as

$$y_t = \begin{cases} \alpha_1 + \beta_1 x_t + \varepsilon_t, & t = 1, 2, \dots, m, \\ \alpha_2 + \beta_2 x_t + \varepsilon_t, & t = m + 1, \dots, n, \end{cases} \quad (1)$$

where ε_t 's are i. i. d. $N(0, \sigma^2)$ random errors with variance $\sigma^2 > 0$, x_t is a nonstochastic explanatory variable, and the regression parameters $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$. The shift point m is such that if $m = n$ there is no shift, but when $m = 1, 2, \dots, n-1$ exactly one shift has occurred.

3. Bayes Estimation

The ML method, as well as other classical approaches are based only on the empirical information provided by the data. However, when there is some technical knowledge on the parameters of the distribution available, a Bayes procedure seems to be an attractive inferential method. The Bayes procedure is based on a posterior density, say, $g(\beta_1, \beta_2, \sigma^{-2}, m | z)$, which is proportional to the product of the likelihood function $L(\beta_1, \beta_2, \sigma^{-2}, m | z)$, with a prior joint density, say, $g(\beta_1, \beta_2, \sigma^{-2}, m)$ representing uncertainty on the parameters values

$$g(\beta_1, \beta_2, \theta_1, \theta_2, m | z) = \frac{L(\beta_1, \beta_2, \sigma^{-2}, m | z) \cdot g(\beta_1, \beta_2, \sigma^{-2}, m)}{\sum_{m=1}^{n-1} \int_{\beta_1} \int_{\beta_2} \int_{\sigma^{-2}} L(\beta_1, \beta_2, \sigma^{-2}, m | z) \cdot \mathfrak{Z}}, \quad (2)$$

where $\mathfrak{Z}$ denotes $g(\beta_1, \beta_2, \sigma^{-2}, m) d\beta_1 d\beta_2 d\sigma^{-2}$.

The likelihood function of $\beta_1, \beta_2, \sigma^{-2}$ and m , given the sample information $Z_t = (x_t, y_t), t = 1, 2, \dots, m, m+1, \dots, n$, is

$$L(\beta_1, \beta_2, \sigma^{-2}, m | z) = \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2} \beta_1^2 \frac{s_{m1}}{\sigma^2} + \beta_1 \left(\frac{s_{m3}}{\sigma^2}\right)\right] \times \exp\left[-\frac{1}{2} \left(\frac{s_{n1} - s_{m1}}{\sigma^2}\right) \beta_2^2 + \beta_2 \left[\frac{s_{m4}}{\sigma^2}\right]\right] \times e^{-A/2\sigma^2} \sigma^{-n}, \quad (3)$$

where

$$s_{k1} = \sum_{i=1}^k x_i^2, \quad s_{k2} = \sum_{i=1}^k x_i y_i, \\ A = \sum_{i=1}^n y_i^2 + n\alpha^2 - 2\alpha \sum_{i=1}^n y_i, \quad (4) \\ S_{m3} = S_{m2} + 2\alpha S_{m1}, \\ S_{m4} = S_{n2} - S_{m2} + 2\alpha(S_{n1} - S_{m1}).$$

3.1. Using Gamma Prior on $1/\sigma^2$ and Conditional Informative Priors on β_1, β_2 with σ^{-2} Unknown. We consider the TPLR model (1) with unknown σ^{-2} . As in Broemeling and Tsurumi [5], we suppose that the shift point m is a priori uniformly distributed over the set $\{1, 2, \dots, n-1\}$ and is independent of β_1 and β_2 . We also suppose that some information on β_1 and β_2 are available that can be expressed in terms of conditional prior probability densities on β_1 and β_2 .

We have conditional prior density on β_1 and β_2 given σ^2 , with $N(0, \sigma^2)$ as

$$g_1(\beta_1 | \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{\beta_1^2}{\sigma^2}\right], \quad -\infty < \beta_1 < \infty, \\ g_1(\beta_2 | \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{\beta_2^2}{\sigma^2}\right], \quad -\infty < \beta_2 < \infty. \quad (5)$$

We also suppose that some information on $1/\sigma^2$ is available and that technical knowledge can be given in terms of prior mean μ and coefficient of variation $\emptyset$. We suppose marginal prior distribution of $1/\sigma^2$ to be gamma (c, d) distribution with mean μ

$$g_1\left(\frac{1}{\sigma^2}\right) = \frac{c^d}{\Gamma d} \left(\frac{1}{\sigma^2}\right)^{d-1} e^{-c/\sigma^2}, \quad \sigma^2 > 0, c, d > 0, \quad (6)$$

where Γd is gamma function same as explained in (8).

