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Research Article

The Optimal Warranty and Preventive Maintenance Policy for the Four-State System

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A complete view for the multistate system considering the four-state system is here introduced. The exponential distribution for failure times and repair times is considered. The steady state availability is established via the Markov process. Different warranty and preventive maintenance policies are introduced, and also the cost of these policies for the manufacturer and the buyer is evaluated.

1. Introduction

All products are unreliable in the sense that they fail. A failure can occur early in an item's life due to manufacturing defects or late in its life due to degradation that is dependent on age and usage.

Most equipment offers some level of warranty to gain some advantages in the highly competitive markets. The warranty assures the buyer that a faulty item will either be repaired or replaced at no cost or at reduced cost.

To the seller, offering a warranty implies additional costs including PM, repair, and replacement that depend on the transaction contract and product characteristics. From the buyer's perspective, however, PM can either lengthen the lifetime of product or effectively reduce its failure rate; well-performing PM during the warranty period may provide the consumer better product service in the postwarranty period and reduce the cost of repairing the deteriorated product.

Kim et al. [1] gave a brief review of the literature linking warranty and preventive maintenance. Furthermore Sheu and Chien [2] obtained the expected total cost (i.e., manufacturing plus warranty costs) under various warranty policies (failure-free policies with and without renewing and rebate policy). Jain and Maheshwari [3] considered renewing pro rata warranty policy to determine the expected maintenance cost for different life-time distributions. Chen and Chien [4] investigated the effect of preventive maintenance carried out by the buyer on items sold under a free-replacement renewing warranty policy. Jung et al. [5] considered a replacement model following the expiration of warranty: renewing warranty and nonrenewing warranty. Huang and Yen [6] analyzed a two-dimensional warranty of both time and usage for deteriorating products; they presented the warranty policy that maximizes the manufacturers profits. Jung et al. [7] studied the system maintenance policy under the renewing warranty, defined the life cycle anew from the user's perspective, and discussed the optimal maintenance policy after the renewing warranty is expired. Yedida and Munavar [8] considered a geometric process maintenance model with preventive repair and warranty using an objective function defined as the average cost per unit of working time.

Some systems are able to perform their task with partial performance; failure of their components leads to degradation of performance. Such systems or components have one working state and two or more different failure states, These systems are called multistate systems (MSSs). There are many examples of MSS-like electrical power generation systems, domestic hot water systems, telecommunication networks systems, and radar systems; see the work of Lindqvist and Doksum in [9] and of Nourelfath and Ait-Kadi in [10].

In our study, Attia et al. [11] introduced the preventive maintenance problem for the multistate system in the case of the three-state system, since we evaluated the optimal numbers of preventive maintenance that maximize the expected profit values.

In this paper, we continue to introduce a complete view for the multistate system considering the four-state system. Also, we assume the exponential distribution for failure times and repair times. We evaluate the steady state availability, introduce different warranty and preventive maintenance policies, and evaluate the cost of these policies for the manufacturer and for the buyer in the multistate system case.

The structure of this study is as follows. In Section 2 the four-state system description is shown and the steady state availability is evaluated using the Markov process technique. In Section 3 different warranty and preventive maintenance policies and the cost of these policies for the multistate system are evaluated. The proposed model is illustrated through two numerical examples in Section 4.

2. The Four-State System Model

We introduce here the multistate system considering the four-state system, and we assume that the system failure times and repair times are exponentially distributed.

The system has four states, $S = \{0, \ldots, 3\}$ such as state 0 represents the system before any failure occurs when the system operates. The intermediate states 1, 2 are ordered to reflect their relative degree of failures, meanwhile in state 3 the system fails completely and cannot operate.

The failure rates are $\lambda_i, i = 1, \ldots, 6$, and the repair rates are $\mu_i, i = 1, \ldots, 6$.

Figure 1 shows the transition rates.

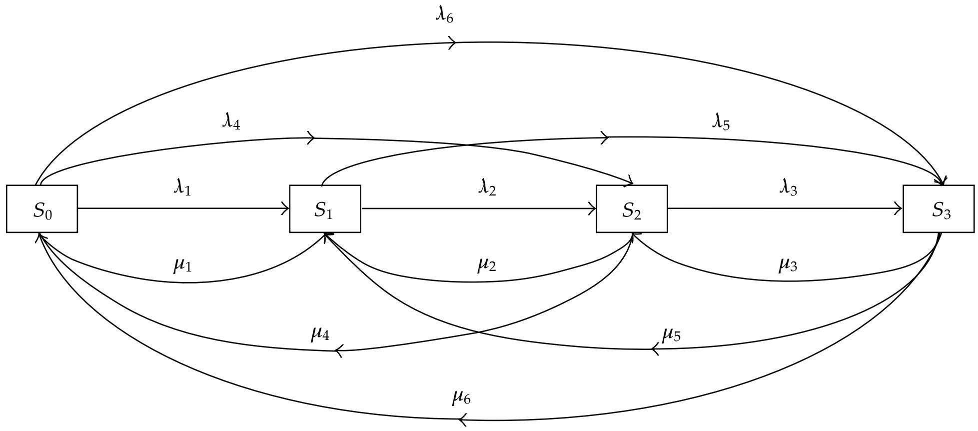

Figure 1: Transition rates.

