Research Article

Positive Solutions for Boundary Value Problem of Nonlinear Fractional q -Difference Equation

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We investigate the existence of multiple positive solutions to the nonlinear q -fractional boundary value problem ${}_c D_q^a u(t) + a(t)f(u(t)) = 0$, $0 \leq t \leq 1$, $2 < a \leq 3$, $u(0) = D_q^2 u(0) = 0$, $\gamma D_q u(1) + \beta D_q^2 u(1) = 0$, by using a fixed point theorem in a cone.

1. Introduction

Fractional differential calculus is a discipline to which many researchers are dedicating their time, perhaps because of its demonstrated applications in various fields of science and engineering [1]. Many researchers studied the existence of solutions to fractional boundary value problems, for example, [2–5].

The q -difference calculus or quantum calculus is an old subject that was initially developed by Jackson [6, 7], basic definitions and properties of q -difference calculus can be found in the book mentioned in [8].

The fractional q -difference calculus had its origin in the works by Al-Salam [9] and Agarwal [10]. More recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q -difference calculus were made, for example, q -analogues of the integral and differential fractional operators properties such as MittagLeffler function [11], just to mention some.

El-Shahed and Hassan [12] studied the existence of positive solutions of the q -difference boundary value problem:

$$\begin{aligned} -D_q^2 u(t) &= a(t)f(u(t)), \quad 0 \leq t \leq 1, \\ \alpha u(0) - \beta D_q u(0) &= 0, \\ \gamma u(1) + \delta D_q u(1) &= 0. \end{aligned} \tag{1.1}$$

Ferreira [13] considered the existence of positive solutions to nonlinear q -difference boundary value problem:

$$\begin{aligned} \left({}_{\text{RL}}D_q^\alpha u\right)(t) &= -f(t, u(t)), \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \\ u(0) &= u(1) = 0. \end{aligned} \quad (1.2)$$

In other paper, Ferreira [14] studied the existence of positive solutions to nonlinear q -difference boundary value problem:

$$\begin{aligned} \left({}_{\text{RL}}D_q^\alpha u\right)(t) &= -f(t, u(t)), \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \\ u(0) &= (D_q u)(0) = 0, \quad (D_q y)(1) = \beta \geq 0. \end{aligned} \quad (1.3)$$

In this paper, we consider the existence of positive solutions to nonlinear q -difference equation:

$${}_CD_q^\alpha u + a(t)f(u(t)) = 0, \quad 0 \leq t \leq 1, \quad 2 < \alpha \leq 3, \quad (1.4)$$

with the boundary conditions

$$\begin{aligned} u(0) &= D_q^2 u(0) = 0, \\ \gamma D_q u(1) + \beta D_q^2 u(1) &= 0, \end{aligned} \quad (1.5)$$

where $\gamma, \beta \geq 0$ and ${}_CD_q^\alpha$ is the fractional q -derivative of the Caputo type.

2. Preliminaries of Fractional q -Calculus

Let $q \in (0, 1)$ and define [8]

$$[\alpha]_q = \frac{q^\alpha - 1}{q - 1} = q^{\alpha-1} + \cdots + 1, \quad \alpha \in \mathfrak{R}. \quad (2.1)$$

The q -analogue of the power $(a - b)^n$ is

$$(a - b)^{(0)} = 1, \quad (a - b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k), \quad a, b \in \mathfrak{R}, \quad n \in \mathbb{N}. \quad (2.2)$$

If α is not a positive integer, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{i=0}^{\infty} \frac{(1 - (b/a)q^i)}{(1 - (b/a)q^{\alpha+i})}. \quad (2.3)$$

Note that if $b = 0$, then $a^{(a)} = a^a$. The q -gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathfrak{R} \setminus \{0, -1, -2, \dots\}, \quad 0 < q < 1 \quad (2.4)$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q -derivative of a function f is here defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x} \quad (2.5)$$

and q -derivatives of higher order by

$$D_q^n f(x) = \begin{cases} f(x) & \text{if } n = 0, \\ D_q D_q^{n-1} f(x) & \text{if } n \in N. \end{cases} \quad (2.6)$$

The q -integral of a function f defined in the interval $[0, b]$ is given by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad 0 \leq |q| < 1, \quad x \in [0, b]. \quad (2.7)$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t. \quad (2.8)$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x), \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in N. \quad (2.9)$$

The fundamental theorem of calculus applies to these operators I_q and D_q , that is,

$$(D_q I_q f)(x) = f(x), \quad (2.10)$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0). \quad (2.11)$$

Basic properties of the two operators can be found in the book mentioned in [8]. We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i) [13]

$$[a(t-s)]^{(\alpha)} = a^\alpha(t-s)^{(\alpha)}, \quad (2.12)$$

$${}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)}, \quad (2.13)$$

$$\left({}_xD_q \int_0^x f(x,t) d_q t \right) (x) = \int_0^x {}_xD_q f(x,t) d_q t + f(qx, x). \quad (2.14)$$

Remark 2.1. We note that if $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$ [13].

