Research Article

Bayes Estimation of Change Point in Discrete Maxwell Distribution

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Received 28 December 2010; Revised 13 May 2011; Accepted 13 May 2011

Academic Editor: Ajit K. Verma

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A sequence of independent lifetimes $X_1, \dots, X_m, X_{m+1}, \dots, X_n$ was observed from Maxwell distribution with reliability $r_1(t)$ at time t but later, it was found that there was a change in the system at some point of time m and it is reflected in the sequence after X_m by change in reliability $r_2(t)$ at time t . The Bayes estimators of m, θ_1, θ_2 are derived under different asymmetric loss functions. The effects of correct and wrong prior information on the Bayes estimates are studied.

1. Introduction

Maxwell distribution plays an important role in physics and other allied sciences. This paper introduces a discrete analogue of the Maxwell distribution, called discrete Maxwell distribution (dMax distribution). This distribution is suggested as a suitable reliability model to fit a range of discrete lifetime data.

In reliability theory many continuous lifetime models have been suggested and studied. However, it is sometimes impossible or inconvenient to measure the life length of a device, on a continuous scale. In practice, we come across situations, where lifetime of a device is considered to be a discrete random variable. For example, in an on/off switching device, the lifetime of the switch is a discrete random variable. Also, the number of voltage fluctuations which an electrical or electronic item can withstand before its failure is a discrete rv.

If the lifetimes of individuals in some population are grouped or when lifetime refers to an integral number of cycles of some sort, it may be desirable to treat it as a discrete rv. When a discrete model is used with lifetime data, it is usually a multinomial distribution, which arises because effectively continuous data have been grouped. Some situations may demand for another discrete distribution, usually over the non negative integers. Such situations are best

treated individually, but generally one tries to adopt one of the standard discrete distributions.

In the last two decades, standard discrete distributions like geometric and negative binomial have been employed to model lifetime data. However, there is a need to find more plausible discrete lifetime distributions to fit to various types of lifetime data. For this purpose, popular continuous lifetime distributions can be helpful in the following manner.

The Maxwell distribution defines the speed of molecules in thermal equilibrium under some conditions as defined in statistical mechanics. For example, this distribution explains many fundamental gas properties in kinetic theory of gases; distribution of energies and moments, and so forth.

Tyagi and Bhattachary ([1], (1989b)) considered Maxwell distribution as a lifetime model for the first time. They obtained minimum variance unbiased estimator (MVUE) and Bayes estimator of the parameter and reliability function of this distribution.

Chaturvedi and Rani [2] studied generalized Maxwell distribution by introducing one more parameter and obtained classical and Bayesian estimation procedures for it. A lifetime model is specified to represent the distribution of lifetimes, and statistical inferences are made on the basis of this model. Physical systems manufacturing the items are often subject to random fluctuations. It may happen that at some point of time instability in the sequence of lifetimes

and reliability is observed. The problem of study is when and where this change has started occurring. This is called change point inference problem. Bayesian ideas may play an important role in the study of such change point problem and has been often proposed as a valid alternative to classical estimation procedure. The monograph of Broemeling and Tsurumi [3] on structural changes, Jani and Pandya [4], Ebrahimi and Ghosh [5] and a survey by Zacks [6], Pandya and Jani [7], Pandya and Bhatt [8], Mayuri Pandya and prbha Jadav [9], Pandya and Jadav [10] are useful references. In this paper we have proposed a discrete Maxwell model to represent the distribution of lifetimes with change point m and have obtained Bayes estimators of m, θ_1, θ_2 .

2. Proposed Change Point Model

In this section we propose change point model on discrete Maxwell distribution. We also derived the Bayes estimates for the model. Let $X_1, X_2, \dots, X_n$ ($n \geq 3$) be a sequence of random lifetimes. First m of them are coming from discrete Maxwell, $dMax(\theta_1)$. So the probability mass function is given by

$$p(x_i) = \frac{4}{\sqrt{\pi}} \frac{1}{\theta_1} Q(x_i, 2, \theta_1), \quad x = 0, 1, \dots, \theta_1 > 0$$

$$i = 1, 2, \dots, m. \quad (1)$$

With reliability $r_1(t)$,

$$r_1(t) = \frac{4}{\sqrt{\pi}} \theta_1^{-3/2} J(t, 2, \theta_1), \quad x = 0, 1, \dots; \theta_1 > 0. \quad (2)$$