The integral representation of Γz is as below [Re $z > 0, \text{Re } x > 0$], Gradshteyn and Ryzhik [8, page 934]

$$\Gamma z = x^z \int_0^\infty e^{-xt} t^{z-1} dt. \quad (7)$$

The gamma function (Euler's integral of the second kind) $\Gamma(z)$ [Re $z > 0$], (Euler) Gradshteyn and Ryzhik [8, page 933], is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \quad (8)$$

If the prior information is given in terms of prior mean μ and coefficient of variation $\emptyset$, then the parameters c and d can be obtained by solving

$$d = \frac{1}{\emptyset^2}, \quad c = \frac{1}{\mu \emptyset^2}. \quad (9)$$

Hence, joint prior pdf of $\beta_1, \beta_2, \sigma^{-2}$ and m , say $g_1(\beta_1, \beta_2, \sigma^{-2}, m)$ is

$$g_1(\beta_1, \beta_2, \sigma^{-2}, m) = k_1 e^{-1/2\sigma^2(\beta_1^2 + \beta_2^2)} \left(\frac{1}{\sigma^2}\right)^d e^{-c/\sigma^2}, \quad (10)$$

where

$$k_1 = \frac{c^d}{2\pi \Gamma d(n-1)}. \quad (11)$$

Joint posterior density of $\beta_1, \beta_2, \sigma^{-2}$, and m say, $g(\beta_1, \beta_2, \sigma^{-2}, m | z)$ is

$$g_1(\beta_1, \beta_2, \sigma^{-2}, m | z) = \frac{L(\beta_1, \beta_2, \sigma^2, m | z) g(\beta_1, \beta_2, m) d\beta_1 d\beta_2 d\sigma^{-2}}{h_1(z)} \\ = k_2 \cdot \exp\left[-\frac{1}{2} \cdot \frac{\beta_1^2}{\sigma^2} [S_{m1} + 1] + \beta_1 \frac{S_{m3}}{\sigma^2}\right] \\ \times \exp\left[-\frac{1}{2} \frac{\beta_2^2}{\sigma^2} [(S_{n1} - S_{m1}) + 1] + \beta_2 \frac{S_{m4}}{\sigma^2}\right] \\ \times e^{-((A/2)+c)(1/\sigma^2)} \left(\frac{1}{\sigma^2}\right)^d \frac{(\sigma^2)^{-n/2}}{h_1(z)}, \quad (12)$$

where

$$k_2 = k_1 \frac{1}{(2\pi)^{n/2}}, \quad (13)$$

S_{m3} , S_{m4} , S_{m1} , S_{m2} and A are as given in (4).
 $h_1(z)$ is the marginal density of z given by

$$\begin{aligned} h_1(z) &= \sum_{m=1}^{n-1} \int_0^\infty \int_{-\infty}^\infty L(\beta_1, \beta_2, \sigma^{-2}, m | z) \\ &\quad \cdot g(\beta_1, \beta_2, \sigma^{-2}, m) d\beta_1 d\beta_2 d\sigma^{-2} \\ &= k_2 \sum_{m=1}^{n-1} T_1(m), \end{aligned} \quad (14)$$

where k_2 is as given in (13)

$$T_1(m) = \frac{\Gamma((n/2) + d - 1)}{((A/2) + C - h_{m1} - h_{m2})^{(n/2)+d-1} \mathfrak{x}}, \quad (15)$$

where $\mathfrak{x}$ denotes $\sqrt{S_{m1} + 1} \cdot \sqrt{S_{n1} - S_{m1} + 1}$,

$$\begin{aligned} h_{m1} &= \frac{S_{m3}^2}{2(S_{m1} + 1)}, \\ h_{m2} &= \frac{S_{m4}^2}{2(S_{n1} - S_{m1} + 1)}, \end{aligned} \quad (16)$$

S_{m3} , S_{m4} , S_{m1} , S_{m2} , S_{n1} , and A are as given in (4).

$\Gamma((n/2) + d - 1)$ is the gamma function as explained in (8).

Marginal posterior density of change point m , say $g(m | z)$ is

$$g_1(m | z) = \frac{T_1(m)}{\sum_{m=1}^{n-1} T_1(m)}, \quad (17)$$

$T_1(m)$ is as given in (15).

3.2. Using Conditional Informative Priors on β_1, β_2 with σ^{-2} Known. We consider the TPLR model (1) with known σ^2 . We also assume same prior consideration of β_1, β_2 , and m as explained in Section 3.1.

Hence, joint prior pdf of β_1, β_2 , and m , say $g_2(\beta_1, \beta_2, m)$ is

$$g_2(\beta_1, \beta_2, m) = k_3 \cdot e^{-1/2((\beta_1^2 + \beta_2^2)/\sigma^2)}, \quad (18)$$

where

$$k_3 = \frac{1}{2\pi(n-1)\sigma^2}. \quad (19)$$

Joint posterior density of β_1, β_2 and m say, $g_2(\beta_1, \beta_2, m | z)$ is

$$\begin{aligned} &g_2(\beta_1, \beta_2, m | z) \\ &= k_4 \exp\left[\frac{-1}{2}\beta_1^2\left[\frac{(S_{m1} + 1)}{\sigma^2}\right] + \beta_1\frac{S_{m3}}{\sigma^2}\right] \\ &\quad \times \exp\left[\frac{-1}{2}\beta_2^2\left[\frac{(S_{n1} - S_{m1} + 1)}{\sigma^2}\right] + \beta_2\frac{S_{m4}}{\sigma^2}\right] / h_2(z), \end{aligned} \quad (20)$$

where

$$k_4 = k_3 \frac{1}{(2\pi)^{n/2}} e^{-A/2\sigma^2} \sigma^{-n}, \quad (21)$$

S_{m3} , S_{m4} , S_{m1} , S_{m2} , and A are as given in (4).

