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Research Article

An Explicit Description of Coxeter Homology Complexes

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Rains (2010) computes the integral homology of real De Concini-Procesi models of subspace arrangements, using some homology complexes whose main ingredients are nested sets and building sets of subspaces. We think that it is useful to provide various different descriptions of these complexes, since they encode relevant information about the homotopy type of the models and there still are interesting open questions about $\mathbb{Z}$-bases of the homology modulo its torsion (see the work by Rains (2010)). In this paper we focus on the case of the Coxeter arrangements: we give an explicit and elementary description, in terms of the combinatorics of the Coxeter groups, of the cells and of the boundary maps of these complexes.

1. Introduction

Let $\mathcal{A}$ be a central subspace arrangement in an euclidean vector space V of dimension n, and let us denote its complement by $\mathcal{M}(\mathcal{A})$. In [1] De Concini and Procesi construct models for $\mathcal{M}(\mathcal{A})$, associated with distinct sets of initial combinatorial data ("building sets," see Section 2) which are subspace arrangements with complement $\mathcal{M}(\mathcal{A})$.

Let $\mathcal{G}$ be a building set as above: in [2] Rains computes the integral homology of the real De Concini-Procesi model $Y_{\mathcal{G}}$, using some homology complexes whose main combinatorial ingredients are the nested sets (again see Section 2) of subspaces in $\mathcal{G}$. In particular, Rains proves the conjecture (formulated in [3] for the particular case of the moduli space $\overline{M}_{0,n}(\mathbb{R})$) about the nonexistence of odd torsion and provides a basis for $H_*(Y_{\mathcal{G}}, \mathbb{F}_2)$.

We think that it is useful to provide various different descriptions of these complexes, since they encode relevant information about the homotopy type of the model and there

still are interesting open questions about $\mathbb{Z}$-bases of the homology modulo its torsion (see Section 6 of [2]).

In this paper we focus on the case of the Coxeter arrangements: we give an explicit and elementary description, in terms of the combinatorics of the Coxeter groups, of the cells and of the boundary maps of the complexes associated to the minimal and to the maximal real De Concini-Procesi model (among the building sets associated to a given subspace arrangement there always are a minimal one and a maximal one with respect to inclusion).

Let then W be a Coxeter group, and let Φ be its root system, which spans the euclidean space V. We denote by $\mathcal{A}(\Phi)$ the arrangement made by the hyperplanes orthogonal to Φ.

As a specialization of a construction in [4], we consider some models for the C^∞ manifold $\mathcal{M}(\mathcal{A}(\Phi))/\mathbb{R}^+$ which are C^∞ compact manifolds with corners. Again they are associated with building sets and their connected components are (diffeomorphic to) polytopes.

Let $\mathcal{G}(\Phi)$ be a building set associated with $\mathcal{A}(\Phi)$, and let $CY_{\mathcal{G}(\Phi)}$ be the related model with corners; according to a "gluing" map described in [4], we obtain the De Concini-Procesi model $Y_{\mathcal{G}(\Phi)}$ as a quotient of $CY_{\mathcal{G}(\Phi)}$ (for different point of views which lead to the same construction, see [5–8]).

The natural CW structure of $CY_{\mathcal{G}(\Phi)}$ arising from the stratification of the polytopes in the boundary induces, gluing in a suitable way the faces of the polytopes, a CW structure on $Y_{\mathcal{G}(\Phi)}$.

We describe in detail the resulting homology complex. In particular, in Section 5 we deal, as a first step, with the minimal De Concini-Procesi model associated to the braid arrangement of dimension n (that is to say, the A_n case), which is isomorphic to the real moduli space of genus 0, stable, $(n+2)$-pointed curves. In Section 6 we study the minimal and maximal models for the general Coxeter groups.

Our description points out (which in fact is the aim of the present paper) how these homology complexes connect the combinatorics of nested sets with the partitions of the Coxeter diagrams and the action of the parabolic subgroups of W.

In the last section, as a concrete example, we focus on some complexes in low-dimensional cases (A_3, A_4, B_3 and F_4); we count cells and write the resulting homology groups which of course are in accordance with the more general results of [2].