Later $n-m$ observations are coming from the discrete Maxwell, $dMax(\theta_2)$. So the probability mass function is given by

$$p(x_i) = \frac{4}{\sqrt{\pi}} \frac{1}{\theta_2} Q(x_i, 2, \theta_2), \quad x = 0, 1, \dots, \theta_2 > 0,$$

$$i = m + 1, \dots, n. \quad (3)$$

With reliability $r_2(t)$,

$$r_2(t) = \frac{4}{\sqrt{\pi}} \theta_2^{-3/2} J(t, 2, \theta_2), \quad x = 0, 1, \dots; \theta_2 > 0, \quad (4)$$

where

$$Q(x, k, \theta_1) = \int_x^{x+1} u^k e^{-u^2/\theta_1} du = J(x, k, \theta_1) - J(x+1, k, \theta_1), \quad (5a)$$

$$J(x, k, \theta_1) = \int_x^\infty u^k e^{-u^2/\theta_1} du, \quad (5b)$$

$$Q(x, k, \theta_2) = \int_x^{x+1} u^k e^{-u^2/\theta_2} du = J(x, k, \theta_2) - J(x+1, k, \theta_2), \quad (5c)$$

$$J(x, k, \theta_2) = \int_x^\infty u^k e^{-u^2/\theta_2} du. \quad (5d)$$

The likelihood function given the sample information $\underline{X} = (X_1, X_2, \dots, X_m, X_{m+1}, \dots, X_n)$

$$L(\theta_1, \theta_2, m | \underline{X}) = \left(\frac{4}{\sqrt{\pi}} \right)^n \theta_1^{-m} G_1(m, \theta_1) \theta_2^{-(n-m)} G_2(m, \theta_2) \quad (6)$$

where

$$G_1(m, \theta_1) = \prod_{i=0}^m Q(x_i, 2, \theta_1),$$

$$G_2(m, \theta_2) = \prod_{i=m+1}^n Q(x_i, 2, \theta_2), \quad (7)$$

$Q(x_i, 2, \theta_1)$ and $Q(x_i, 2, \theta_2)$ are explained in (5a) and (5c).

3. Bayes Estimation

3.1. The Conjugate Analysis Using Inverted Gamma Prior Distribution. The ML method, as well as other classical approaches, is based only on the empirical information provided by the data. However, when there is some technical knowledge on the parameters of the distribution available, a Bayes procedure seems to be an attractive inferential method. The Bayes procedure is based on a posterior density, say $g(\theta_1, \theta_2, m | \underline{X})$, which is proportional to the product of the likelihood function $L(\theta_1, \theta_2, m | \underline{X})$ with a prior joint density, say $g(\theta_1, \theta_2, m)$ representing the uncertainty on the parameters values.

We also suppose that some information on θ_1 and θ_2 is available and that this technical knowledge can be given in terms of prior mean values μ_1, μ_2 and variance σ_1 and σ_2 , respectively. Suppose that the marginal prior density of θ_1 and θ_2 is the inverted Gamma with respective mean μ_1 and μ_2 :

$$g(\theta_1) = \frac{a_1 b_1}{\Gamma b_1} (\theta_1)^{-(b_1+1)} e^{-a_1/\theta_1},$$

$$g(\theta_2) = \frac{a_2 b_2}{\Gamma b_2} (\theta_2)^{-(b_2+1)} e^{-a_2/\theta_2}, \quad (8)$$

$$a_i b_i > 0, \quad \theta_i > 0, \quad i = 1, 2,$$

where the parameters $a_i, b_i, i = 1, 2$ are obtained by solving

$$b_i = 2 + \frac{\mu_i^2}{\sigma_i^2}, \quad (9)$$

$$a_i = \mu_i(b_i - 1), \quad i = 1, 2.$$

Following Calabria and Pulcini [13] we assume the prior information to be correct if the true value of $\theta_1(\theta_2)$ is close to prior mean $\mu_1(\mu_2)$ and is assumed to be wrong if $\theta_1(\theta_2)$ is far from $\mu_1(\mu_2)$.

As in the study by Broemeling and Tsurumi [3], we suppose the marginal prior distribution of m to be discrete uniform over the set $\{1, 2, \dots, n-1\}$

$$g(m) = \frac{1}{(n-1)}. \quad (10)$$

The joint prior density of θ_1, θ_2, m is

$$g_1(\theta_1, \theta_2, m) = \frac{1}{n-1} \frac{a_1^{b_1}}{\Gamma b_1} \frac{a_2^{b_2}}{\Gamma b_2} \theta_1^{-(b_1+1)} e^{-a_1/\theta_1} \theta_2^{-(b_2+1)} e^{-a_2/\theta_2} \\ = k \theta_1^{-(b_1+1)} e^{-a_1/\theta_1} \theta_2^{-(b_2+1)} e^{-a_2/\theta_2}, \quad (11)$$

where

$$k = \frac{1}{n-1} \frac{a_1^{b_1}}{\Gamma b_1} \frac{a_2^{b_2}}{\Gamma b_2}. \quad (12)$$

The joint posterior density of θ_1, θ_2, m is

$$g_1(\theta_1, \theta_2, m | \underline{x}) = \frac{L(\theta_1, \theta_2, m | \underline{x}) g_1(\theta_1, \theta_2, m)}{h_1(\underline{x})}, \\ g_1(\theta_1, \theta_2, m | \underline{x}) = k_2 \theta_1^{-(m+b_1+1)} e^{-a_1/\theta_1} G_1(m, \theta_1) \theta_2^{-(n-m+b_2+1)} \\ \times e^{-a_2/\theta_2} G_2(m, \theta_2) \cdot h_1^{-1}(\underline{x}), \quad (13)$$

$$k_2 = k \cdot k_1 = k \left(\frac{4}{\sqrt{\pi}} \right)^n, \quad (14)$$

$G_1(m, \theta_1)$ and $G_2(m, \theta_2)$ are explained in (7).