The following definition was considered first in [10].

Definition 2.2. Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional q -integral of the Riemann-Liouville type is $({}_RLI_q^\alpha f)(x) = f(x)$ and

$$\left({}_RLI_q^\alpha f \right) (x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x-qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha \in \mathfrak{R}^+, \quad x \in [0, 1]. \quad (2.15)$$

Definition 2.3 (see [15]). The fractional q -derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $({}_RLD_q^\alpha f)(x) = f(x)$ and

$$\left({}_RLD_q^\alpha f \right) (x) = \left(D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f \right) (x), \quad \alpha > 0, \quad (2.16)$$

where $[\alpha]$ is the smallest integer greater than or equal to α .

Definition 2.4 (see [15]). The fractional q -derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$\left({}_CD_q^\alpha f \right) (x) = \left(I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f \right) (x), \quad \alpha > 0, \quad (2.17)$$

where $[\alpha]$ is the smallest integer greater than or equal to α .

Lemma 2.5. Let $\alpha, \beta \geq 0$ and let f be a function defined on $[0, 1]$. Then, the next formulas hold:

- (1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x),$
- (2) $(D_q^\alpha I_q^\alpha f)(x) = f(x).$

The next result is important in the sequel. It was proved in a recent work by the author of [13].

Theorem 2.6. Let $\alpha > 0$ and $n \in \mathbb{N}$. Then, the following equality holds:

$$\left({}_RLI_q^\alpha {}_RLD_q^n f \right) (x) = {}_RLD_q^n {}_RLI_q^\alpha f(x) - \sum_{k=0}^{\alpha-1} \frac{x^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} \left(D_q^k f \right) (0). \quad (2.18)$$

Theorem 2.7 (see [15]). *Let $\alpha > 0$ and $n \in \mathfrak{R}^+ \setminus N$. Then, the following equality holds:*

$$\left(I_q^\alpha {}_C D_q^\alpha f\right)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{x^k}{\Gamma_q(k+1)} \left(D_q^k f\right)(0). \quad (2.19)$$

3. Fractional Boundary Value Problem

We will consider now the question of existence of positive solutions to the following problem:

$$\begin{aligned} {}_C D_q^\alpha u(t) + a(t)f(u(t)) &= 0, \quad 0 \leq t \leq 1, \quad 2 < \alpha \leq 3, \\ u(0) &= D_q^2 u(0) = 0, \\ \gamma D_q u(1) + \beta D_q^2 u(1) &= 0, \end{aligned} \quad (3.1)$$

where $\gamma, \beta \geq 0$, ${}_C D_q^\alpha$ is the fractional q -derivative of the Caputo type, and $f \in C([0, \infty), [0, \infty))$, $a \in C([0, 1], [0, \infty))$ such that $\int_0^1 a(s) d_q s > 0$. To that end, we need the following theorem.

Theorem 3.1 (see [16, 17]). *Let X be a Banach space, and $P \subset X$ is a cone in X . Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$.*

Let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow P$ be completely continuous operator. In addition suppose either:

P1: $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$ or

P2: $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$ and $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$,

holds. Then, T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 3.2. *Let $y \in C[0, 1]$; then the boundary value problem*

$$\begin{aligned} {}_C D_q^\alpha u(t) + y(t) &= 0, \quad 0 \leq t \leq 1, \quad 2 < \alpha \leq 3, \\ u(0) &= D_q^2 u(0) = 0, \\ \gamma D_q u(1) + \beta D_q^2 u(1) &= 0 \end{aligned} \quad (3.2)$$

has a unique solution

$$u(t) = \int_0^1 G(t, qs) y(s) d_q s, \quad (3.3)$$

where

$$G(t, s) = \begin{cases} \frac{t(1-s)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} + \frac{\beta t(1-s)^{(\alpha-3)}}{\gamma \Gamma_q(\alpha-2)} & 0 \leq t \leq s \leq 1, \\ \frac{t(1-s)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} + \frac{\beta t(1-s)^{(\alpha-3)}}{\gamma \Gamma_q(\alpha-2)} - \frac{(t-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} & 0 \leq s \leq t \leq 1. \end{cases} \quad (3.4)$$