2. Building Sets and Nested Sets

Let us rewrite in our euclidean case some definitions from [1]. We start by a (central) subspace arrangement $\mathcal{A}$ in the euclidean space V. It is convenient to deal also with its "dual" object: let us denote by $\mathcal{A}^\perp$ the arrangement made by the subspaces orthogonal to the subspaces of $\mathcal{A}$:

$$\mathcal{A}^\perp = \left\{ B^\perp \mid B \in \mathcal{A} \right\}. \tag{2.1}$$

Then we denote by $\mathcal{C}_{\mathcal{A}}$ the dual of the lattice $\mathcal{L}(\mathcal{A})$ of intersections of the subspaces in $\mathcal{A}$; in other words, $\mathcal{C}_{\mathcal{A}}$ is the closure, under the sum, of $\mathcal{A}^\perp$.

In the sequel building, arrangements will play a crucial role.

Definition 2.1. The subspace arrangement $\mathcal{A}$ in V is called "building set" or "building arrangement" if every element C of $\mathcal{C}_{\mathcal{A}}$ is the direct sum $C = G_1 \oplus G_2 \oplus \cdots \oplus G_k$ of the set of the maximal elements $G_1, G_2, \ldots, G_k$ of $\mathcal{A}^\perp$ contained in C.

For instance, the arrangement in $\mathbb{R}^2$ given by three distinct lines $\{l_1, l_2, l_3\}$ is not building, while $\{l_1, l_2\}$ is building.

Let $\mathcal{B}$ be any subspace arrangement in V; the family of building arrangements that have the same intersection lattice as $\mathcal{B}$ (in particular, all these arrangements have the same complement in V) is not empty. Furthermore, in this family there is a minimum and a maximum element with respect to inclusion (which may eventually coincide in trivial cases, see [1]). The elements of the minimum building arrangement are the "irreducible subspaces" of $\mathcal{L}(\mathcal{B})$, while the maximum building set is $\mathcal{L}(\mathcal{B})$ itself.

We can now recall the notion of "nested set" (see [1]) which generalizes the one introduced by Fulton and MacPherson in their paper [9] on models of configuration spaces.

Definition 2.2. Let $\mathcal{K}$ be a building arrangement of subspaces in V. A subset $\mathcal{S} \subset \mathcal{K}$ is called "nested relative to $\mathcal{K}$," or $\mathcal{K}$-nested, if, given any of its subset $\{U_1, \ldots, U_k\}$, $k \geq 2$, of pairwise noncomparable elements, we have that $\bigcap_{i=1}^{k} U_i \notin \mathcal{K}$ (or equivalently, $\sum_{i=1}^{k} U_i^{\perp} \notin \mathcal{K}^{\perp}$).

3. Wonderful Models: constructions over $\mathbb{R}$

A model for the complement $\mathcal{M}(\mathcal{G})$ of a subspace arrangement $\mathcal{G}$ in a real or complex vector space V, from the point of view of algebraic geometry, is a smooth irreducible variety $Y_{\mathcal{G}}$ equipped with a proper map $\pi : Y_{\mathcal{G}} \to V$ such that

(i) π is an isomorphism on the preimage of $\mathcal{M}(\mathcal{G})$;

(ii) the complement of this preimage is a divisor with normal crossings.

In their paper [1], De Concini and Procesi constructed models of this type, provided that the set of subspaces $\mathcal{G}$ is building, and computed their cohomology in the complex case.

In [1] arrangements of linear subspaces in the projective space $\mathbf{P}(V)$ have also been studied: the associated compact models are constructed in the following way.

Let $\mathcal{G}$ be a building set (we can suppose that it contains $\{0\}$), and let $\mathbf{P}(\mathcal{M}(\mathcal{G}))$ be the complement in $\mathbf{P}(V)$ of the projective subspaces $\mathbf{P}(A)$ $(A \in \mathcal{G})$. Then one considers the map

$$i : \mathbf{P}(\mathcal{M}(\mathcal{G})) \longrightarrow \mathbf{P}(V) \times \prod_{D \in \mathcal{G} - \{0\}} \mathbf{P}\left(\frac{V}{D}\right), \tag{3.1}$$

where in the first coordinate we have the inclusion and the map from $\mathcal{M}(\mathcal{G})$ to $\mathbf{P}(V/D)$ is the restriction of the canonical projection $(V - D) \to \mathbf{P}(V/D)$.

Definition 3.1. The compact model $Y_{\mathcal{G}}$ is obtained by taking the closure of the image of i.

De Concini and Procesi proved that the complement $\mathfrak{D}$ of $\mathbf{P}(\mathcal{M}(\mathcal{A}))$ in $Y_{\mathcal{G}}$ is the union of smooth irreducible divisors $\mathfrak{D}_G$ indexed by the elements $G \in \mathcal{G} - \{0\}$.