Where $h_2(z)$ is the marginal density of z given by

$$\begin{aligned} h_2(z) &= k_4 \sum_{m=1}^{n-1} \int_{-\infty}^\infty \exp\left[-\frac{1}{2}\beta_1^2\left[\frac{(S_{m1} + 1)}{\sigma^2}\right] + \beta_1\frac{S_{m3}}{\sigma^2}\right] d\beta_1 \\ &\quad \times \int_{-\infty}^\infty \exp\left[-\frac{1}{2}\beta_2^2\left[\frac{(S_{n1} - S_{m1} + 1)}{\sigma^2}\right] + \beta_2\frac{S_{m4}}{\sigma^2}\right] d\beta_2. \end{aligned} \quad (22)$$

And the integrals,

$$\begin{aligned} G_{1m} &= \int_{-\infty}^\infty \exp\left[-\frac{1}{2}\beta_2^2\left[\frac{(S_{n1} - S_{m1} + 1)}{\sigma^2}\right] + \beta_2\frac{S_{m4}}{\sigma^2}\right] d\beta_2 \\ &= \frac{\sqrt{2\pi} \cdot e^{(S_{m4}^2/2\sigma^2)(S_{n1} - S_{m1} + 1)}}{\sqrt{(S_{n1} - S_{m1} + 1)/\sigma}}, \end{aligned} \quad (23)$$

$$\begin{aligned} G_{2m} &= \int_{-\infty}^\infty \exp\left[-\frac{1}{2}\beta_1^2\left[\frac{(S_{m1} + 1)}{\sigma^2}\right] + \beta_1\frac{S_{m3}}{\sigma^2}\right] d\beta_1 \\ &= \frac{\sqrt{2\pi} \cdot e^{(S_{m3}^2/2\sigma^2)(S_{m1} + 1)}}{\sqrt{(S_{m1} + 1)/\sigma}}. \end{aligned} \quad (24)$$

So using (23) and (24) results in (22), it reduces to

$$\begin{aligned} h_2(z) &= k_4 \sum_{m=1}^{n-1} T_2(m), \\ T_2(m) &= G_{1m} G_{2m}, \end{aligned} \quad (25)$$

where k_4 is as given in (21). G_{1m} and G_{2m} are as given in (23) and (24).

S_{m3} , S_{m4} , S_{m1} , and S_{m2} are as given in (4).

Marginal posterior density of change point m , β_1 , and β_2 is

$$g_2(m | z) = \frac{T_2(m)}{\sum_{m=1}^{n-1} T_2(m)}, \quad (26)$$

$$g_2(\beta_1 | z) = \frac{k_4}{h_2(z)} \left[\sum_{m=1}^{n-1} e^{-1/2}\beta_1^2\left(\frac{S_{m1} + 1}{\sigma^2}\right) + \beta_1\left(\frac{S_{m3}}{\sigma^2}\right) \right] G_{1m}, \quad (27)$$

$g_2(\beta_2 | z)$

$$= \frac{k_4}{h_2(z)} \left[\sum_{m=1}^{n-1} e^{-1/2}\beta_2^2\left(\frac{S_{n1} - S_{m1} + 1}{\sigma^2}\right) + \beta_2\frac{S_{m4}}{\sigma^2} \right] G_{2m}, \quad (28)$$

k_4 and G_{1m} , G_{2m} , $h_2(z)$ are as given in (21), (23), (24), and (25), respectively.

S_{m3} , S_{m4} , S_{m1} , S_{m2} are as given in (4).

4. Bayes Estimates under Symmetric Loss Function

The Bayes estimator of a generic parameter (or function thereof) α based on a squared error loss (SEL) function:

$$L_1(\alpha, d) = (\alpha - d)^2, \quad (29)$$

where d is a decision rule to estimate α , is the posterior mean. As a consequence, the SEL function relative to an integer parameter,

$$L_1(m, v) = (m - v)^2, \quad m, v = 0, 1, 2, \dots \quad (30)$$

Hence, the Bayesian estimate of an integer-valued parameter under the SEL function $L_1(m, v)$ is no longer the posterior mean and can be obtained by numerically minimizing the corresponding posterior loss. Generally, such a Bayesian estimate is equal to the nearest integer value to the posterior mean. So, we consider the nearest value to the posterior mean as Bayes Estimate.

The Bayes estimator of m under SEL is

$$m^* = \frac{\sum_{m=1}^{n-1} m T_1(m)}{\sum_{m=1}^{n-1} T_1(m)}, \quad (31)$$

$$m^{**} = \frac{\sum_{m=1}^{n-1} m T_2(m)}{\sum_{m=1}^{n-1} T_2(m)}, \quad (32)$$

where $T_1(m)$ and $T_2(m)$ are as given in (15) and (25).

Other Bayes estimators of α based on the loss functions

$$L_2(\alpha, d) = |\alpha - d|, \quad (33)$$

$$L_3(\alpha, d) = \begin{cases} 0, & \text{if } |\alpha - d| < \epsilon, \epsilon > 0, \\ 1, & \text{otherwise,} \end{cases}$$

is the posterior median and posterior mode, respectively.

5. Asymmetric Loss Function

The Loss function $L(\alpha, d)$ provides a measure of the financial consequences arising from a wrong decision rule d to estimate an unknown quantity (a generic parameter or function thereof) α . The choice of the appropriate loss function depends on financial considerations only and is independent from the estimation procedure used. The use of symmetric loss function was found to be generally inappropriate, since for example, an overestimation of the reliability function is usually much more serious than an underestimation.

A useful asymmetric loss, known as the Linex loss function was introduced by Varian [9]. Under the assumption that the minimal loss occurs at α , the Linex loss function can be expressed as

$$L_4(\alpha, d) = \exp[q_1(d - \alpha)] - q_1(d - \alpha) - I, \quad q_1 \neq 0. \quad (34)$$

The sign of the shape parameter q_1 reflects the direction of the asymmetry, $q_1 > 0$ if overestimation is more serious

than underestimation, and vice versa, and the magnitude of q_1 reflects the degree of asymmetry.