And $h_1(\underline{x})$ is the marginal posterior density of $\underline{X}$:

$$h_1(\underline{x}) = \sum_{m=1}^{n-1} \int_0^\infty \int_0^\infty L(\theta_1, \theta_2, m | \underline{x}) g_1(\theta_1, \theta_2, m) d\theta_1 d\theta_2 \\ = k_2 \sum_{m=1}^{n-1} I_1(m) I_2(m), \quad (15)$$

where

$$I_1(m) = \int_0^\infty \theta_1^{-(m+b_1+1)} e^{-a_1/\theta_1} G_1(m, \theta_1) d\theta_1, \quad (16)$$

$$I_2(m) = \int_0^\infty \theta_2^{-(n-m+b_2+1)} e^{-a_2/\theta_2} G_2(m, \theta_2) d\theta_2, \quad (17)$$

$G_1(m, \theta_1)$ and $G_2(m, \theta_2)$ are explained in (7).

Marginal posterior density of θ_1 and of θ_2 is obtained by integrating the joint posterior density of θ_1, θ_2, m given in (13) with respect to θ_2 and with respect to θ_1 , respectively, and summing over m ,

$$g_1(\theta_1 | \underline{x}) = k_2 \sum_{m=1}^{n-1} \theta_1^{-(m+b_1+1)} e^{-a_1/\theta_1} G_1(m, \theta_1) \\ \times \int_0^\infty \theta_2^{-(n-m+b_2+1)} e^{-a_2/\theta_2} G_2(m, \theta_2) d\theta_2 h_1^{-1}(\underline{x}), \quad (18)$$

$$g_1(\theta_2 | \underline{x}) = k_2 \sum_{m=1}^{n-1} \theta_2^{-(n-m+b_2+1)} e^{-a_2/\theta_2} G_2(m, \theta_2) \\ \times \int_0^\infty \theta_1^{-(m+b_1+1)} e^{-a_1/\theta_1} G_1(m, \theta_1) d\theta_1 h_1^{-1}(\underline{x}), \quad (19)$$

$G_1(m, \theta_1)$ and $G_2(m, \theta_2)$ are explained in (7), and $h_1(\underline{x})$ is the same as in (15).

The marginal posterior density of change point m , say $g_1(m | \underline{x})$, is obtained by integrating the joint posterior density of θ_1, θ_2, m (13) with respect to θ_1 and θ_2

$$g_1(m | \underline{x}) = \int_0^\infty \int_0^\infty g_1(\theta_1, \theta_2, m | \underline{x}) d\theta_1 d\theta_2 \\ = \int_0^\infty k_2 \theta_1^{-(m+b_1+1)} e^{-a_1/\theta_1} G_1(m, \theta_1) \theta_2^{-(n-m+b_2+1)} \\ \times e^{-a_2/\theta_2} G_2(m, \theta_2) d\theta_1 d\theta_2 h_1^{-1}(\underline{x}) \\ = \frac{I_1(m) I_2(m)}{\sum_{m=1}^{n-1} I_1(m) I_2(m)}. \quad (20)$$

The Bayes estimator of a generic parameter (or function thereof) α , based on a squared error loss (SEL) function:

$$L_1(\alpha, d) = (\alpha - d)^2, \quad (21)$$

where d is a decision rule to estimate and α , is the posterior mean.

Bayes estimator of m under SEL and the inverted Gamma prior is

$$m_1^* = \frac{\sum_{m=1}^{n-1} m I_1(m) I_2(m)}{\sum_{m=1}^{n-1} I_1(m) I_2(m)}. \quad (22)$$

$I_1(m)$ and $I_2(m)$ are the same as in (16).

Other Bayes estimators of α based on the loss functions

$$L_2(\alpha, d) = |\alpha - d|,$$

$$L_3(\alpha, d) = \begin{cases} 0, & \text{if } |\alpha - d| < \epsilon, \epsilon > 0, \\ 1, & \text{otherwise,} \end{cases} \quad (23)$$

are the posterior median and posterior mode, respectively.