To be more precise, let us introduce the following notation.

Definition 3.2. Given a subspace $C \subset V$, we define the following two (possibly empty) subspace arrangements:

(1) $\mathcal{A}_C = \{H \in \mathcal{A} C \subset H\}$,

(2) $\mathcal{A}^C = \{B \cap C \mid B \in \mathcal{A} - \mathcal{A}_C\}$.

Furthermore, given two subspaces $H, C \subset V$, we will denote by $\mathcal{A}_H^C$ the subspace arrangement $\mathcal{A}_H^C = \{B \cap C \mid B \in \mathcal{A}_H - (\mathcal{A}_C \cap \mathcal{A}_H)\}$.

If we now denote by π the projection onto the first component $\mathbf{P}(\mathrm{V})$, $\mathfrak{D}_G$ is equal to the closure of

$$\pi^{-1}\left(\mathbf{P}(G) - \bigcup_{L \in \mathcal{A}^G} \mathbf{P}(L)\right). \tag{3.2}$$

It can also be characterized as the unique irreducible component such that $\pi(\mathfrak{D}_G) = \mathbf{P}(G)$. A complete characterization of the boundary is provided by the observation that if we consider a collection $\mathcal{T}$ of subspaces in $\mathcal{G} - \{0\}$, then

$$\mathfrak{D}_{\mathcal{T}} \equiv \bigcap_{A \in \mathcal{T}} \mathfrak{D}_A \tag{3.3}$$

is nonempty if and only if $\mathcal{T}$ is nested, and in this case $\mathfrak{D}_{\mathcal{T}}$ is a smooth irreducible subvariety.

From the point of view of differentiable geometry, the compact differentiable models of configuration spaces which appear in Kontsevich's paper [10] on deformation quantization of the Poisson manifolds raised the interest in the construction of differentiable models with corners of real subspace arrangements.

Kontsevich's compactifications have been shown in [4] (see also [11]) to be particular cases of the following more general construction.

Let us denote by $S(\mathbb{R}^n)$ the $n-1$th dimensional unit sphere in $\mathbb{R}^n$, and, for every subspace $A \subset \mathbb{R}^n$, let $S(A) = A \cap S(\mathbb{R}^n)$. Then we can consider the compact manifold

$$K = S(\mathbb{R}^n) \times \prod_{A \in \mathcal{A} - \{0\}} S\left(A^{\perp}\right) \tag{3.4}$$

and notice that there is an open embedding

$$\phi : \frac{\mathcal{M}(\mathcal{A})}{\mathbb{R}^+} \longrightarrow K. \tag{3.5}$$

This is obtained as a composition of the section $s : \mathcal{M}(\mathcal{A})/\mathbb{R}^+ \to \mathcal{M}(\mathcal{A})$ provided by

$$s([p]) = \frac{p}{|p|} \in S(\mathbb{R}^n) \cap \mathcal{M}(\mathcal{A}) \tag{3.6}$$

with the map

$$\mathcal{M}(\mathcal{A}) \longrightarrow S(\mathbb{R}^n) \times \prod_{A \in \mathcal{A} - \{0\}} S\left(A^{\perp}\right), \tag{3.7}$$

where on each factor we have a well-defined projection.

Definition 3.3. We define $CY_{\mathcal{A}}$ as the closure in K of $\phi(\mathcal{M}(\mathcal{A})/\mathbb{R}^+)$.

In [4] it has been proven that when $\mathcal{A}$ is a building set, $CY_{\mathcal{A}}$ is a smooth manifold with corners.

It is a differentiable model for $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$ in the following sense: if we denote by $c\pi$ the projection onto the first component $S(\mathbb{R}^n)$, then $c\pi$ is surjective and it is an isomorphism on the preimage of $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$. Furthermore, $c\pi$ establishes a bijective correspondence between the (closures of) codimension 1 open strata in the boundary of $CY_{\mathcal{A}}$ and the elements of $\mathcal{A} - \{0\}$.

More precisely, if $A \in \mathcal{A} - \{0\}$, its associated boundary component is

$$C\mathfrak{D}_A = \overline{c\pi^{-1}\left(S(A) - \bigcup_{B\in\mathcal{A}^A} S(B)\right)}. \tag{3.8}$$

We notice that the combinatorial structure of the boundary mimicks the one of complex De Concini-Procesi models (see [4]).