Proof. We may apply Lemma 2.5 and Theorem 2.7; we see that

$$u(t) = u(0) + \frac{D_q u(0)}{\Gamma_q(2)} t + \frac{D_q^2 u(0)}{\Gamma_q(3)} t^2 - I_q^\alpha y(t). \quad (3.5)$$

By using the boundary conditions $u(0) = D_q^2 u(0) = 0$, we get

$$u(t) = At - \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_qs. \quad (3.6)$$

Differentiating both sides of (3.6), one obtains, with the help of (2.13), and (2.14)

$$\begin{aligned} (D_q u)(t) &= A - \int_0^t \frac{[\alpha - 1]_q (t - qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} y(s) d_qs, \\ (D_q^2 u)(t) &= - \int_0^t \frac{[\alpha - 1]_q [\alpha - 2]_q (t - qs)^{(\alpha-3)}}{\Gamma_q(\alpha)} y(s) d_qs. \end{aligned} \quad (3.7)$$

Then, by the condition $\gamma D_q u(1) + \beta D_q^2 u(1) = 0$, we have

$$A = \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} y(s) d_qs + \frac{\beta}{\gamma} \int_0^1 \frac{(1 - qs)^{(\alpha-3)}}{\Gamma_q(\alpha - 2)} y(s) d_qs. \quad (3.8)$$

The proof is complete. $\square$

Lemma 3.3. *Function G defined above satisfies the following conditions:*

$$G(t, qs) \geq 0, \quad G(1, qs) \geq G(t, qs), \quad 0 \leq t, s \leq 1, \quad (3.9)$$

$$G(t, qs) \geq g(t)G(1, qs), \quad 0 \leq t, s \leq 1 \quad \text{with } g(t) = t. \quad (3.10)$$

Proof. We start by defining two functions

$$\begin{aligned} g_1(t, s) &= \frac{t(1 - s)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} + \frac{\beta}{\gamma} \frac{t(1 - s)^{(\alpha-3)}}{\Gamma_q(\alpha - 2)} - \frac{(t - s)^{(\alpha-1)}}{\Gamma_q(\alpha)}, \quad 0 \leq s \leq t \leq 1, \\ g_2(t, s) &= \frac{t(1 - s)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} + \frac{\beta}{\gamma} \frac{t(1 - s)^{(\alpha-3)}}{\Gamma_q(\alpha - 2)}, \quad 0 \leq t \leq s \leq 1. \end{aligned} \quad (3.11)$$

It is clear that $g_2(t, qs) \geq 0$. Now, $g_1(0, qs) = 0$ and, in view of Remark 2.1, for $t \neq 0$

$$\begin{aligned}
 g_1(t, qs) &= \frac{t(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} + \frac{\beta t(1-qs)^{(\alpha-3)}}{\gamma \Gamma_q(\alpha-2)} - \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
 &\geq \frac{t(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} + \frac{\beta t(1-qs)^{(\alpha-3)}}{\gamma \Gamma_q(\alpha-2)} - \frac{t^{\alpha-1}(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
 &\geq \frac{t(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} + \frac{\beta t(1-qs)^{(\alpha-3)}}{\gamma \Gamma_q(\alpha-2)} - \frac{t(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
 &= \frac{t}{\Gamma_q(\alpha)} \left[[\alpha-1]_q (1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)} \right] \geq 0.
 \end{aligned} \tag{3.12}$$

Therefore, $G(t, qs) \geq 0$. Moreover, for fixed $s \in [0, 1]$,

$$\begin{aligned}
 {}_tD_q g_1(t, qs) &= \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} + \frac{\beta (1-qs)^{(\alpha-3)}}{\gamma \Gamma_q(\alpha-2)} - \frac{[\alpha-1]_q (t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} \\
 &\geq \frac{1}{\Gamma_q(\alpha)} \left[[\alpha-1]_q (1-qs)^{(\alpha-2)} - [\alpha-1]_q (t-qs)^{(\alpha-2)} \right] \\
 &= \frac{1}{\Gamma_q(\alpha-1)} \left[(1-qs)^{(\alpha-2)} - (t-qs)^{(\alpha-2)} \right] \geq 0,
 \end{aligned} \tag{3.13}$$

that is, $g_1(t, qs)$ is an increasing function of t . Obviously, $g_2(t, qs)$ is increasing in t , therefore; $G(t, qs)$ is an increasing function of t for fixed $s \in [0, 1]$. This concludes the proof of (3.9).