3.2. Posterior Distribution Functions Using Noninformative Prior. A noninformative prior is a prior that adds no information to that contained in the empirical data. Thus, a Bayes inference based upon noninformative prior has generally a theoretical interest only, since, from an engineering view point, the Bayes approach is very attractive for it allows incorporating expert opinion or technical knowledge in the estimation procedure. Let the joint noninformative prior density of θ_1, θ_2 , and m be given by,

$$g_2(\theta_1, \theta_2, m) = \frac{1}{(n-1)\theta_1\theta_2}. \quad (24)$$

The joint posterior density using noninformative prior $g_2(\theta_1, \theta_2, m)$, say $g_2(\theta_1, \theta_2, m | \underline{x})$, is

$$g_2(\theta_1, \theta_2, m | \underline{x}) = k_2 \sum_{m=1}^{n-1} \theta_1^{-(m+1)} e^{-1/\theta_1} G_1(m, \theta_1) \\ \times \theta_2^{-(n-m+1)} e^{-1/\theta_2} G_2(m, \theta_2) \cdot h_2^{-1}(\underline{x}), \quad (25)$$

where $G_1(m, \theta_1)$, $G_2(m, \theta_2)$ and k_2 are the same as in (7) and (14) and $h_2(\underline{x})$ is the marginal density of $\underline{x}$ under the noninformative priors and is obtained by,

$$\begin{aligned} h_2(\underline{x}) &= \sum_{m=1}^{n-1} \int_0^\infty L(\theta_1, \theta_2, m | \underline{x}) g_2(\theta_1, \theta_2, m) d\theta_1 d\theta_2 \\ &= k_2 \sum_{m=1}^{n-1} I_3(m) I_4(m), \end{aligned} \quad (26)$$

where

$$\begin{aligned} I_3(m) &= \int_0^\infty \theta_1^{-(m+1)} e^{-1/\theta_1} G_1(m, \theta_1) \theta_2^{-(n-m+1)} d\theta_1, \\ I_4(m) &= \int_0^\infty e^{-1/\theta_2} G_2(m, \theta_2) d\theta_2. \end{aligned} \quad (27)$$

Marginal posterior density of θ_1 , θ_2 and change point m under the noninformative prior (25) is obtained as

$$\begin{aligned} g_2(\theta_1 | \underline{x}) &= k_2 \sum_{m=1}^{n-1} \theta_1^{-(m+1)} e^{-1/\theta_1} G_1(m, \theta_1) \\ &\quad \times \int_0^\infty \theta_2^{-(n-m+1)} e^{-1/\theta_2} G_2(m, \theta_2) d\theta_2 \cdot h_2^{-1}(\underline{x}), \\ g_2(\theta_2 | \underline{x}) &= k_2 \sum_{m=1}^{n-1} e^{-1/\theta_2} G_2(m, \theta_2) \theta_2^{-(n-m+1)} \\ &\quad \times \int_0^\infty \theta_1^{-(m+1)} e^{-1/\theta_1} G_1(m, \theta_1) d\theta_1 \cdot h_2^{-1}(\underline{x}), \end{aligned} \quad (28)$$

$$\quad (29)$$

where $G_1(m, \theta_1)$, $G_2(m, \theta_2)$, and k_2 are same as in (7) and (14). $h_2(\underline{x})$ is same as in (26).

The marginal posterior density of change point m is, say $g_2(m | \underline{x})$ is obtained by integrating the joint posterior density of θ_1, θ_2, m (25) with respect to θ_1 and θ_2 ,

$$\begin{aligned} g_2(m | \underline{x}) &= \iint_0^\infty g_2(\theta_1, \theta_2, m | \underline{x}) d\theta_1 d\theta_2 \\ &= \iint_0^\infty k_2 \sum_{m=1}^{n-1} \theta_1^{-(m+1)} e^{-1/\theta_1} G_1(m, \theta_1) \theta_2^{-(n-m+1)} \\ &\quad \times e^{-1/\theta_2} G_2(m, \theta_2) \cdot h_2^{-1}(\underline{x}) d\theta_1 d\theta_2 \\ &= \frac{I_3(m) I_4(m)}{\sum_{m=1}^{n-1} I_3(m) I_4(m)} \end{aligned} \quad (30)$$

where $I_3(m)$ and $I_4(m)$ are the same as in (27).

Bayes estimator of m under SEL and noninformative prior is

$$m_2^* = \frac{\sum_{m=1}^{n-1} m I_3(m) I_4(m)}{\sum_{m=1}^{n-1} I_3(m) I_4(m)}. \quad (31)$$

4. Bayes Estimates of Change Point and Other Parameters under Asymmetric Loss Functions

The loss function $L(\alpha, d)$ provides a measure of the financial consequences arising from a wrong decision rule d to estimate an unknown quantity α . The choice of the appropriate loss function depends on financial considerations only and is independent of the estimation procedure used. The use of symmetric loss function was found to be generally inappropriate, since, for example, an overestimation of the reliability function is usually much more serious than an underestimation.

In this section, we derive Bayes estimators of change point m under different asymmetric loss functions using both prior considerations explained in Sections 3.1 and 3.2. A useful asymmetric loss, known as the Linex loss function was introduced by Varian [12]. Under the assumption that the minimal loss occurs at d , the Linex loss function can be expressed as

$$L_4(\alpha, d) = \exp[q_1(d - \alpha)] - q_1(d - \alpha) - 1, \quad q_1 \neq 0. \quad (32)$$

The sign of the shape parameter q_1 reflects the deviation of the asymmetry, $q_1 > 0$, if over, estimation is more serious than underestimation, and vice versa, and the magnitude of q_1 reflects the degree of asymmetry.