Theorem 3.4. *$C\mathfrak{D}_A$ is a manifold with corners of the following type:*

$$C\mathfrak{D}_A \cong CY_{\mathcal{A}_A^{A^\perp}} \times CY_{\mathcal{A}^A}. \tag{3.9}$$

Let $\mathcal{T}$ be a subset of $\mathcal{A}$ which includes $\{0\}$; then:

$$C\mathfrak{D}_{\mathcal{T}} = \bigcap_{B\in\mathcal{T}-\{0\}} C\mathfrak{D}_B \tag{3.10}$$

is nonempty if and only if $\mathcal{T}$ is nested in $\mathcal{A}$.

The relations between the algebraic-geometric and the differentiable construction of models have been studied in [12] by describing the combinatorial properties of a surjective map $F : CY_{\mathcal{A}} \to Y_{\mathcal{A}}$.

Let us recall the definition of F: the model $CY_{\mathcal{A}}$ is embedded in

$$K = S(\mathbb{R}^n) \times \prod_{A\in\mathcal{A}-\{0\}} S\left(A^\perp\right) \tag{3.11}$$

while $Y_{\mathcal{A}}$ is embedded inside

$$K' = \mathbf{P}(\mathbb{R}^n) \times \prod_{D\in\mathcal{A}-\{0\}} \mathbf{P}\left(\frac{\mathbb{R}^n}{D}\right). \tag{3.12}$$

Now, given any $A \in \mathcal{A}$, we can consider the natural isomorphism between $A^\perp$ and $\mathbb{R}^n/A$ provided by the projection.

Remark 3.5. As a consequence of this identification, there is a map F' from K to K' whose restriction to each factor $S(A^{\perp})$ is the $2 \mapsto 1$ projection $S(A^{\perp}) \mapsto \mathbf{P}(\mathbb{R}^n/A)$ (in particular this means that on the first factor we are considering the projection $S(\mathbb{R}^n) \mapsto \mathbf{P}(\mathbb{R}^n)$).

Theorem 3.6 (see [12]). *If one restricts F' to $CY_{\mathcal{A}}$, one obtains a surjective map*

$$F : CY_{\mathcal{A}} \longrightarrow Y_{\mathcal{A}}. \tag{3.13}$$

Let $\mathcal{S}$ be a $\mathcal{A}$-nested set which contains 0. Then F restricted to the internal points of $C\mathfrak{D}_{\mathcal{S}}$ is a $2^{|\mathcal{S}|}$-sheeted covering of the open part of the boundary component $\mathfrak{D}_{\mathcal{S}}$ in $Y_{\mathcal{A}}$.

Remark 3.7. In particular, when $\mathcal{S} = \{0\}$, this statement reduces to the obvious observation that F restricted to $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$ is a 2-sheeted covering of $\mathbf{P}(\mathcal{M}(\mathcal{A}))$.

4. The Coxeter Arrangements

Let us specialize the results described in the preceding sections to the case of the Coxeter arrangements.

Let W be a Coxeter group, and let Φ be its root system, which spans the euclidean space V.

The arrangement $\mathcal{A}(\Phi)$ provided by the hyperplanes orthogonal to the roots is not building in general. In this paper we will restrict our attention to the minimal and maximal building arrangements associated to it: $\mathcal{A}_{m\Phi}$ and $\mathcal{A}_{M\Phi}$.

The arrangement $\mathcal{A}_{m\Phi}$ is made by the "irreducible" subspaces, that is to say, its elements are the subspaces which are orthogonal to the irreducible root subsystems of Φ (see [13, 14]):

$$\mathcal{A}_{m\Phi} = \left\{ \langle J \rangle^{\perp} \mid J \subseteq \Phi \text{ and } J \text{ irreducible} \right\}, \tag{4.1}$$

where $\langle J \rangle$ is the linear span of J.

The maximal building arrangement $\mathcal{A}_{M\Phi}$ is equal to the full lattice of intersections of the hyperplanes orthogonal to the roots. Then, with a slight abuse of notation, we will denote by $Y_{m\Phi}$, $CY_{m\Phi}$, $Y_{M\Phi}$, and $CY_{M\Phi}$ (instead of by $Y_{\mathcal{A}_{m\Phi}}$ and $CY_{\mathcal{A}_{m\Phi}}$, etc.) the associated models.

We notice that there is a bijective correspondence between the connected components of $CY_{m\Phi}$, $CY_{M\Phi}$ (this is true in general for any building set $\mathcal{G}(\Phi)$ associated to Φ, not just for the minimal and maximal building sets) and the Weyl chambers. In fact, if C is a Weyl chamber, then the closure $\overline{C}$ of the embedding of $C/\mathbb{R}^+$ into $CY_{m\Phi}$ (resp., $CY_{M\Phi}$) is a connected component of $CY_{m\Phi}$ (resp., $CY_{M\Phi}$).