The posterior expectation of the Linex loss function is

$$E_\alpha\{L_4(\alpha, d)\} = \exp(q_1 d) E_\alpha\{\exp(-q_1 \alpha)\} - q_1(d - E_\alpha\{\alpha\}) - I, \quad (35)$$

where $E_\alpha\{f(\alpha)\}$ denotes the expectation of $f(\alpha)$ with respect to the posterior density $g(\alpha | z)$. The Bayes estimate α_L^* is the value of d that minimizes $E_\alpha\{L_4(\alpha, d)\}$

$$\alpha_L^* = -\frac{1}{q_1} \ln[E_\alpha\{\exp(-q_1 \alpha)\}] \quad (36)$$

provided that $E_\alpha\{\exp(-q_1 \alpha)\}$ exists and is finite.

5.1. Assuming σ^{-2} Unknown. Minimizing the posterior expectation of the Linex loss function $E_m[L_4(m, d)]$. Where $E_m[L_4(m, d)]$ denotes the expectation of $L_4(m, d)$ with respect to the posterior density $g_1(m | z)$ given in (17), we get the Bayes estimate of m by means of the nearest integer value to (37), say m_L^* , as under. We get the Bayes estimators of m using Linex loss function, respectively, as

$$m_L^* = -\frac{1}{q_1} \ln[E_m\{\exp(-q_1 m)\}] \quad (37)$$

$$= -\frac{1}{q_1} \ln\left[\frac{\sum_{m=1}^{n-1} e^{-q_1 m} T_1(m)}{\sum_{m=1}^{n-1} T_1(m)}\right].$$

Another loss function, called general entropy (GE) loss function, proposed by Calabria and Pulcini [10], is given by

$$L_5(\alpha, d) = \left(\frac{d}{\alpha}\right)^{q_3} - q_3 \ln\left(\frac{d}{\alpha}\right) - I. \quad (38)$$

The Bayes estimate α_E^* is the value of d that minimizes $E_\alpha[L_5(\alpha, d)]$:

$$\alpha_E^* = [E_\alpha(\alpha^{-q_3})]^{-1/q_3} \quad (39)$$

provided that $E_\alpha(\alpha^{-q_3})$ exists and is finite.

Combining the General Entropy Loss with the posterior density (17), we get the estimate m by means of the nearest integer value to (40), say m_E^* , as below. We get the Bayes estimates m_E^* of m using General Entropy loss function as

$$m_E^* = [E_m[m^{-q_3}]]^{-1/q_3} = \left[\frac{\sum_{m=1}^{n-1} m^{-q_3} T_1(m)}{\sum_{m=1}^{n-1} T_1(m)}\right]^{-1/q_3}, \quad (40)$$

where $T_1(m)$ is as given in (15).

5.2. Assuming σ^{-2} Known. Combining the Linex loss with posterior density (26), we get the Bayes estimate of m by means of the nearest integer value to (41), say m_L^{**} as below.

$$m_L^{**} = -\frac{1}{q_1} \ln[E_m\{\exp(-q_1 m)\}] \quad (41)$$

$$= -\frac{1}{q_1} \ln\left[\frac{\sum_{m=1}^{n-1} e^{-q_1 m} T_1(m)}{\sum_{m=1}^{n-1} T_1(m)}\right].$$

Combining the Linex loss with the posterior distributions (27) and (28), respectively, we get the Bayes estimators of β_1 and β_2 using Linex loss function as

$$\begin{aligned}\beta_{1L}^{**} &= -\frac{1}{q_1} \ln \left[E_{\beta_1} e^{-q_1 \beta_1} \right] \\ &= -\frac{1}{q_1} \ln \frac{k_4}{h_2(z)} \sum_{m=1}^{n-1} \frac{\sqrt{2\pi} \cdot e^{(S_{m3}-q_1\sigma^2)^2/2\sigma^2(S_{m1}+1)}}{\sqrt{(S_{m1}+1)/\sigma}} G_{1m},\end{aligned}\quad (42)$$

$$\begin{aligned}\beta_{2L}^{**} &= -\frac{1}{q_1} \ln \frac{k_4}{h_2(z)} \\ &\times \left[\sum_{m=1}^{n-1} \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2} \beta_1^2 \left(\frac{S_{n1} - S_{m1} + 1}{\sigma^2} \right) \right. \right. \\ &\quad \left. \left. + \beta_1 \left(\frac{S_{m4}}{\sigma^2} \right) - q_1 \right\} d\beta_1 \right] G_{2m},\end{aligned}\quad (43)$$

where k_4 , $h_2(z)$, G_{1m} , and G_{2m} are same as given in (21), (22), (23), and (24), respectively. S_{m3} , S_{m4} , S_{m1} , S_{m2} are as given in (4).

Minimizing expectation $E[L_5(m, d)]$ and then taking expectation with respect to posterior density $g_2(m | z)$, we get the estimate m by means of the nearest integer value to (44) say m_E^* , as below. We get the Bayes estimates m_E^* of m using General Entropy loss function as

$$m_E^{**} = [E_1[m^{-q_3}]]^{-1/q_3} = \left[\frac{\sum_{m=1}^{n-1} m^{-q_3} T_2(m)}{\sum_{m=1}^{n-1} T_2(m)} \right]^{-1/q_3}, \quad (44)$$

where $T_2(m)$ is same as given in (25).