The posterior expectation of the Linex loss function is

$$E_\alpha \{L_4(\alpha, d)\} = \exp(q_1 d) E_\alpha \{\exp(-q_1 \alpha)\} - q_1(d - E\{\alpha\}) - 1, \quad (33)$$

where $E_\alpha \{f(a)\}$ denotes the expectation of $f(a)$ with respect to the posterior density $g(\alpha | \underline{x})$. The Bayes estimate a_l^* is the value of d that minimizes $E_\alpha \{L_4(\alpha, d)\}$

$$a_l^* = -\frac{1}{q_1} \ln[E_\alpha \{\exp(-q_1 \alpha)\}] \quad (34)$$

provided that $E_\alpha \{\exp(-q_1 \alpha)\}$ exists and is finite.

Minimizing expected loss function $E_m[L_4(m, d)]$ and using posterior distributions (20) and (30), we get the Bayes estimators of m using Linex loss function, respectively, as

$$\begin{aligned} m_L^* &= -\frac{1}{q_1} \ln \left[\frac{\sum_{m=1}^{n-1} e^{-mq_1} I_1(m) I_2(m)}{\sum_{m=1}^{n-1} I_1(m) I_2(m)} \right], \\ m_L^{**} &= -\frac{1}{q_1} \ln \left[\frac{\sum_{m=1}^{n-1} e^{-mq_1} I_3(m) I_4(m)}{\sum_{m=1}^{n-1} I_3(m) I_4(m)} \right], \end{aligned} \quad (35)$$

where $I_1(m) I_2(m)$ and $I_3(m) I_4(m)$ are the same as in (16) and (27).

Minimizing expected loss function $E_{\theta_1}[L_4(\theta_1, d)]$ and using posterior distributions (18) and (28), we get the Bayes estimators of θ_1 using Linex loss function, respectively, as

$$\begin{aligned}\theta_{1L}^* &= -\frac{1}{q_1} \ln \left[k_2 \sum_{m=1}^{n-1} \int_0^\infty \theta_2^{-(n-m+b_2+1)} e^{-a_2/\theta_2} G_2(m, \theta_2) d\theta_2 \right. \\ &\quad \times \int_0^\infty \theta_1^{-(m+b_1+1)} e^{-a_1/\theta_1 - \theta_1 q_1} \\ &\quad \times G_1(m, \theta_1) d\theta_1 h_1^{-1}(\underline{x}) \left. \right], \\ \theta_{1L}^{**} &= \frac{1}{q_1} \ln \left[\int_0^\infty k_2 \sum_{m=1}^{n-1} \theta_1^{-(m+1)} e^{-1/\theta_1 - \theta_1 q_1} G_1(m, \theta_1) d\theta_1 \right. \\ &\quad \times \int_0^\infty \theta_2^{-(n-m+1)} e^{-1/\theta_2} G_2(m, \theta_2) d\theta_2 \cdot h_2^{-1}(\underline{x}) \left. \right],\end{aligned}\quad (36)$$

where $G_1(m, \theta_1)$, $G_2(m, \theta_2)$, and k_2 are the same as in (7) and (14). and $h_1(\underline{x})$ and $h_2(\underline{x})$ are the same as in (15) and (26).

Minimizing expected loss function $E_{\theta_2}[L_4(\theta_2, d)]$ and using posterior distribution (19) and (29), we get the Bayes estimators of θ_2 using Linex loss function, respectively, as

$$\begin{aligned}\theta_{2L}^* &= \frac{1}{q_1} \ln \left[k_2 \sum_{m=1}^{n-1} \int_0^\infty \theta_1^{-(m+b_1+1)} e^{-a_1/\theta_1} G_1(m, \theta_1) d\theta_1 \right. \\ &\quad \times \int_0^\infty \theta_2^{-(n-m+b_2+1)} e^{-a_2/\theta_2 - \theta_2 q_1} \\ &\quad \times G_2(m, \theta_2) d\theta_2 h_1^{-1}(\underline{x}) \left. \right], \\ \theta_{2L}^{**} &= -\frac{1}{q_1} \ln \left[k_2 \int_0^\infty e^{-1/\theta_2 - \theta_2 q_1} G_2(m, \theta_2) \theta_2^{-(n-m+1)} d\theta_2 \right. \\ &\quad \times \int_0^\infty \theta_1^{-(m+1)} e^{-1/\theta_1} G_1(m, \theta_1) d\theta_1 \cdot h_2^{-1}(\underline{x}) \left. \right],\end{aligned}\quad (37)$$

where $G_1(m, \theta_1)$, $G_2(m, \theta_2)$, and k_2 are the same as in (7) and (14) and $h_1(\underline{x})$ and $h_2(\underline{x})$ are the same as in (15) and (26).