We also notice that, in general for any building set $\mathcal{G}(\Phi)$ associated to the arrangement $\mathcal{A}(\Phi)$, the map F of Theorem 3.6 is injective when restricted to $\overline{C}$ and $F(\overline{C})$ (and therefore $\overline{C}$) is diffeomorphic to a convex polytope (see [5, 6, 12, 13, 15]). For instance, in the A_n case, the polytope associated to the minimal building arrangement is a Stasheff's associahedron (see [16]) while the one associated to the maximum building is a permutohedron. In general for any Φ and any building set $\mathcal{G}(\Phi)$, this polytope is a nestohedron (see [17–19] and also [20]).

As an immediate consequence, we have the following algebraic-topological corollary of Theorem 3.6, which for simplicity of notation we state for minimal models but which holds for any model.

Corollary 4.1. *Let W be a Coxeter group with root system Φ, and let $Y_{m\Phi}$ and $CY_{m\Phi}$ be as before its associated minimal models. Let us equip $CY_{m\Phi}$ with the CW structure provided by the connected components of the open boundary strata; then $Y_{m\Phi}$, with the structure given by the images via F of these components, is a CW complex and F is a map of CW complexes.*

5. Cellular Complexes for A_n

Let us first focus on the essential braid arrangement of dimension n: it consists of the hyperplanes $\{x_i = x_j\}$ $(1 \le i < j \le n+1)$ in $V = \mathbb{R}^{n+1}/\mathbb{R}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$. These hyperplanes are orthogonal to the roots of the root system A_n.

In this section we will describe the minimal spherical model CY_{mA_n} and the minimal real model Y_{mA_n} associated to this root system. This example has another well-known geometric interpretation, as Y_{mA_n} can be viewed as the real moduli space of genus 0, stable, $(n+2)$-pointed curves (see [7, 8, 12, 21]). In Section 6 we will see that this construction can be generalized to any Coxeter arrangement. Since the model Y_{mA_n} is a quotient of CY_{mA_n}, we first give a description of CY_{mA_n} as a cell complex, and then we will present the identification map.

5.1. The Model CY_{mA_n}

In the model CY_{mA_n}, the maximal cells are in correspondence with the elements of the Coxeter group of type A_n, and we denote them by means of the permutation representation on the set $\{1,\dots,n+1\}$. So we write $c = (\sigma_1,\dots,\sigma_{n+1})$ for the $(n-1)$-cell corresponding to the element σ, where $\sigma_i = \sigma(i)$. If we denote by C the open chamber in $V = \mathbb{R}^{n+1}/\mathbb{R}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$ containing the (class of the) vector $(\sigma_1,\dots,\sigma_{n+1})$, we can think of c as the closure in CY_{mA_n} of the embedding of $C/\mathbb{R}^+$.

An irreducible subspace is given by the equation $x_{j_1} = \cdots = x_{j_k}$ and has nontrivial intersection with the closure of the chamber C if and only if it is in the form $\{x_{\sigma_i} = x_{\sigma_{i+1}} = \cdots = x_{\sigma_j}\}$ with $i < j$ and $j - i < n$. It follows that we can denote the corresponding cell in the boundary of c including into (a couple of) parentheses the numbers $\sigma_i,\dots,\sigma_j$. Finally, given some cells $d_1,\dots,d_k$ in the boundary of c, their intersection is nonempty if and only if the corresponding subspaces form a nested set. This means that the corresponding parentheses are pairwise disjoint or ordered by inclusion.

For example in the spherical model CY_{mA_3}, $(2,1,3,4)$ is a maximal cell and it has dimension 2. The 1-cells in its boundary are

$$((2,1),3,4),((2,1,3),4),(2,(1,3),4),(2,(1,3,4)),(2,1,(3,4)). \tag{5.1}$$

The 0-cells are

$$(((2,1),3),4),((2,(1,3)),4),(2,((1,3),4)),(2,(1,(3,4))),((2,1),(3,4)). \tag{5.2}$$

Now we need to fix an orientation on cells. We can do this on the maximal cells by endowing the sphere S^{n-1} with the positive orientation and (denoting by $\mathcal{M}$ the complement of the arrangement) requiring the projection $S(\mathcal{M}) \to CY_{mA_n}$ to be orientation preserving. For the lower-dimension cell we need to fix an ordering in the set of parentheses. Given a cell c, we can order its parentheses in the following way:

(a) if parentheses p_1 are included in parentheses p_2 (for example, $(2,1) \subset (2,1,3)$), we say that $p_1 < p_2$;

(b) if p_1 and p_2 are disjoint, we say that $p_1 < p_2$ if and only if the greatest number contained in p_1 is smaller than the greatest number contained in p_2 (for example, $(2,3) < (1,4)$).