Note 1. The confluent hypergeometric function of the first kind ${}_1F_1(a, b; x)$ [11] is a degenerate form of the hypergeometric function ${}_2F_1(a, b, c; x)$ which arises as a solution to the confluent hypergeometric differential equation. It is also known as Kummer's function of the first kind and denoted by ${}_1F_1$, defined as follows:

$${}_1F_1(a, b; x) = \sum_{m=0}^{\infty} \frac{(a, m) x^m}{(b, m) m!}, \quad \text{for } |x| < 1. \quad (45)$$

With Pochhammer coefficients $(a, m) = \Gamma(a + m)/\Gamma(a)$ for $m \geq 1$ and $(a, 0) = 1$ [12, page 755], also has an integral representation

$${}_1F_1(a, b; x) = \int_0^1 \frac{e^{xu} u^{a-1} (1-u)^{b-a-1}}{B(a, b-a)} du. \quad (46)$$

The symbols Γ and B denoting the usual functions gamma and beta, respectively.

When a and b are both integer, some special results are obtained. If $a < 0$, and either $b > 0$ or $b < a$, the series yields a polynomial with a finite number of terms. If integer $b \leq 0$, the function is undefined.

Note 2. ${}_pF_q[\{(a_1), \dots, (a_p)\}, \{(b_1), \dots, (b_q)\}, z]$ is called a generalized hypergeometric series and defined as (Gradshteyn and Ryzhik [8, page 1045]).

${}_pF_q[\{(a_1), \dots, (a_p)\}, \{(b_1), \dots, (b_q)\}, z]$ has series expansion

$$\begin{aligned}{}_pF_q \left[\{(a_1), \dots, (a_p)\}, \{(b_1), \dots, (b_q)\}, z \right] \\ = \sum_{m=0}^{\infty} \frac{(a_1)_m, \dots, (a_p)_m}{(b_1)_m, \dots, (b_q)_m} \frac{z^m}{m!}.\end{aligned}\quad (47)$$

In many special cases hypergeometric ${}_pF_q$ is automatically converted to other functions.

For $p = q + 1$, hypergeometric ${}_pF_q$ [a list, b list, z] has a branch cut discontinuity in the complex z plane running from 1 to ∞ .

Hypergeometric ${}_pF_q$ (Regularized) is finite for all finite values of its argument so long as $p \leq q$.

Note 3. $\beta(x, y)$ is the beta function Euler's integral of the first kind defined as Gradshteyn and Ryzhik [8, pages 948, 950],

$$\begin{aligned}\beta(x, y) &= \int_0^1 t^{x-1} (1-t)^{y-1} dt, \\ \beta(x, y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.\end{aligned}\quad (48)$$

The gamma function is as explained in (8).

Minimizing expected loss function $E[L_5(\beta_i, d)]$ and using posterior distributions (27) and (28), we get the Bayes estimates of β_i , $i = 1, 2$ using General Entropy loss function, respectively, as

$$\begin{aligned}\beta_{iE}^{**} &= \left\{ E \left(\beta_i^{-q_3} \right) \right\}^{-1/q_3}, \quad i = 1, 2, \\ \beta_{1E}^{**} &= \left\{ \frac{k_4}{h_2(z)} \sum_{m=1}^{n-1} J_{1m} G_{1m} \right\}^{-1/q_3}, \\ J_{1m} &= \int_{-\infty}^{\infty} \beta_1^{-q_3} e^{-(1/2\sigma^2)\beta_1^2(S_{m1}+1) + \beta_1 S_{m3}/\sigma^2} d\beta_1 \\ &= \frac{1}{((S_{m1}+1)/\sigma^2)(-1+q_3)} \\ &\times \left\{ k_{5m} \Gamma \left(\frac{1-q_3}{2} \right) {}_1F_1 \left(\frac{q_3}{2}, \frac{1}{2}, -\frac{s_{m3}^2/\sigma^2}{2(S_{m1}+1)} \right) \right\} \\ &+ \left\{ G_{3m} \Gamma \left(1 - \frac{q_3}{2} \right) \text{Hypergeometric } {}_1F_1 \right. \\ &\quad \left. \times \left(\frac{1+q_3}{2}, \frac{3}{2}, -\frac{s_{m3}^2/\sigma^2}{2(S_{m1}+1)\sigma^2} \right) \right\}\end{aligned}$$