The transition probabilities of the Markov process are given by

$$\begin{aligned}
P_0(t+\Delta t) &= P_0(t)[1-(\lambda_1+\lambda_4+\lambda_6)\Delta t] + P_1(t)\mu_1\Delta t + P_2(t)\mu_4\Delta t + P_3(t)\mu_6\Delta t,\\
P_1(t+\Delta t) &= P_0(t)\lambda_1\Delta t + P_1(t)\left[1-\left(\lambda_2+\lambda_5+\mu_1\right)\Delta t\right] + P_2(t)\mu_2\Delta t + P_3(t)\mu_5\Delta t,\\
P_2(t+\Delta t) &= P_0(t)\lambda_4\Delta t + P_1(t)\lambda_2\Delta t + P_2(t)\left[1-\left(\lambda_3+\mu_2+\mu_4\right)\Delta t\right] + P_3(t)\mu_3\Delta t,\\
P_3(t+\Delta t) &= P_0(t)\lambda_6\Delta t + P_1(t)\lambda_5\Delta t + P_2(t)\lambda_3\Delta t + P_3(t)\left[1-\left(\mu_3+\mu_5+\mu_6\right)\Delta t\right],
\end{aligned} \tag{2.1}$$

where $P_j(t+\Delta t)$ is the probability that the system is in state j $(j=0,\ldots,3)$, at time $(t+\Delta t)$.

Lisnianski and Levitin [12] established a system of differential equations for the general case of k states. In the following, we take the four-state system as a special case and use the Laplace transformation technique to have the exact solution.

The differential equations for our system are as follows:

$$\begin{aligned}
P_0'(t) &= -(\lambda_1+\lambda_4+\lambda_6)P_0(t) + \mu_1P_1(t) + \mu_4P_2(t) + \mu_6P_3(t),\\
P_1'(t) &= \lambda_1P_0(t) - \left(\lambda_2+\lambda_5+\mu_1\right)P_1(t) + \mu_2P_2(t) + \mu_5P_3(t),\\
P_2'(t) &= \lambda_4P_0(t) + \lambda_2P_1(t) - \left(\lambda_3+\mu_2+\mu_4\right)P_2(t) + \mu_3P_3(t),\\
P_3'(t) &= \lambda_6P_0(t) + \lambda_5P_1(t) + \lambda_3P_2(t) - \left(\mu_3+\mu_5+\mu_6\right)P_3(t).
\end{aligned} \tag{2.2}$$

Taking the Laplace transformations of (2.2) yields the following linear equations in terms of $P_j^*(s)$ for $j=0,1,2,3$:

$$\begin{aligned}
sP_0^*(s) - P_0(0) &= -(\lambda_1+\lambda_4+\lambda_6)P_0^*(s) + \mu_1P_1^*(s) + \mu_4P_2^*(s) + \mu_6P_3^*(s),\\
sP_1^*(s) - P_1(0) &= \lambda_1P_0(s) - \left(\lambda_2+\lambda_5+\mu_1\right)P_1^*(s) + \mu_2P_2^*(s) + \mu_5P_3^*(s),\\
sP_2^*(s) - P_2(0) &= \lambda_4P_0^*(s) + \lambda_2P_1^*(s) - \left(\lambda_3+\mu_2+\mu_4\right)P_2^*(s) + \mu_3P_3^*(s),\\
sP_3^*(s) - P_3(0) &= \lambda_6P_0^*(s) + \lambda_5P_1^*(s) + \lambda_3P_2^*(s) - \left(\mu_3+\mu_5+\mu_6\right)P_3^*(s).
\end{aligned} \tag{2.3}$$