Now we notice that, for any parentheses p that we can add to c, the corresponding cell is in the boundary of c. Let $c(p)$ be the cell obtained from c adding the parentheses p and suppose that $p_1 < \cdots < p_k$ are the parenthesis of c. If $p_i < p < p_{i+1}$, we define the number $\nu(c,p) = i$ as the position (eventually 0) of the last parentheses before p in the ordering of the parentheses of c. We define the orientation on the cell $c(p)$ as $(-1)^{\nu(c,p)}$ times the natural orientation induced by c on its boundary. So the boundary of the cell c is given by

$$\partial c = \sum_p (-1)^{\nu(c,p)} c(p), \tag{5.3}$$

where the sum is taken over all the possible parentheses p that can be added to c.

5.2. The Model Y_{mA_n}

Our next step is to define an identification between cells of the model CY_{mA_n}, in order to get Y_{mA_n} as a quotient complex.

Let c be a cell, and let p be (a couple of) parentheses of c. In view of Remark 3.5 it suffices to describe the identifying relation between c and the cell c' obtained from c by inverting the order of the numbers contained in the parentheses p (and so by inverting the order of the numbers of all parentheses contained in p). We say that

$$c \simeq (-1)^{k+1} c', \tag{5.4}$$

where k is the number of elements in parentheses p. More explicitly,

$$(\cdots(a_{i_1}, \ldots, a_{i_k})\cdots) \simeq (-1)^{k+1}(\cdots(a_{i_k}, \ldots, a_{i_1})\cdots). \tag{5.5}$$

Since the ordering relation between parentheses depends only on the elements in the parentheses, it follows immediately that the identification relation is compatible with the boundary map. These relations, according to Corollary 4.1, describe the cellular complex for the model Y_{mA_n} as a quotient of the cellular complex for CY_{mA_n}.

Remark 5.1. We can associate to a cell c the ordered set of its elements $s(c) = (\sigma_1, \ldots, \sigma_{n+1})$ (forgetting the parentheses data). Since a cell $\hat{c}$ in Y_{mA_n} corresponds to an equivalence class

$[c]$ of cells in CY_{mA_n}, we can choose as a representative for $[c]$ the cell $c' \in [c]$ with the smaller associated set $s(c')$, according to the lexicographical order.

6. Cellular Complexes for a Coxeter Arrangement

Let (W, Φ) be a Coxeter system. Let $\Delta \subset \Phi$ be the set of simple roots. We suppose we realize W as a reflection group in the real vector space $V = \mathbb{R}^n$ spanned by the roots in Φ and consider the corresponding minimal and maximal building arrangements $\mathcal{A}_{m\Phi}$ and $\mathcal{A}_{M\Phi}$. We give in the next two subsections a description of the cell complexes for the minimal models $CY_{m\Phi}$ and $Y_{m\Phi}$. Again we first give a description of the model $CY_{m\Phi}$, and then we obtain $Y_{m\Phi}$ as a quotient. In the last subsection we discuss the changes needed to study the case of the maximal models $CY_{M\Phi}$ and $Y_{M\Phi}$.

6.1. The Minimal Model $CY_{m\Phi}$

The maximal cells of $CY_{m\Phi}$ are in correspondence with the open chambers C of the space $\mathcal{M}(\mathcal{A}_{m\Phi})$ (which coincides with the complement of the union of the hyperplanes orthogonal to the roots in Φ). We now choose a set of simple roots Δ and therefore a fundamental C_e whose walls are in correspondence with Δ. Then we can fix a point x in the fundamental chamber and associate to the element $w \in W$ the chamber C_w containing the point $w(x)$. So maximal cells for $CY_{m\Phi}$ are in correspondence with the elements of the group W.