$$\begin{aligned}
& + \left\{ G_{4m} \text{ Hypergeometric } p^F q, \right. \\
& \quad \times \left(\left\{ \frac{1}{2}, 1 \right\}, \left\{ 1 - \frac{q_3}{2}, \frac{3}{2} - \frac{q_3}{2} \right\}, \right. \\
& \quad \quad \left. \left. - \frac{s_{m3}^2/\sigma^2}{2(S_{m1} + 1)\sigma^2} \right) \right\}, \\
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
k_{5m} &= (-1)^{-q_3} 2^{(1/2)(-1-q_3)} \left\{ - \frac{(S_{m1} + 1)^2}{s_{m3}^2} \right\}^{-q_3} \\
&\times \left\{ - \frac{s_{m3}^2}{(S_{m1} + 1)^2} \right\}^{-q_3} \exp \left\{ \frac{s_{m3}^2/\sigma^2}{2(S_{m1} + 1)} \right\} \\
&\times \left[\left(\frac{S_{m1} + 1}{\sigma^2} \right)^{q_3/2} (-1 + q_3) \right. \\
&\quad \times \left. \left\{ \sqrt{\frac{S_{m1} + 1}{\sigma^2}} \left((-1)^{q_3} \left(- \frac{S_{m1} + 1}{S_{m3}} \right)^{q_3} \right. \right. \right. \\
&\quad \quad \left. \left. \times \left(- \frac{S_{m3}}{S_{m1} + 1} \right)^{q_3} + 1 \right) \right\} \right], \\
G_{3m} &= \sqrt{2} \frac{S_{m3}}{\sigma^2} \left[(-1)^{q_3} \left(- \frac{S_{m1} + 1}{S_{m3}} \right)^{q_3} \left(- \frac{S_{m3}}{S_{m1} + 1} \right)^{q_3} - 1 \right], \\
G_{4m} &= 2^{(1+q_3)/2} \left[\left\{ - \frac{(S_{m1} + 1)^2}{s_{m3}^2} \right\}^{q_3} \cdot \frac{S_{m3}}{\sigma^2} \right. \\
&\quad \times \left. \left\{ (-1)^{q_3} \left(- \frac{S_{m3}}{S_{m1} + 1} \right)^{q_3} - \left(\frac{S_{m3}}{S_{m1} + 1} \right)^{q_3} \right\} \right]. \\
\end{aligned} \tag{50}$$

Hypergeometric $1^F 1((1 + q_3)/2, 3/2, -(s_{m3}^2/\sigma^2)/2(S_{m1} + 1)\sigma^2)$ and hypergeometric $p^F q(\{1/2, 1\}, \{1 - q_3/2, 3/2 - q_3/2\}, -(s_{m3}^2/\sigma^2)/2(S_{m1} + 1)\sigma^2)$ are hypergeometric functions same as explained in Notes 1 and 2, respectively. k_{5m} , G_{3m} , G_{4m} , and G_{1m} are as explained in (23) and (50), respectively. k_4 is as given in (21). $\Gamma((1 - q_3)/2)$ and $\Gamma(1 - q_3/2)$ are gamma functions same as explained in (8)'. S_{m3} , S_{m4} , S_{m1} , and S_{m2} are as given in (4)

$$\begin{aligned}
\beta_{2E}^{**} &= \left\{ \frac{k_4}{h_2(z)} \sum_{m=1}^{n-1} J_{2m} G_{2m} \right\}^{-1/q_3}, \\
J_{2m} &= \int_{-\infty}^{\infty} \beta_2^{-q_3} e^{-(1/2\sigma^2) \beta_2^2 (S_{n1} - S_{m1} + 1) + \beta_2 (S_{m4}/\sigma^2)} d\beta_2 \\
&= \frac{1}{((S_{n1} - S_{m1} + 1)/\sigma^2)(-1 + q_3)} \\
&\times \left\{ k_{6m} \cdot \Gamma\left(\frac{1 - q_3}{2}\right) \text{ Hypergeometric } 1^F 1 \right. \\
&\quad \times \left. \left(\frac{q_3}{2}, \frac{1}{2}, - \frac{s_{m4}^2/\sigma^2}{2(S_{n1} - S_{m1} + 1)} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left\{ G_{5m} \Gamma\left(1 - \frac{q_3}{2}\right) \text{ Hypergeometric } 1^F 1 \right. \\
& \quad \times \left(\frac{1 + q_3}{2}, \frac{3}{2}, - \frac{s_{m4}^2/\sigma^2}{2(S_{n1} - S_{m1} + 1)} \right) \Big\} \\
& + \left\{ G_{6m} \text{ Hypergeometric } p^F q \right. \\
& \quad \times \left(\left\{ \frac{1}{2}, 1 \right\}, \left\{ 1 - \frac{q_3}{2}, \frac{3}{2} - \frac{q_3}{2} \right\}, \right. \\
& \quad \quad \left. \left. - \frac{s_{m4}^2/\sigma^2}{2(S_{n1} - S_{m1} + 1)} \right) \right\}, \\
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
k_{6m} &= (-1)^{-q_3} 2^{(1/2)(-1-q_3)} \\
&\times \left\{ - \frac{(S_{n1} - S_{m1} + 1)^2}{s_{m4}^2} \right\}^{-q_3} \\
&\times \left\{ - \frac{s_{m4}^2}{(S_{n1} - S_{m1} + 1)^2} \right\}^{-q_3} \\
&\times \exp \left\{ \frac{s_{m4}^2/\sigma^2}{2(S_{n1} - S_{m1} + 1)} \right\} \\
&\times \left[\left(\frac{S_{n1} - S_{m1} + 1}{\sigma^2} \right)^{q_3/2} (-1 + q_3) \right. \\
&\quad \times \left. \left\{ \sqrt{\frac{S_{n1} - S_{m1} + 1}{\sigma^2}} \left((-1)^{q_3} \left(- \frac{S_{n1} - S_{m1} + 1}{S_{m4}} \right)^{q_3} \right. \right. \right. \\
&\quad \quad \left. \left. \times \left(- \frac{S_{m4}}{S_{n1} - S_{m1} + 1} \right)^{q_3} + 1 \right) \right\} \right], \\
G_{5m} &= \sqrt{2} \frac{S_{m4}}{\sigma^2} \left[(-1)^{q_3} \left(- \frac{S_{n1} - S_{m1} + 1}{S_{m4}} \right)^{q_3} \right. \\
&\quad \times \left. \left(- \frac{S_{m4}}{S_{n1} - S_{m1} + 1} \right)^{q_3} - 1 \right], \\
G_{6m} &= 2^{(1+q_3)/2} \left[\left\{ - \frac{(S_{n1} - S_{m1} + 1)^2}{s_{m4}^2} \right\}^{q_3} \frac{S_{m4}}{\sigma^2} \right. \\
&\quad \times \left\{ (-1)^{q_3} \left(- \frac{S_{m4}}{S_{n1} - S_{m1} + 1} \right)^{q_3} \right. \\
&\quad \quad \left. \left. - \left(\frac{S_{m4}}{S_{n1} - S_{m1} + 1} \right)^{q_3} \right\} \right]. \\
\end{aligned} \tag{52}$$