Suppose now that $t \geq qs$. Then,

$$\begin{aligned}
 \frac{G(t, qs)}{G(1, qs)} &= \frac{\gamma [\alpha-1]_q t(1-qs)^{(\alpha-2)} + \beta [\alpha-1]_q [\alpha-2]_q t(1-qs)^{(\alpha-3)} - \gamma (t-qs)^{(\alpha-1)}}{\gamma [\alpha-1]_q (1-qs)^{(\alpha-2)} + \beta [\alpha-1]_q [\alpha-2]_q (1-qs)^{(\alpha-3)} - \gamma (1-qs)^{(\alpha-1)}} \\
 &\geq \frac{\gamma [\alpha-1]_q t(1-qs)^{(\alpha-2)} + \beta [\alpha-1]_q [\alpha-2]_q t(1-qs)^{(\alpha-3)} - \gamma t(1-qs)^{(\alpha-1)}}{\gamma [\alpha-1]_q (1-qs)^{(\alpha-2)} + \beta [\alpha-1]_q [\alpha-2]_q (1-qs)^{(\alpha-3)} - \gamma (1-qs)^{(\alpha-1)}} \\
 &= t.
 \end{aligned} \tag{3.14}$$

If $t \leq qs$, Then

$$\frac{G(t, qs)}{G(1, qs)} = t, \tag{3.15}$$

and this finishes the proof of (3.10). $\square$

Remark 3.4. If we let $0 < \tau < 1$, then

$$\min_{t \in [\tau, 1]} G(t, qs) \geq \tau G(1, qs), \quad \text{for } s \in [0, 1]. \quad (3.16)$$

Let $X = C[0, 1]$ be the Banach space endowed with norm $\|u\| = \sup_{t \in [\tau, 1]} |u(t)|$. Let $\tau = q^n$ [18] for a given $n \in \mathbb{N}$ and define the cone $P \subset X$ by

$$P = \left\{ u \in X : u(t) \geq 0, \min_{t \in [\tau, 1]} u(t) \geq \tau \|u\| \right\}, \quad (3.17)$$

and the operator $T : P \rightarrow X$ is defined by

$$Tu(t) = \int_0^1 G(t, qs) a(s) f(u(s)) d_qs. \quad (3.18)$$

Remark 3.5. It follows from the nonnegativeness and continuity of G, a , and f that the operator $T : P \rightarrow X$ is completely continuous [3]. Moreover, for $u \in P, Tu(t) \geq 0$ on $[0, 1]$ and

$$\begin{aligned} \min_{t \in [\tau, 1]} Tu(t) &= \min_{t \in [\tau, 1]} \int_0^1 G(t, qs) a(s) f(u(s)) d_qs \\ &\geq \tau \int_0^1 G(1, qs) a(s) f(u(s)) d_qs \\ &= \tau \|Tu\| \end{aligned} \quad (3.19)$$

that is, $T(P) \subset P$.

Throughout this section, we will use the following notations:

$$\Lambda_1 = \left(\int_0^1 G(1, qs) a(s) d_qs \right)^{-1}, \quad \Lambda_2 = \left(\tau^2 \int_\tau^1 G(1, qs) a(s) d_qs \right)^{-1}. \quad (3.20)$$

It is obvious that $\Lambda_1, \Lambda_2 > 0$. Also, we define

$$f_0 = \lim_{u \rightarrow 0^+} \max_{0 \leq t \leq 1} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \rightarrow +\infty} \max_{0 \leq t \leq 1} \frac{f(u)}{u}. \quad (3.21)$$

Theorem 3.6. Let $\tau = q^n$ with $n \in \mathbb{N}$. And assume that the following assumptions are satisfied:

(H1) $f_0 = f_\infty = 0$;