In the minimal building set every irreducible subspace is the invariant set of a parabolic subgroup. Given a subset $\Lambda \subset \Delta$ such that the corresponding graph Γ_Λ is a connected subgraph of the Dynkin diagram Γ_Δ, we call I_Λ the invariant subspace of the parabolic subgroup W_Λ generated by Λ. Since a generic parabolic subgroup is conjugated to a parabolic subgroup of type W_Λ, we can write a generic invariant subspace in the form $I(w, \Lambda) = wI_\Lambda$ for an element $w \in W$ and for a subset $\Lambda \subset \Delta$ such that the graph Γ_Λ is connected. Notice that the couple (w, Λ) is not unique.

We will denote a cell in the boundary of the maximal fundamental cell by a couple $(e, \mathcal{L})$, where e is the identity in W and $\mathcal{L}$ is an *admissible* set of subsets of Δ, that is:

(a) every set $\Lambda \in \mathcal{L}$ is a proper subset of Δ such that Γ_Λ is connected;

(b) for any two sets $\Lambda, \Lambda' \in \mathcal{L}$, either one is included in the other or the two subsets are disjoint and the corresponding subgroups W_Λ and $W_{\Lambda'}$ commute.

Notice that the admissible sets correspond to the fundamental nested sets described in [1].

In analogy with the previous section, we can think of the set $\Lambda \in \mathcal{L}$ as a couple of "parentheses" in the graph Γ (a "tubing," see [5]).

We will denote by $(w, \mathcal{L})$ the cell in CY_Φ which is equal to $w((e, \mathcal{L}))$.

Now we want to give an orientation to the cells CY_Φ; we start by fixing an ordering on the set of roots Φ. Then we consider a cell $(w, \mathcal{L})$: we want to fix an ordering on the elements of $\mathcal{L}$ which depends on w. Given two sets $\Lambda, \Lambda' \in \mathcal{L}$, we say that $\Lambda < \Lambda'$ if one of the following cases occurs:

(a) $\Lambda \subset \Lambda'$;

(b) $\max(w\Lambda) < \max(w\Lambda')$.

Table 1: Description of the fundamental cells.

Models	mA_3 mB_3	MA_3 MB_3	mA_4 mB_4 mF_4	mD_4	MA_4, MB_4 MD_4 MF_4
# 0-cells	5	6	14	16	24
# 1-cells	5	6	21	24	36
# 2-cells	1	1	9	10	14
# 3-cells	0	0	1	1	1

Table 2: Minimal models: number of cells.

Model	A_3	A_4	B_3	B_4	D_4	F_4
Rank C_0	15	105	30	336	192	1008
Rank C_1	30	315	60	1008	576	3024
Rank C_2	12	270	24	864	480	2592
Rank C_3	0	60	0	192	96	576

Now let $\Lambda \notin \mathcal{L}$, and let $\Lambda_1 < \cdots < \Lambda_k$ be the elements of $\mathcal{L}$ written according to the above described ordering. Suppose that $\Lambda_i < \Lambda < \Lambda_{i+1}$. We define the integer $\nu_w(\mathcal{L}, \Lambda) = i$.

We are now ready to give an orientation to the cells in $CY_{m\Phi}$. For the maximal cells $(w, \emptyset)$, we do this identifying $\mathcal{M}(\mathcal{A}_{m\Phi})/\mathbb{R}^+$ with its embedding $S(\mathcal{M}(\mathcal{A}_{m\Phi})) \subset \mathbb{R}^n$ and requiring the map $S(\mathcal{M}(\mathcal{A}_{m\Phi})) \to CY_{m\Phi}$ to be orientation preserving. If we suppose we have oriented a cell $c = (w, \mathcal{L})$, we can orient a cell $c' = (w, \mathcal{L} \cup \{\Lambda\})$ with $(-1)^{\nu_w(\mathcal{L},\Lambda)}$ times the orientation induced by c on its boundary.

So the boundary of c is

$$\partial c = \sum_{\Lambda} (-1)^{\nu_w(\mathcal{L},\Lambda)} (w, \mathcal{L} \cup \{\Lambda\}), \tag{6.1}$$

where the sum is taken over all $\Lambda \subset \Delta$ such that $\mathcal{L} \cup \{\Lambda\}$ is still admissible.