Hypergeometric ${}_1F_1((1 + q_3)/2, 3/2, -(s_{m3}^2/\sigma^2)/2(S_{m1} + 1)\sigma^2)$ and hypergeometric ${}_pF_q(\{1/2, 1\}, \{1 - q_3/2, (3/2) - q_3/2\}, -(s_{m3}^2/\sigma^2)/2(S_{m1} + 1)\sigma^2)$ are hypergeometric functions same as explained in Notes 1 and 2, respectively. k_{6m} , G_{5m} , G_{6m} , and G_{2m} are as explained in (23) and (50), respectively. k_4 is as given in (21).

k_4 is as given in (21). $\Gamma((1 - q_3)/2)$ and $\Gamma(1 - q_3/2)$ are gamma functions same as explained in (8)'. S_{m3} , S_{m4} , S_{m1} , and S_{m2} are as given in (4).

Remark 1. Putting $q_3 = -1$ in (40) and (44), we get the Bayes estimators of m , posterior means under the squared error loss as given in (31) and (32). Note that for $q_3 = -1$, the GE loss function reduces to the squared error loss function.

6. Numerical Study

6.1. Illustration. Let us consider the two-phase regression model

$$\begin{aligned} y_t &= 3x_t + \varepsilon_t, & t &= 1, 2, 3, 4, \\ y_t &= 3.5x_t + \varepsilon_t, & t &= 5, 6, \dots, 15, \end{aligned} \quad (53)$$

where ε_t 's are i.i.d. $N(0, 1)$ random errors. We take the first 15 values of x_t and ε_t from Table 4.1 of Zellner [13] to generate 15 sample values (x_t, y_t) $t = 1, 2, \dots, 15$. The generated sample values are given in Table 1. The β_1, β_2 , and σ^2 themselves were random observations. β_1 and β_2 were from standard normal distribution and precision $1/\sigma^2$ was from gamma distribution with $\mu = 1$ and coefficient of variation $\phi = 1.4$, respectively, in $c = 0.5$, $d = 0.5$.

We have calculated posterior mean, posterior median and posterior mode of m . The results are shown in Table 2.

We also compute the Bayes estimators m_E of m using (40) for unknown σ^2 and (44) for known σ^2 and m_L using (37) for unknown σ^2 and (41) for known σ^2 for data given in Table 1. The results are shown in Table 3.

Table 3 shows that for small values of $|q|$, $q = 0.9, 0.5, 0.2, 0.1$ Linex loss function is almost symmetric and nearly quadratic and the values of the Bayes estimate under such a loss is not far from the posterior mean. Table 3 also shows that for $q_1 = q_3 = 1.5, 1.2$, Bayes estimates are less than actual value of $m = 4$.

It can be seen from Table 3 that positive sign of shape parameter of loss functions reflects overestimation is more serious than underestimation. Thus, problem of overestimation can be solved by taking the value of shape parameter of Linex and General Entropy loss functions positive and high.

For $q_1 = q_3 = -1, -2$, Bayes estimates are quite large than actual value $m = 4$. It can be seen from Table 3 that the negative sign of shape parameter of loss functions reflects underestimation is more serious than overestimation. Thus, problem of underestimation can be solved by taking the value of shape parameters of Linex and General Entropy loss functions negative.

We get Bayes estimators β_{1L}^{**} , β_{2L}^{**} , β_{1E}^{**} , and β_{2E}^{**} of β_1 and β_2 using (42), (43), (49), and (51), respectively, for

the data given in Table 1 and for different value of shape parameter q_1 and q_3 . The results are shown in Table 4.

Tables 3 and 4 show that the values of the shape parameters of Linex and General Entropy loss functions increase, the values of Bayes estimates decrease.

7. Sensitivity of Bayes Estimates

In this section, we study the sensitivity of the Bayes estimator, obtained in Sections 4 and 5 with respect to change in the prior of parameters. The mean μ of gamma prior on σ^{-2} has been used as prior information in computing the parameters c, d of the prior. We have computed posterior mean m^* using (31) and m^{**} using (32) for the data given in Table 1 considering different sets of values of (μ) . Following Calabria and Pulcini [10], we also assume the prior information to be correct if the true value of σ^{-2} is closed to prior mean μ and is assumed to be wrong if σ^{-2} is far from μ . We observed that the posterior mode m^* appears to be robust with respect to the correct choice of the prior density of σ^{-2} and also with a wrong choice of the prior density of σ^{-2} . This can be seen from Table 5.