Under the initial conditions, $P_0(0) = 1$, $P_1(0) = 0$, $P_2(0) = 0$, and $P_3(0) = 0$, these means that the system is initially assumed to be in the working state, and we can rewrite the equations (2.3) in the following form:

$$\begin{aligned}
P_0^*(s) &= \frac{s^3 + a_1 s^2 + b_1 s + c_1}{s\left(s^3 + ds^2 + es + f\right)},\\
P_1^*(s) &= \frac{a_2 s^2 + b_2 s + c_2}{s\left(s^3 + ds^2 + es + f\right)},\\
P_2^*(s) &= \frac{a_3 s^2 + b_3 s + c_3}{s\left(s^3 + ds^2 + es + f\right)},\\
P_3^*(s) &= \frac{a_4 s^2 + b_4 s + c_4}{s\left(s^3 + ds^2 + es + f\right)},
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
a_1 &= \lambda_2 + \lambda_3 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6,\\
b_1 &= \mu_4\lambda_5 + \mu_5\lambda_3 + \mu_5\mu_2 + \mu_5\mu_4 + \mu_6\lambda_3 + \mu_6\mu_2 + \mu_6\mu_4 + \mu_3\mu_1 + \mu_3\lambda_5 + \lambda_2\mu_5 + \mu_3\lambda_2\\
&\quad + \mu_3\mu_2 + \mu_6\lambda_5 + \mu_6\lambda_2 + \mu_5\mu_1 + \mu_6\mu_1 + \lambda_3\lambda_2 + \lambda_3\mu_1 + \mu_2\mu_1 + \mu_2\lambda_5 + \lambda_3\lambda_5 + \mu_4\mu_1\\
&\quad + \mu_4\lambda_2 + \mu_3\mu_4,\\
c_1 &= \lambda_2\mu_5\mu_4 + \mu_6\lambda_2\mu_4 + \mu_6\lambda_5\lambda_3 + \mu_6\lambda_5\mu_2 + \mu_6\lambda_5\mu_4 + \mu_6\lambda_2\lambda_3 + \mu_3\mu_4\lambda_5 + \mu_3\mu_4\lambda_2 + \mu_1\mu_5\lambda_3\\
&\quad + \mu_1\mu_5\mu_2 + \mu_1\mu_5\mu_4 + \mu_1\mu_6\lambda_3 + \mu_1\mu_6\mu_2 + \mu_1\mu_6\mu_4 + \mu_1\mu_3\mu_2 + \mu_1\mu_3\mu_4,\\
d &= \lambda_2 + \mu_6 + \mu_3 + \mu_2 + \lambda_6 + \lambda_4 + \lambda_3 + \mu_4 + \lambda_1 + \mu_1 + \mu_5 + \lambda_5,\\
e &= \lambda_5\lambda_1 + \lambda_5\lambda_4 + \lambda_5\lambda_6 + \mu_1\lambda_6 + \mu_1\lambda_4 + \lambda_2\lambda_1 + \lambda_2\lambda_6 + \mu_4\lambda_5 + \mu_5\lambda_3 + \mu_5\mu_2 + \mu_5\mu_4 + \mu_6\lambda_3\\
&\quad + \mu_6\mu_2 + \lambda_3\lambda_1 + \mu_2\lambda_4 + \mu_2\lambda_6 + \mu_2\lambda_1 + \mu_4\lambda_6 + \mu_6\mu_4 + \mu_5\lambda_1 + \mu_5\lambda_6 + \mu_6\lambda_1 + \mu_4\lambda_1 + \lambda_3\lambda_4\\
&\quad + \lambda_3\lambda_6 + \mu_3\mu_1 + \mu_3\lambda_5 + \lambda_2\mu_5 + \mu_3\lambda_2 + \lambda_2\lambda_4 + \mu_3\mu_2 + \lambda_4\mu_6 + \mu_3\lambda_4 + \mu_3\lambda_6 + \mu_3\lambda_1 + \mu_6\lambda_5\\
&\quad + \mu_6\lambda_2 + \mu_5\mu_1 + \mu_6\mu_1 + \lambda_4\mu_5 + \lambda_3\lambda_2 + \lambda_3\mu_1 + \mu_2\mu_1 + \mu_2\lambda_5 + \lambda_3\lambda_5 + \mu_4\mu_1 + \mu_4\lambda_2 + \mu_3\mu_4,\\
f &= \mu_2\mu_1\lambda_6 + \mu_4\lambda_2\lambda_6 + \mu_4\lambda_5\lambda_1 + \mu_5\lambda_3\lambda_4 + \lambda_2\mu_5\mu_4 + \mu_6\lambda_2\mu_4 + \mu_6\lambda_5\lambda_3 + \mu_6\lambda_5\mu_2 + \mu_6\lambda_5\mu_4\\
&\quad + \mu_4\lambda_5\lambda_6 + \mu_4\mu_1\lambda_6 + \lambda_3\lambda_2\lambda_1 + \lambda_3\lambda_2\lambda_4 + \lambda_3\lambda_2\lambda_6 + \lambda_3\lambda_5\lambda_1 + \lambda_3\lambda_5\lambda_4 + \lambda_3\lambda_5\lambda_6 + \lambda_3\mu_1\lambda_4\\
&\quad + \lambda_3\mu_1\lambda_6 + \mu_6\lambda_2\lambda_3 + \mu_5\lambda_1\mu_4 + \mu_5\lambda_1\lambda_3 + \mu_5\lambda_1\mu_2 + \mu_3\lambda_4\mu_2 + \lambda_2\mu_5\lambda_1 + \mu_3\lambda_2\lambda_1 + \mu_3\lambda_2\lambda_4\\
&\quad + \mu_3\lambda_2\lambda_6 + \mu_3\lambda_5\lambda_1 + \mu_3\lambda_5\lambda_4 + \mu_3\lambda_5\lambda_6 + \mu_3\mu_1\lambda_4 + \mu_3\mu_1\lambda_6 + \lambda_4\mu_6\lambda_2 + \lambda_4\mu_6\lambda_5 + \lambda_4\mu_6\mu_1\\
&\quad + \lambda_2\mu_5\lambda_4 + \lambda_4\mu_6\mu_2 + \mu_6\lambda_1\lambda_3 + \mu_6\lambda_1\mu_2 + \mu_6\lambda_1\mu_4 + \mu_3\mu_4\lambda_5 + \mu_3\mu_2\lambda_1 + \mu_3\mu_4\lambda_1 + \mu_3\mu_4\lambda_2\\
&\quad + \mu_5\lambda_3\lambda_6 + \mu_3\lambda_6\mu_2 + \mu_5\mu_4\lambda_6 + \mu_2\lambda_5\lambda_1 + \mu_2\lambda_5\lambda_4 + \mu_2\lambda_5\lambda_6 + \lambda_2\mu_5\lambda_6 + \lambda_2\mu_6\lambda_1 + \mu_1\lambda_4\mu_5\\
&\quad + \mu_5\mu_2\lambda_6 + \lambda_4\mu_5\mu_2 + \mu_1\mu_5\lambda_3 + \mu_1\mu_5\mu_2 + \mu_1\mu_5\mu_4 + \mu_1\mu_6\lambda_3 + \mu_1\mu_6\mu_2 + \mu_1\mu_6\mu_4 + \mu_1\mu_3\mu_2\\
&\quad + \mu_1\mu_3\mu_4,\\
a_2 &= \lambda_1,\\
b_2 &= \lambda_3\lambda_1 + \mu_2\lambda_4 + \mu_2\lambda_1 + \mu_5\lambda_1 + \mu_5\lambda_6 + \mu_6\lambda_1 + \mu_4\lambda_1 + \mu_3\lambda_1,\\
c_2 &= \mu_5\lambda_3\lambda_4 + \mu_5\lambda_1\mu_4 + \mu_5\lambda_1\lambda_3 + \mu_5\lambda_1\mu_2 + \mu_3\lambda_4\mu_2 + \lambda_4\mu_6\mu_2 + \mu_6\lambda_1\lambda_3 + \mu_6\lambda_1\mu_2 + \mu_6\lambda_1\mu_4\\
&\quad + \mu_3\mu_2\lambda_1 + \mu_3\mu_4\lambda_1 + \mu_5\lambda_3\lambda_6 + \mu_3\lambda_6\mu_2 + \mu_5\mu_4\lambda_6 + \mu_5\mu_2\lambda_6 + \mu_4\mu_5\mu_2,
\end{aligned}$$