(H2) There exist constant $k > 0$ and $M \in (\Lambda_2, \infty)$ such that

$$f(u) \geq Mk, \quad \text{for } u \in [\tau k, k]. \quad (3.22)$$

Then, the BVP (3.1) has at least two positive solutions u_1, u_2 , such that

$$r < \|u_1\| < k < \|u_2\| < R. \quad (3.23)$$

Proof. First, since $f_0 = 0$, for any $\varepsilon \in (0, \Lambda_1)$ there exists $r \in (0, k)$ such that

$$f(u) \leq \varepsilon u, \quad \text{for } u \in [0, r]. \quad (3.24)$$

Letting $\Omega_1 = \{u \in P : \|u\| < r\}$, for any $u \in \partial\Omega_1$, we get

$$\begin{aligned} \|Tu\| &= \max_{t \in [0,1]} \int_0^1 G(t, qs) a(s) f(u(s)) d_qs \\ &\leq \varepsilon r \int_0^1 G(1, qs) a(s) d_qs < r = \|u\|, \end{aligned} \quad (3.25)$$

which yields

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_1. \quad (3.26)$$

Thus,

$$i(T, \Omega_1, P) = 1. \quad (3.27)$$

Second, in view of $f_\infty = 0$, then for any $\varepsilon \in (0, \Lambda_1)$, there exists $l > k$ such that

$$f(u) \leq \varepsilon u, \quad \text{for } u \in [l, \infty) \quad (3.28)$$

and we consider two cases.

Case 1. Suppose that $f(u)$ is unbounded; then from $f \in C([0, \infty), [0, \infty))$, there is $R > l$ such that

$$f(u) \leq f(R), \quad \text{for } u \in [0, R], \quad (3.29)$$

then, from (3.28) and (3.29), one has

$$f(u) \leq f(R) \leq \varepsilon R, \quad \text{for } u \in [0, R]. \quad (3.30)$$

For $u \in \partial\Omega_2$, we get

$$\begin{aligned}
 \|Tu\| &= \max_{t \in [0,1]} \int_0^1 G(t, qs) a(s) f(u(s)) d_qs \\
 &\leq \int_0^1 G(1, qs) a(s) f(R) d_qs \\
 &\leq \varepsilon R \int_0^1 G(1, qs) a(s) d_qs \\
 &< R = \|u\|.
 \end{aligned} \tag{3.31}$$

Case 2. Suppose that $f(u)$ is bounded and $L = \max_{0 \leq t \leq 1, 0 \leq u \leq R} |f(t, u(t))|$. Taking $R = \max\{L/\varepsilon, k\}$, for $u \in \partial\Omega_2$, we get

$$\begin{aligned}
 \|Tu\| &= \max_{t \in [0,1]} \int_0^1 G(t, qs) a(s) f(u(s)) d_qs \\
 &\leq L \int_0^1 G(1, qs) a(s) d_qs \\
 &\leq \varepsilon R \int_0^1 G(1, qs) a(s) d_qs \\
 &< R = \|u\|.
 \end{aligned} \tag{3.32}$$

Hence, in either case, we always may set $\Omega_2 = \{u \in P : \|u\| < R\}$ such that

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_2. \tag{3.33}$$

Thus,

$$i(T, \Omega_2, P) = 1. \tag{3.34}$$

Finally, set $\Omega = \{u \in P : \|u\| < k\}$, for $u \in \partial\Omega$, since $\min_{t \in [\tau, 1]} u(t) \geq \tau \|u\| = \tau k$ for $u \in P$, and hence, for any $u \in \partial\Omega$, from (H2), we can get

$$\begin{aligned}
 \|Tu\| &= \max_{t \in [0,1]} \int_0^1 G(t, qs) a(s) f(u(s)) d_qs \\
 &\geq \tau \int_0^1 G(1, qs) a(s) f(u(s)) d_qs \\
 &\geq \tau^2 M k \int_\tau^1 G(1, qs) a(s) d_qs > k = \|u\|,
 \end{aligned} \tag{3.35}$$

which implies

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega. \quad (3.36)$$

Thus,

$$i(T, \Omega, P) = 0. \quad (3.37)$$

Hence, since $r < k < R$ and from (3.27), (3.34), and (3.37), it follows from the additivity of the fixed point index that

$$\begin{aligned} i(T, \Omega_2 \setminus \overline{\Omega}, P) &= 1, \\ i(T, \Omega \setminus \overline{\Omega}_1, P) &= -1. \end{aligned} \quad (3.38)$$

Thus, T has two positive solutions u_1, u_2 such that $r < \|u_1\| < k < \|u_2\| < R$ for $t \in (0, 1]$. So, the proof is complete. $\square$

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