Another loss function, called General Entropy (GE) loss function, proposed by Calabria and Pulcini [11], is given by,

$$L_5(\alpha, d) = \left(\frac{d}{\alpha}\right)^{q_3} - q_3 \ln\left(\frac{d}{\alpha}\right) - I. \quad (38)$$

The Bayes estimate α_E^* is the value of d that minimizes $E_\alpha[L_5(\alpha, d)]$:

$$\alpha_E^* = [E_\alpha(\alpha^{-q_3})]^{-1/q_3}, \quad (39)$$

provided that $E_\alpha(\alpha^{-q_3})$ exists and is finite.

Minimizing expectation $E_m[L_5(m, d)]$ and using posterior distributions (20) and (30), we get the Bayes estimators m_E^* , m_E^{**} of m , respectively, as

$$\begin{aligned}m_E^* &= [E_m[m^{-q_3}]]^{-1/q_3} = \left[\frac{\sum_{m=1}^{n-1} m^{-q_3} I_1(m) I_2(m)}{\sum_{m=1}^{n-1} I_1(m) I_2(m)} \right]^{-1/q_3}, \\ m_E^{**} &= \left[\frac{\sum_{m=1}^{n-1} m^{-q_3} I_3(m) I_4(m)}{\sum_{m=1}^{n-1} I_3(m) I_4(m)} \right]^{-1/q_3},\end{aligned}\quad (40)$$

where $I_1(m)I_2(m)$ and $I_3(m)I_4(m)$ are the same as in (16) and (27).

Minimizing expected loss function $E_{\theta_1}[L_5(\theta_1, d)]$ and using posterior distributions (18) and (28), we get the Bayes estimates of θ_1 using General Entropy loss function and informative and noninformative prior, respectively, as

$$\begin{aligned}\theta_{1E}^* &= \left[k_2 \sum_{m=1}^{n-1} \int_0^\infty \theta_2^{-(n-m+b_2+q_3+1)} e^{-a_2/\theta_2} G_2(m, \theta_2) d\theta_2 \right. \\ &\quad \times \int_0^\infty \theta_1^{-(m+b_1+1)} e^{-a_1/\theta_1} G_1(m, \theta_1) d\theta_1 h_1^{-1}(\underline{x}) \left. \right]^{-1/q_3}, \\ \theta_{1E}^{**} &= \left[\int_0^\infty k_2 \sum_{m=1}^{n-1} \theta_1^{-(m+q_3+1)} e^{-1/\theta_1} G_1(m, \theta_1) d\theta_1 \right. \\ &\quad \times \int_0^\infty \theta_2^{-(n-m+1)} e^{-1/\theta_2} G_2(m, \theta_2) d\theta_2 \cdot h_2^{-1}(\underline{x}) \left. \right]^{-1/q_3},\end{aligned}\quad (41)$$

where $G_1(m, \theta_1)$, $G_2(m, \theta_2)$ and k_2 are same as in (7) and (14). $h_1(\underline{x})$ and $h_2(\underline{x})$ are same as in (15) and (26).

Minimizing expected loss function $E_{\theta_2}[L_5(\theta_2, d)]$ and using posterior distributions (19) and (29), we get the Bayes estimates of θ_2 using General Entropy loss function as

$$\begin{aligned}\theta_{2E}^* &= \left[k_2 \sum_{m=1}^{n-1} \int_0^\infty \theta_1^{-(m+b_1+1)} e^{-a_1/\theta_1} G_1(m, \theta_1) d\theta_1 \right. \\ &\quad \times \int_0^\infty \theta_2^{-(n-m+b_2+q_3+1)} e^{-a_2/\theta_2} \\ &\quad \times G_2(m, \theta_2) d\theta_2 h_1^{-1}(\underline{x}) \left. \right]^{-1/q_3}, \\ \theta_{2E}^{**} &= \left[k_2 \int_0^\infty e^{-1/\theta_2} G_2(m, \theta_2) \theta_2^{-(n-m+q_3+1)} d\theta_2 \right. \\ &\quad \times \int_0^\infty \theta_1^{-(m+1)} e^{-1/\theta_1} G_1(m, \theta_1) d\theta_1 \cdot h_2^{-1}(\underline{x}) \left. \right]^{-1/q_3},\end{aligned}\quad (42)$$

where, $G_1(m, \theta_1)$, $G_2(m, \theta_2)$ and k_2 are same as in (7) and (14) and $h_1(\underline{x})$ and $h_2(\underline{x})$ are the same as in (15) and (26).

Remark 1. Putting $q_3 = -1$ in (40), (41), and (42), we get the Bayes estimators of m , θ_1 , and θ_2 , posterior means under the squared error loss.

TABLE 1: Generated samples from dMax distribution.