$$
\begin{aligned}
a_3 &= \lambda_4,\\
b_3 &= \lambda_5\lambda_4 + \mu_1\lambda_4 + \lambda_2\lambda_1 + \lambda_2\lambda_4 + \lambda_4\mu_6 + \mu_3\lambda_4 + \mu_3\lambda_6 + \lambda_4\mu_5,\\
c_3 &= \lambda_2\mu_5\lambda_1 + \mu_3\lambda_2\lambda_1 + \mu_3\lambda_2\lambda_4 + \mu_3\lambda_2\lambda_6 + \mu_3\lambda_5\lambda_1 + \mu_3\lambda_5\lambda_4 + \mu_3\lambda_5\lambda_6 + \mu_3\mu_1\lambda_4 + \mu_3\mu_1\lambda_6\\
&\quad + \lambda_4\mu_6\lambda_2 + \lambda_4\mu_6\lambda_5 + \lambda_4\mu_6\mu_1 + \lambda_2\mu_5\lambda_4 + \lambda_2\mu_5\lambda_6 + \lambda_2\mu_6\lambda_1 + \mu_1\lambda_4\mu_5,\\
a_4 &= \lambda_6,\\
b_4 &= \lambda_5\lambda_1 + \lambda_5\lambda_6 + \mu_1\lambda_6 + \lambda_2\lambda_6 + \mu_2\lambda_6 + \mu_4\lambda_6 + \lambda_3\lambda_4 + \lambda_3\lambda_6,\\
c_4 &= \mu_2\mu_1\lambda_6 + \mu_4\lambda_2\lambda_6 + \mu_4\lambda_5\lambda_1 + \mu_4\lambda_5\lambda_6 + \mu_4\mu_1\lambda_6 + \lambda_3\lambda_2\lambda_1 + \lambda_3\lambda_2\lambda_4 + \lambda_3\lambda_2\lambda_6 + \lambda_3\lambda_5\lambda_1\\
&\quad + \lambda_3\lambda_5\lambda_4 + \lambda_3\lambda_5\lambda_6 + \lambda_3\mu_1\lambda_4 + \lambda_3\mu_1\lambda_6 + \mu_2\lambda_5\lambda_1 + \mu_2\lambda_5\lambda_4 + \mu_2\lambda_5\lambda_6.
\end{aligned}
\tag{2.5}
$$