6.2. The Minimal Model $Y_{m\Phi}$

Now we define the identification of the cells of the model $CY_{m\Phi}$. Let w_Δ be the longest element of the Coxeter group W. In general we will write w_Λ for the longest element of the parabolic subgroup W_Λ. Let $c = (w, \mathcal{L})$, and let $c' = (w', \mathcal{L}')$ be cells in $CY_{m\Phi}$: the identifying relation is given by

$$c \simeq (-1)^{\dim(\overline{\Lambda})} c', \tag{6.2}$$

where $w' = w w_{\overline{\Lambda}}$ for a set $\overline{\Lambda} \in \mathcal{L}$, and the sets $\{I(w, \Lambda) \mid \Lambda \in \mathcal{L}\}$ and $\{I(w w_{\overline{\Lambda}}, \Lambda) \mid \Lambda \in \mathcal{L}'\}$ are equal. Notice that these sets are the nested sets associated with the cells c and c', respectively, and that they are equal if and only if the sets $\{I(e, \Lambda) \mid \Lambda \in \mathcal{L}\}$ and $\{I(w_{\overline{\Lambda}}, \Lambda) \mid \Lambda \in \mathcal{L}'\}$ are equal. We notice that the above-described identification relations are compatible with the boundary map ∂.

Table 3: Minimal models: integral homology.

Model	A_3	A_4	B_3	B_4	D_4	F_4
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	$\mathbb{Z}^4 \oplus \mathbb{Z}_2$	$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^6$	$\mathbb{Z}^7 \oplus \mathbb{Z}_2$	$\mathbb{Z}^{22} \oplus \mathbb{Z}_2^{13}$	$\mathbb{Z}^{16} \oplus \mathbb{Z}_2^{13}$	$\mathbb{Z}^{50} \oplus \mathbb{Z}_2^{15}$
H_2	0	$\mathbb{Z}^9 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^{21} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{49} \oplus \mathbb{Z}_2$
H_3	0	0	0	0	0	0

Table 4: Maximal models: number of cells.

Model	A_3	A_4	B_3	B_4	D_4
Rank C_0	18	180	36	576	288
Rank C_1	36	540	72	1728	864
Rank C_2	12	420	24	1344	672
Rank C_3	0	60	0	192	96

Remark 6.1. If two cells c and c' are antipodal, the above relation means

$$c \simeq (-1)^n c'. \tag{6.3}$$

Remark 6.2. In order to perform explicit computations, it is useful to choose a standard representative c for every cell $[c] \in Y_{m\Phi}$. This can be done for instance by fixing a total ordering on the group W and, given a class $[c]$, by choosing the representative $\overline{c} = (w, \mathcal{L}) \in [c]$ such that w is the smallest possible.

6.3. The Maximal Models $CY_{M\Phi}$ and $Y_{M\Phi}$

In the maximal case (maximal models appear for instance in [6]; see also [22] for some further references), we will denote a cell in $CY_{M\Phi}$ by a couple $(w, \mathcal{L})$, where as before w is an element in W and $\mathcal{L}$ is an *admissible* set of subsets of Δ, but this time the definition of *admissible* is the following:

(a) every set $\Lambda \in \mathcal{L}$ is a proper subset of Δ (notice that Γ_Λ does not need to be connected);

(b) the sets in $\mathcal{L}$ are totally ordered by inclusion.

Let $\Lambda \notin \mathcal{L}$, and let $\Lambda_1 < \cdots < \Lambda_k$ be the elements of $\mathcal{L}$ written according to the inclusion ordering. Suppose that $\Lambda_i < \Lambda < \Lambda_{i+1}$. Then we define the integer $\nu(\mathcal{L}, \Lambda) = i$ (notice that this time it does not depend on w).

Now the boundary map can be defined by the same procedure as in the minimal case.

Also the identification of the cells of the model $CY_{M\Phi}$ can be done following the same rules of the preceding subsection.

7. Some Low-Dimensional Examples

As a concrete example of the combinatorics involved in these homology complexes, we describe by Tables 1, 2, 3, 4, and 5 the minimal and maximal models for the root systems of type A_3, A_4, B_3, B_4 and the minimal model of F_4.

Table 5: Maximal models: integral homology.

Model	A_3	A_4	B_3	B_4	D_4
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	$\mathbb{Z}^7 \oplus \mathbb{Z}_2$	$\mathbb{Z}^{25} \oplus \mathbb{Z}_2^{16}$	$\mathbb{Z}^{13} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{58} \oplus \mathbb{Z}_2^{41}$	$\mathbb{Z}^{34} \oplus \mathbb{Z}_2^{25}$
H_2	0	$\mathbb{Z}^{24} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^{57} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{33} \oplus \mathbb{Z}_2$
H_3	0	0	0	0	0

We list the total number of cells in the model, the cells in a foundamental chamber, and we compute (we have been assisted by the computer algebra systems Axiom and Aldor) the resulting homology groups. Of course the listed groups are in accordance with the more general results of [2] (for the rational cohomology of the minimal models see also [23]).

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