Sample no.	N	M	Sample		Actual reliability	
			$\theta_1 = 1.0$ (first m observation)	$\theta_2 = 0.5$ (last $(n-m)$ observation)	$R_1(t)$	$R_2(t)$
1	30	15	$0 \times 5, 1 \times 10$	$0 \times 11, 1 \times 4$	0.0460	0.0011
2	50	25	$0 \times 9, 1 \times 16$	$0 \times 18, 1 \times 7$		
3	50	35	$0 \times 11, 1 \times 24$	$0 \times 25, 1 \times 10$		
			$\theta_1 = 5.0$	$\theta_2 = 2.0$		
4	30	15	$1 \times 6, 2 \times 4, 3 \times 2, 4 \times 2$	$0, 1 \times 10, 2 \times 4$	0.8495	0.6594
5	50	25	$0, 1 \times 9, 2 \times 6, 3 \times 6, 4 \times 3$	$0 \times 3, 1 \times 16, 2 \times 6$		
6	50	35	$0 \times 3, 1 \times 9, 2 \times 11, 3 \times 8, 4 \times 4$	$0 \times 5, 1 \times 21, 2 \times 8, 3$		

TABLE 2: Bayes estimate of m , θ_1 , and θ_2 under SEL.

Sample no.	N	Bayes estimates of m (posterior mean)	Bayes Estimates of θ_1 and θ_2 (posterior mean)	
			θ_1^*	θ_2^*
1	30	15	1.0	0.5
2	50	25	1.0	0.5
3	50	35	1.0	0.5
4	30	15	5.0	2.0
5	50	25	5.0	2.0
6	50	35	5.0	2.0

Note that, for $q_3 = -1$, the GE loss function reduces to the squared error loss function.

5. Numerical Study

We have generated 30 random observations from dmax distribution involving change point discussed in Section 2. The first 15 observations from dMax with $\theta_1 = 1.0$ and at $t = 5.0$, $R_{1t} = 0.0460$ and next 15 observations from dMax distribution with $\theta_2 = 0.5$ and $R_{2t} = 0.0011$, θ_1 and θ_2 themselves were random observation from inverted Gamma prior distributions with prior means $\mu_1 = 1.0$, $\mu_2 = 0.5$ and variance $\sigma_1 = 1.0$ and $\sigma_2 = 1.0$, respectively, resulting in $a_1 = 2.0$, $b_1 = 3.0$, and $a_2 = .5$, $b_2 = .75$. These observations are given in Table 1 first row.

We also compute the Bayes estimators m_L^*, m_E^* of m , $\theta_{1E}^*, \theta_{1L}^*$ and $\theta_{2E}^*, \theta_{2L}^*$ of θ_1 and θ_2 using the results given in Section 4 for the data given in Table 1 and for different values of shape parameter q_1 and q_3 . The results are shown in Tables 3 and 4.

We have generated 6 random samples from discrete Maxwell distribution involving change point discussed in Section 2 with $n = 30, 50, 50$, $m = 15, 25, 35$, and $\theta_1 = 1.0, 5.0$ and $\theta_2 = 0.5$ and 2.0 . As explained in Section 3.1, θ_1 and θ_2 themselves were random observations from inverted Gamma prior distributions with prior means μ_1, μ_2 , respectively. These observations are given in Table 1. We have calculated posterior means of m , θ_1 , and θ_2 under both the prior for all samples, and the results are shown in Table 2.

Table 3 shows that for small values of $|q|$, $q_1 = 0.09, 0.2, 0.1$ Linex loss function is almost symmetric and nearly quadratic and the values of the Bayes estimate under such

a loss is not far from the posterior mean. Table 3 also shows that, for $q_1 = 1.5, 1.2$, Bayes estimates are less than actual value of $m = 15$.

For $q_1 = q_3 = -1, -2$, Bayes estimates are quite large than actual value $m = 15$. It can be seen from Tables 3 and 4 that the negative sign of shape parameter of loss functions reflecting underestimation is more serious than overestimation. Thus, problem of underestimation can be solved by taking the value of shape parameters of Linex and General Entropy loss functions negative.

Table 4 shows that, for small values of $|q|$, $q_3 = 0.09, 0.2, 0.1$ General Entropy loss function, the values of the Bayes estimate under such a loss is not far from the posterior mean. Table 4 also shows that, for $q_3 = 1.5, 1.2$, Bayes estimates are less than actual value of $m = 15$.

It can be seen from Tables 3 and 4 that positive sign of shape parameter of loss functions reflecting overestimation is more serious than under estimation. Thus, problem of over estimation can be solved by taking the value of shape parameter of Linex and General Entropy loss functions positive and high.

5.1. Sensitivity of Bayes Estimates. In this section, we study the sensitivity of the Bayes estimator, obtained in Section 3 with respect to change in the prior of parameters. The means μ_1 and μ_2 of inverted Gamma prior on θ_1 and θ_2 have been used as prior information in computing the parameters a_1, a_2, b_1 , and b_2 of the prior. We have computed posterior mean m^* for the data given in Table 1, considering different sets of values of (μ_1, μ_2) . Following Calabria and Pulcini [13], we also assume the prior information to be correct if the true value of θ_1 and θ_2 is closed to prior mean $\mu_1(\mu_2)$ and

TABLE 3: The Bayes estimates using Linex loss function. ($n = 30$, $m = 15$, $\theta_1 = 1.0$, $\theta_2 = 0.5$).