The probability that the system is in working state at time t is called the point availability of the system.

The system point availability is given by

$$A(t) = P_0(t) + P_1(t) + P_2(t). \tag{2.6}$$

The Laplace transform of $A(t)$ is given by

$$
\begin{aligned}
A^*(s) &= P_0^*(s) + P_1^*(s) + P_2^*(s),\\
A^*(s) &= \frac{s^3 + a_1 s^2 + b_1 s + c_1}{s(s^3 + ds^2 + es + f)} + \frac{a_2 s^2 + b_2 s + c_2}{s(s^3 + ds^2 + es + f)} + \frac{a_3 s^2 + b_3 s + c_3}{s(s^3 + ds^2 + es + f)}.
\end{aligned}
\tag{2.7}
$$

The steady state availability of the system is given by

$$A = \lim_{t\to\infty} A(t) = \lim_{s\to 0} sA^*(s), \tag{2.8}$$

$$A = \frac{c_1 + c_2 + c_3}{f}, \tag{2.9}$$

where $c_1, c_2, c_3,$ and f are defined as before.

Kim et al. [1] introduced different choices for the warranty and preventive maintenance. In the next section we develop these policies and apply them to the multistate system, present different warranty and preventive maintenance policies, and continue to evaluate the cost of these policies for the manufacturer and the buyer and the total cost.

3. The Warranty and Preventive Maintenance Policy for the Multistate System

All malfunctions of a product in the warranty period $[0, W)$ are repairable by manufacturer with a free warranty policy and charges no fee to the buyer. The lifetime of each product is L. In the postwarranty period $[W, L)$, the cost of repairing deteriorated product is borne by the buyer. In the warranty period, we assume that the cost of PM should be borne by the seller.

In the postwarranty period $[W, L)$, the cost of PM is borne by the buyer. We present here different options of warranty and PM policies as follows.

3.1. Policy A1

We assume that this policy have no PM action over the whole useful life of the system. Suppose that the useful life of the system is L and the warranty period $[0, W)$ includes the first state, in this case we compute the warranty cost for the manufacturer and for the buyer as follows.

(i) The warranty cost for the manufacturer can be represented by

$$\begin{aligned} C_{MA1} &= C_R \int_0^W (\text{failure rate for the first state}) dt \\ &= C_R \int_0^W \lambda_1 dt = C_R \lambda_1 W, \end{aligned} \tag{3.1}$$

where C_{MA1} is the cost for the manufacturer in policy A1 and C_R is the repair cost.

(ii) The warranty cost for the buyer can be represented by

$$\begin{aligned} C_{BA1} &= C_R \int_W^{1/\lambda_2} (\text{failure rate for the second state}) dt \\ &\quad + C_R \int_{1/\lambda_2}^{L} (\text{failure rate for the third state}) dt, \\ C_{BA1} &= C_R \int_W^{1/\lambda_2} \lambda_2 dt + C_R \int_{1/\lambda_2}^{L} \lambda_3 dt = C_R \left(1 - \lambda_2 W + \lambda_3 L - \frac{\lambda_3}{\lambda_2}\right), \end{aligned} \tag{3.2}$$

where C_{BA1} is the cost for the buyer in policy A1.

3.2. Policy A2

This policy also assumes no PM action over the whole useful life of the system, and the warranty includes the first and the second states.

(i) The warranty cost for the manufacturer can be represented by

$$C_{MA2} = C_R \int_0^{1/\lambda_1} \lambda_1 dt + C_R \int_{1/\lambda_1}^{W} \lambda_2 dt = C_R \left(1 + \lambda_2 W - \frac{\lambda_2}{\lambda_1}\right). \tag{3.3}$$

(ii) The warranty cost for the buyer can be represented by

$$C_{BA2} = C_R \int_W^L \lambda_3 dt = C_R \lambda_3 \ (L - W). \tag{3.4}$$

3.3. *Policy B1*

We assume that this policy have continuous PM action over the whole useful life of the system, and the warranty includes the first state.