Table 5 shows that when prior mean $\mu = 1 =$ actual value of σ^{-2} , it means correct choice of prior of σ^{-2} . The values of Bayes estimator posterior mode is 4. It gives correct estimation of change point. Now, when $\mu = 0.5$ and 1.5 (far from true value of $\sigma^{-2} = 1$), it means wrong choice of prior of σ^{-2} . The value of Bayes estimator of posterior mode remains 4. But, posterior mean and posterior median do not remain same for wrong choice of prior of σ^{-2} . Thus, posterior mode is not sensitive with wrong choice of prior density of σ^{-2} . While, posterior mean and posterior median are sensitive with wrong choice of prior density of σ^{-2} .

8. Simulation Study

In Sections 4 and 5, we have obtained Bayes estimates of m on the basis of the generated data given in Table 1 for given values of parameters. To justify the results, we have generated 10,000 different random samples with $m = 4$, $n = 15$, $\beta_1 = 3.2, 3.3, 3.4$, $\beta_2 = 3.5, 3.6, 3.7$ and obtained the frequency distributions of posterior mean, median of m, m_L^*, m_E^* with the correct prior consideration. The result is shown in Tables 2 and 3. The value of shape parameter of the general entropy loss and Linex loss used in simulation study for shift point is taken as 0.1.

We have also simulated several standard normal samples. For each β_1, β_2 , and m and n , 1000 pseudorandom samples from two-phase linear regression model discussed in Section 2 have been simulated and Bayes estimators of change point m has been computed using $q_3 = 0.9$ and for different prior mean μ .

Table 6 leads to conclusion that performance of m_L^*, m_F^* , posterior mode's and posterior median's has better performance than that of posterior mean of change point explained in Sections 4 and 5. 46% values of posterior mean are closed to actual value of change point with correct choice of prior. 62% values of posterior median are closed to actual value

TABLE 1: Zellener data for two-phase linear regression model.

T	x_t	y_t	T	x_t	y_t	T	x_t	y_t
1	1.7	4.6	6	4.1	11.27	11	5.1	16.90
2	2.2	6.4	7	4.2	13.90	12	5.2	17.32
3	2.7	6.9	8	4.5	14.18	13	6.0	17.99
4	3.3	8.0	9	4.7	15.45	14	6.2	18.92
5	3.9	13.75	10	4.8	16.40	15	6.5	19.36

TABLE 2: The values of Bayes estimates of shift point, regression coefficients and precision.

Prior	Bayes estimates of m		
	Posterior Mean	Posterior Median	Posterior Mode
Unknown σ^{-2}	4	5	4
Known σ^{-2}	4	5	4

TABLE 3: The Bayes estimates of change point using asymmetric loss functions.

Prior	Shape parameter		Bayes estimates of change point	
	q_1	q_3	m_L^*	m_E^*
Unknown σ^{-2}	1.5	1.5	2	2
	1.2	1.2	3	3
	0.9	0.9	4	4
	0.5	0.5	4	4
	0.2	0.2	4	4
	0.1	0.1	4	4
	-1.0	-1.0	6	4
	-2.0	-2.0	7	6
Known σ^{-2}	q_1	q_3	m_L^{**}	m_E^{**}
	1.5	1.5	2	2
	1.2	1.2	3	3
	0.9	0.9	4	4
	0.5	0.5	4	4
	0.2	0.2	4	4
	0.1	0.1	4	4
	-1.0	-1.0	5	4
	-2.0	-2.0	6	5

of change point with correct choice of prior. 70% values of posterior mode are close to correct values of change point with correct prior considerations. 65% values of m_L^* are closed to actual values of m . 66% values of m_E^* are closed to actual values of m .

9. Conclusions

In this study, we are discussing the Bayes estimator of shift point, the integer parameter, posterior mean is less appealing. Posterior median and posterior mode appear as better estimators as they would be always integer. Our numerical study showed that the Bayes estimators posterior

TABLE 4: The Bayes estimates of Bayes estimates of regression coefficients using asymmetric loss functions.

	Shape parameter	Bayes estimates of regression co-efficients	
	q_1	β_1 β_{1L}^{**}	β_2 β_{2L}^{**}
Known σ^{-2}	1.5	1.3	2.2
	1.2	1.8	2.4
	0.9	2.5	3.2
	0.5	3.0	3.5
	0.2	3.2	3.7
	0.1	3.4	3.9
Known σ^{-2}	q_3	β_{1E}^{**}	β_{2E}^{**}
	1.5	1.0	2.6
	1.2	1.5	2.8
	0.9	2.9	3.3
	0.5	3.1	3.4
	0.2	3.3	3.6
	0.1	3.5	3.8

TABLE 5: The values of Posterior mean of change point for different values of prior means.

Values of prior mean μ	Bayes estimates of change point		
	Posterior mean	Posterior median	Posterior mode
0.5	5	6	4
1	4	5	4
1.5	6	6	4

TABLE 6: Frequency distributions of the Bayes estimates of the change point.

Bayes estimate	% Frequency for		
	01-02	03-05	06-15
Posterior mean	35	46	19
Posterior median	23	62	15
Posterior mode	12	70	18
m_L^*	20	65	15
m_F^*	22	66	12

mode of m is robust with respect to the correct choice of the prior specifications on σ^{-2} and wrong choice of the prior specifications on σ^{-2} , posterior median and posterior mode are sensitive in case prior specifications on $1/\sigma^2$ deviate

simultaneously from the true values. Here, we discussed regression model with one change point, in practice it may have two or more change point models. One can apply these models to econometric data such as poverty and irrigation.

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