	q_1	m_L^*	θ_{1L}^*	θ_{2L}^*
Informative prior	0.09	15	1.0	0.53
	0.10	15	1.0	0.52
	0.20	15	1.0	0.51
	1.2	14	0.97	0.48
	1.5	13	0.90	0.44
	-1.0	16	1.5	0.57
	-2.0	17	1.7	0.59
Noninformative Prior	0.09	14	0.93	0.54
	0.10	14	0.92	0.51
	0.20	14	0.91	0.50
	1.2	13	0.85	0.46
	1.5	12	0.82	0.41
	-1.0	16	1.5	0.56
	-2.0	17	1.7	0.57

TABLE 4: The Bayes estimates using General Entropy loss function. ($n = 30$, $m = 15$, $\theta_1 = 1.0$, $\theta_2 = 0.5$).

	q_3	θ_{1E}^*	m_E^*	θ_{2E}^*
Informative Prior	0.09	1.3	15	0.53
	0.10	1.3	15	0.51
	0.20	1.2	15	0.50
	1.20	1.0	13	0.47
	1.50	0.93	12	0.45
	-1.0	1.5	16	0.55
	-2.0	1.7	17	0.58
Noninformative Prior	0.09	1.2	14	0.53
	0.10	1.2	14	0.53
	0.20	1.0	14	0.52
	1.2	0.90	12	0.47
	1.5	0.84	11	0.44
	-1.0	1.4	17	0.58
	-2.0	1.8	19	0.61

is assumed to be wrong if θ_1 and θ_2 are far from $\mu_1(\mu_2)$. We observed that the posterior mean m^* appears to be robust with respect to the correct choice of the prior density of $\theta_1(\theta_2)$ and a wrong choice of the prior density of $\theta_2(\theta_1)$.

This can be seen from Tables 5, 6, and 7.

Table 5 shows that, when prior mean $\mu_1 = 1$ = actual value of θ_1 , $\mu_2 = 0.3$ and 0.7 (far from true value of $\theta_2 = 0.5$), it means correct choice of prior of θ_1 and wrong choice of prior of θ_2 , the values of Bayes estimator posterior mean remains the same and is 15. It gives correct estimation of m . Thus, posterior mean is not sensitive with wrong choice of prior density of $\theta_2(\theta_1)$ and correct choice of prior of $\theta_1(\theta_2)$.

6. Simulation Study

In Section 5, we have obtained Bayes estimates of m on the basis of the generated data given in Table 1 for given

TABLE 5: Bayes estimate of m for Sample 1.

μ_1	μ_2	m^*
1.0	0.3	15
1.0	0.5	15
1.0	0.7	15
0.7	0.5	15
0.9	0.5	15
1.4	0.5	15

TABLE 6: Bayes estimate of m for Sample 5.

μ_1	μ_2	m^*
5.0	1.0	25
5.0	3.0	25
5.0	4.0	25
4.0	2.0	25
6.0	2.0	25
6.5	2.0	25

TABLE 7: Bayes Estimate of m for Sample 6.

μ_1	μ_2	m^*
1.0	0.3	35
1.0	0.5	35
1.0	0.7	35
0.7	0.5	35
0.9	0.5	35
1.4	0.5	35

values of parameters. To justify the results, we have generated 10,000 different random samples with $m = 10, n = 30, 50$, $\theta_1 = 1$, $\theta_2 = 0.5$ and obtained the frequency distributions of posterior mean m_L^* , m_E^* of m with the correct prior consideration. The result is shown in Table 8. The value of shape parameter of the General Entropy loss and Linex loss used in simulation study for change point is taken as 0.1. We have also simulated several dMax samples with $m = 15, 25, 35$; $n = 30, 50$, and $\theta_1 = 0.15, 0.11, 0.10$; $\theta_2 = 0.55, 0.45, 0.35$. For each m, n, θ_1 , and θ_2 1000 pseudorandom samples have been simulated and Bayes estimators of change point m using $q_1 = q_3 = 0.9$ have been computed for same value of a_1, a_2 and for different prior means μ_1 and μ_2 . We observed that the posterior mean m^* appears to be robust with respect to the correct choice of the prior density of $\theta_1(\theta_2)$ and a wrong choice of the prior density of $\theta_2(\theta_1)$.for each combination of prior means μ_1 and μ_2 .

The value of Bayes estimator of change point m , based on Linex loss and general Entropy loss, using $q_1 = q_3 = 0.9$ is 10.

Table 8 leads to conclusion that performance of posterior means has better performance than that of m_L^* , m_E^* , of change point. 78% values of posterior mean are closed to actual value of change point with correct choice of prior. 65% values of m_L^* are closed to actual value of change point with correct

TABLE 8: Frequency distributions of the Bayes estimates of the change point.

Bayes estimate	% frequency for		
	01–08	09–11	12–20
Posterior mean	12	78	10
m_L^*	20	65	15
m_E^*	22	66	12

choice of prior. 66% values of m_E^* are close to correct values of change point with correct prior considerations.

Acknowledgment

The authors would like to thank the editor and the referee for their valuable suggestions which improved the earlier version of the paper.

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