(i) The warranty cost for the manufacturer is computed as follows:

$$C_{MB1} = C_R \int_0^W \lambda_1 dt + C_{\mathrm{PM}} W = C_R \lambda_1 W + C_{\mathrm{PM}} W, \tag{3.5}$$

where C_{PM} is the cost for the PM.

(ii) The warranty cost for the buyer is computed as follows:

$$\begin{aligned} C_{BB1} &= C_R \int_W^{1/\lambda_2} \lambda_2 dt + C_R \int_{1/\lambda_2}^L \lambda_3 dt + C_{\mathrm{PM}}(L - W) \\ &= C_R \left(1 - \lambda_2 W + \lambda_3 L - \frac{\lambda_3}{\lambda_2}\right) + C_{\mathrm{PM}}(L - W). \end{aligned} \tag{3.6}$$

3.4. *Policy B2*

We assume that this policy have continuous PM action over the whole useful life of the system, and the warranty includes the first state and the second state.

(i) The warranty cost for the manufacturer

$$\begin{aligned} C_{MB2} &= C_R \int_0^{1/\lambda_1} \lambda_1 dt + C_R \int_{1/\lambda_1}^W \lambda_2 dt + C_{\mathrm{PM}} W \\ &= C_R \left(1 + \lambda_2 W - \frac{\lambda_2}{\lambda_1}\right) + C_{\mathrm{PM}} W. \end{aligned} \tag{3.7}$$

(ii) The warranty cost for the buyer

$$C_{BB2} = C_R \int_W^L \lambda_3 dt + C_{\mathrm{PM}}(L - W) = C_R \lambda_3 (L - W) + C_{\mathrm{PM}}(L - W). \tag{3.8}$$

3.5. Policy C1

We assume that this policy have no PM action over the warranty period and continuous PM over the postwarranty period, and the warranty includes the first state.

(i) The warranty cost for the manufacturer is computed as follows:

$$C_{MC1} = C_R \int_0^W \lambda_1 dt = C_R \lambda_1 W. \tag{3.9}$$

(ii) The warranty cost for the buyer is computed as follows:

$$\begin{aligned} C_{BC1} &= C_R \int_W^{1/\lambda_2} \lambda_2 dt + C_R \int_{1/\lambda_2}^{L} \lambda_3 dt + C_{\mathrm{PM}}(L - W) \\ &= C_R \left(1 - \lambda_2 W + \lambda_3 L - \frac{\lambda_3}{\lambda_2}\right) + C_{\mathrm{PM}}(L - W). \end{aligned} \tag{3.10}$$

3.6. Policy C2

We assume that this policy have no PM action over the warranty period and continuous PM over the postwarranty period, and the warranty includes the first state and the second state.

(i) The warranty cost for the manufacturer is given as:

$$C_{MC2} = C_R \int_0^{1/\lambda_1} \lambda_1 dt + C_R \int_{1/\lambda_1}^{W} \lambda_2 dt = C_R \left(1 + \lambda_2 W - \frac{\lambda_2}{\lambda_1}\right). \tag{3.11}$$

(ii) The warranty cost for the buyer is given as:

$$C_{BC2} = C_R \int_W^L \lambda_3 dt + C_{\mathrm{PM}}(L - W) = C_R \lambda_3 \ (L - W) + C_{\mathrm{PM}}(L - W). \tag{3.12}$$

3.7. Policy D1

We assume that this policy have continuous PM action over the warranty period and no PM over the postwarranty period, and the warranty includes the first state.

(i) The warranty cost for the manufacturer is computed as follows:

$$C_{MD1} = C_R \int_0^W \lambda_1 dt + C_{\mathrm{PM}} W = C_R \lambda_1 W + C_{\mathrm{PM}} W. \tag{3.13}$$

(ii) The warranty cost for the buyer is computed as follows:

$$C_{BD1} = C_R \int_W^{1/\lambda_2} \lambda_2 dt + C_R \int_{1/\lambda_2}^{L} \lambda_3 dt = C_R\left(1 - \lambda_2 W + \lambda_3 L - \frac{\lambda_3}{\lambda_2}\right). \tag{3.14}$$

3.8. Policy D2

We assume that this policy have continuous PM action over the warranty period and no PM over the postwarranty period, and the warranty includes the first state and the second state.

(i) The warranty cost for the manufacturer is given as:

$$\begin{aligned} C_{MD2} &= C_R \int_0^{1/\lambda_1} \lambda_1 dt \ + C_R \int_{1/\lambda_1}^{W} \lambda_2 dt + C_{\mathrm{PM}} W \\ &= C_R\left(1 + \lambda_2 W - \frac{\lambda_2}{\lambda_1}\right) + C_{\mathrm{PM}} W. \end{aligned} \tag{3.15}$$

(ii) The warranty cost for the buyer is given as:

$$C_{BD2} = C_R \int_W^L \lambda_3 dt = C_R \lambda_3 (L - W). \tag{3.16}$$

In the next section, we will show the cost of these polices by two numerical examples.

4. Numerical Examples

4.1. Example 1

We assume that the exponential distribution for the failure time and repair time for the four-state system.

The parameters values of the failure rates and repair rates were given as (Lisnianski and Levitin [12]):

$$\begin{gathered} \lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = 0.7, \quad \lambda_4 = 0.3, \quad \lambda_5 = 0.4, \quad \lambda_6 = 0.1, \\ \mu_1 = 100, \quad \mu_2 = 80, \quad \mu_3 = 50, \quad \mu_4 = 45, \quad \mu_5 = 40, \quad \mu_6 = 32. \end{gathered} \tag{4.1}$$

Then the steady state availability will be 0.9991.

Now to compare between the different policies, we introduce four values for repair cost: 100, 200, 1000, and 3000 and two values for preventive maintenance cost: 20, 10. Table 1 illustrates the total cost, the manufacturer cost, and buyer cost resulting from using these policies.

Table 1

Choices	1	2	3	4	5	6	7	8
C_R	100	100	200	200	1000	1000	3000	3000
C_{PM}	20	10	20	10	20	10	20	10
C_{MA1}	600	600	1200	1200	6000	6000	18000	18000
C_{BA1}	430	430	860	860	4300	4300	12900	12900
Tot. C_{A1}	1030	1030	2060	2060	10300	10300	30900	30900
C_{MA2}	350	350	700	700	3500	3500	10500	10500
C_{BA2}	490	490	980	980	4900	4900	14700	14700
Tot. C_{A2}	840	840	1680	1680	8400	8400	25200	25200
C_{MB1}	660	630	1260	1230	6060	6030	18060	18030
C_{BB1}	570	500	1000	930	4440	4370	13040	12970
Tot. C_{B1}	1230	1130	2260	2160	10500	10400	31100	31000
C_{MB2}	410	380	760	730	3560	3530	10560	10530
C_{BB2}	630	560	1120	1050	5040	4970	14840	14770
Tot. C_{B2}	1040	940	1880	1780	8600	8500	25400	25300
C_{MC1}	600	600	1200	1200	6000	6000	18000	18000
C_{BC1}	570	500	1000	930	4440	4370	13040	12970
Tot. C_{C1}	1170	1100	2200	2130	10440	10370	31040	30970
C_{MC2}	350	350	700	700	3500	3500	10500	10500
C_{BC2}	630	560	1120	1050	5040	4970	14840	14770
Tot. C_{C2}	980	910	1820	1750	8540	8470	25340	25270
C_{MD1}	660	630	1260	1230	6060	6030	18060	18030
C_{BD1}	430	430	860	860	4300	4300	12900	12900
Tot. C_{D1}	1090	1060	2120	2090	10360	10330	30960	30930
C_{MD2}	410	380	760	730	3560	3530	10560	10530
C_{BD2}	490	490	980	980	4900	4900	14700	14700
Tot. C_{D2}	900	870	1740	1710	8460	8430	25260	25230

From Table 1, comparing the suggestion policies by regarding the expected total cost, we find that policy A2 is the optimal policy, where no PM action over the whole useful life of the system is assumed and the warranty includes the first and the second states.

If we order these policies according to the least total cost, we find that A2 policy is the least-costing policy for choice 1, followed by policies D2, C2, A1, B2, D1, C1, B1 in an ascending order regarding the cost. But the rest of choices (2–8), although A2 is still the least-costing policy for each, are of a bit different sequence (D2, C2, B2, A1, D1, C1, B1) also in an ascending order regarding the cost.

But from the manufacturer point of view, if we order these policies according to the least cost, we find that the policies can be ordered in an ascending order regarding the cost as follows, (A2 or C2), (B2 or D2), (A1 or C1), (B1 or D1). So we find that the least-costing policy for the manufacturer is A2 or C2.

From the buyer point of view, we find that for the choices (1–4), the policies can be ordered as follows: (A1 or D1), (A2 or D2), (B1 or C1), (B2 or C2). But for the choices (5–8) the policies can be ordered as follows: (A1 or D1), (B1 or C1), (A2 or D2), (B2 or C2) in an ascending order regarding the cost. So the least-costing policy for the buyer is A1 or D1.

Table 2

Choices	1	2	3	4	5	6	7	8
C_R	100	100	200	200	1000	1000	3000	3000
C_{PM}	40	20	40	20	40	20	40	20
C_{MA1}	150	150	300	300	1500	1500	4500	4500
C_{BA1}	283.1	283.1	566.2	566.2	2831.2	2831.2	8493.7	8493.7
Tot. C_{A1}	433.1	433.1	866.2	866.2	4331.2	4331.2	12994	12993.7
C_{MA2}	122.2	122.2	244.3	244.3	1221.7	1221.7	3665	3665
C_{BA2}	300	300	600	600	3000	3000	9000	9000
Tot. C_{A2}	422.2	422.2	844.3	844.3	4221.7	4221.7	12665	12665
C_{MB1}	350	250	500	400	1700	1600	4700	4600
C_{BB1}	883.1	583.1	1166.2	866.2	3431.2	3131.2	9093.7	8793.7
Tot. C_{B1}	1233.1	833.1	1666.2	1266.2	5131.2	4731.2	13793.7	13393.7
C_{MB2}	322.2	222.2	444.3	344.3	1421.7	1321.7	3865	3765
C_{BB2}	900	600	1200	900	3600	3300	9600	9300
Tot. C_{B2}	1222.2	822.2	1644.3	1244.3	5021.7	4621.7	13465	13065
C_{MC1}	150	150	300	300	1500	1500	4500	4500
C_{BC1}	883.1	583.1	1166.2	866.2	3431.2	3131.2	9093.7	8793.7
Tot. C_{C1}	1033.1	733.1	1466.2	1166.2	4931.2	4631.2	13593.7	13293.7
C_{MC2}	122.2	122.2	244.3	244.3	1221.7	1221.7	3665	3665
C_{BC2}	900	600	1200	900	3600	3300	9600	9300
Tot. C_{C2}	1022.2	722.2	1444.3	1144.3	4821.7	4521.7	13265	12965
C_{MD1}	350	250	500	400	1700	1600	4700	4600
C_{BD1}	283.1	283.1	566.2	566.2	2831.2	2831.2	8493.7	8493.7
Tot. C_{D1}	633.1	533.1	1066.2	966.2	4531.2	4431.2	13193.7	13093.7
C_{MD2}	322.2	222.2	444.3	344.3	1421.7	1321.7	3865	3765
C_{BD2}	300	300	600	600	3000	3000	9000	9000
Tot. C_{D2}	622.2	522.2	1044.3	944.3	4421.7	4321.7	12865	12765

4.2. Example 2

In this example we assume the failure rates and repair rates are as follows:

$$\begin{gathered}\lambda_1 = 0.3, \quad \lambda_2 = 0.133, \quad \lambda_3 = 0.2, \quad \lambda_4 = 0.333, \quad \lambda_5 = 0.15, \quad \lambda_6 = 0.25, \\ \mu_1 = 0.467, \quad \mu_2 = 0.326, \quad \mu_3 = 0.256, \quad \mu_4 = 0.167, \quad \mu_5 = 0.526, \quad \mu_6 = 0.356.\end{gathered} \tag{4.2}$$

Then the steady state availability will be 0.8522.

To compare between the different policies, we introduce four values for repair cost: 100, 200, 1000, and 3000 and two values for preventive maintenance cost: 40, 20. Table 2 illustrates the total cost, the manufacturer cost, and buyer cost resulting from using these policies.

From Table 2, comparing the suggestion policies by regarding the expected total cost, we find that policy A2 is the optimal policy.

For choices (1–5) we find that the policies can be ordered as follows: A2, A1, D2, D1, C2, C1, B2, B1, arranged in an ascending order regarding the cost.

But for choices (6, 7) the policies can be ordered as follows: A2, D2, A1, D1, C2, B2, C1, B1, and for the choice (8) the policies can be ordered as follows: A2, D2, C2, A1, B2, D1, C1, B1, also arranged in an ascending order regarding the cost.

But from the manufacturer point of view, if we order these policies according to the least cost, we find that for choices (1–4) the policies can be arranged as follows: (A2 or C2), (A1 or C1), (B2 or D2), (B1 or D1).

For choices (5–8) the policies can be ordered as follows: (A2 or C2), (B2 or D2), (A1 or C1), (B1 or D1).

From the buyer point of view, we find that for choices (1–7) the policies can be ordered as follows: (A1 or D1), (A2 or D2), (B1 or C1), (B2 or C2). And for the choice (8) the policies can be ordered as follows: (A1 or D1), (B1 or C1), (A2 or D2), (B2 or C2).

For both the manufacturer and the buyer, the policies are also arranged in an ascending order regarding the cost.

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