Review Article

Exponential Stability and Guaranteed Cost of Switched Linear Systems with Mixed Time-Varying Delays

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This paper deals with the problems of exponential stability and guaranteed cost of switched linear systems with mixed time delays. Based on Lyapunov functional method, we present new delay-dependent conditions that guarantee both the exponential stability and an upper bound for some performance index. The criteria are delay-dependent conditions and are given in terms of linear matrix inequalities. Numerical examples are provided to illustrate the effectiveness of the results.

1. Introduction

A switched system is a dynamical system that includes several subsystems and a logical rule that orchestrates switching between these subsystems at each instant of time [1]. The logical rule generates switching signals to determine which subsystem will be activated on a certain time interval. In fact, switching systems arise in many practical processes that cannot be described by exclusively continuous or exclusively discrete models in manufacturing, communication networks automotive engine control, chemical processes, and so on see, for example, [2, 3]. There have been many studies on stability of switching systems [4–6].

The stability analysis of switched time delay has attracted a lot of attention [7–9]. The main approach for stability analysis relies on the use of Lyapunov Krasovskii functionals and LMI approach for constructing common Lyapunov function and switching rules [10–12]. In [13], the asymptotic stability of switched linear time delay symmetric systems has been studied. In [11], a switching system composed by a finite number of linear point time delay differential equations has been studied, and it has been shown that the asymptotic stability may be achieved by using a common Lyapunov function method and minimum switching rule. The results of [11] have been extended in [14] to linear switching system with

discrete and distributed delays. The delays considered are time invariant, and the conditions are given in terms of maximal and minimal eigenvalues of certain matrices. The exponential stability problem was considered in [15] for switching linear systems with impulsive effects by using the matrix measure concept. The exponential stability of time delay systems has been considered in [16, 17], and switching linear systems with mixed delays have been studied in [18]. In [19], sufficient stability conditions have been obtained by using a generalisation of Halanay's inequality.

In this paper, we will focus on exponential stability of switched linear systems with mixed time delays. The subsystems considered are continuous with time-varying delays. The derived conditions are delay dependent and are given in terms of LMIs. The conditions guarantee both the exponential stability and an upper bound of a performance index. The approach is based on Lyapunov Krasovskii functional and allows us to avoid explicitly substitution of the dynamic system equation in the derivative of the Lyapunov function. Furthermore, we use an extended variable to completely avoid bounding treatment of the weighted cross-product of the instantaneous state and the delayed state. As a consequence, the conservatism of the conditions is expected to be reduced with respect to earlier works (e.g. [10, 16, 20]). The conditions allow us to compute easily both the stability factor and the decay rate of the solution. Numerical computations are performed for illustration.

2. Preliminaries

Consider a class of switched linear systems with time-varying delay of the form

$$\begin{aligned}\dot{x}(t) &= A_\sigma x(t) + D_\sigma x(t - h(t)) + E_\sigma \int_{t-\tau(t)}^t x(s) ds, \quad t > 0, \\ x(t) &= \phi(t), \quad t \in [-r, 0],\end{aligned}\tag{2.1}$$

where $x \in R^n$ is the state, $\sigma(x) : R^n \rightarrow I := \{1, 2, \dots, N\}$ is the switching rule which is piece-wise constant function depending on the state in each time, $\phi \in C([-h, 0], R^n)$ is the initial function, with norm $\|\phi\| = \sup_{\theta \in [-h, 0]} \|\phi(\theta)\|$, $A_\sigma, D_\sigma, E_\sigma \in \{[A_i, D_i, E_i], i = 1, 2, \dots, N\}$, A_i , D_i , and E_i are given matrices. Moreover $\sigma(x) = i$ implies that the i th subsystem is activated, and we have the following subsystem:

$$\dot{x}(t) = A_i x(t) + D_i x(t - h(t)) + E_i \int_{t-\tau(t)}^t x(s) ds, \quad t > 0.\tag{2.2}$$

$h(t)$ and $\tau(t)$ are unknown time-varying delay terms, but bounded by

$$0 \leq h(t) \leq \bar{h}, \quad \dot{h}(t) \leq \mu_h,\tag{2.3}$$

$$0 \leq \tau(t) \leq \bar{\tau}, \quad \tau(t) \leq \mu_\tau,\tag{2.4}$$

where $\bar{h}$, $\bar{\tau}$, μ_h , and μ_τ are given nonnegative constants, $r = \max(\bar{h}, \bar{\tau})$.

In this paper, we are interested in establishing conditions guaranteeing the asymptotic stability of switched system (2.1) and finding the least upper bound for the cost function given by

$$J = \int_0^{+\infty} x^T(t) Y x(t) dt, \quad (2.5)$$

where $Y > 0$. Assume that there exists a Hurwitz linear convex combination $\tilde{A}$ of A_i , that is,

$$\tilde{A} = \sum_{i=1}^N \lambda_i A_i, \quad 0 < \lambda_i < 1, \quad \sum_{i=1}^N \lambda_i = 1. \quad (2.6)$$

Then for a positive definite matrix Z_1 , there exists a positive definite matrix R_{11} satisfying

$$\tilde{A}^T R_{11} + R_{11} \tilde{A} = -Z_1. \quad (2.7)$$

Hence for $x \neq 0$, we can write

$$x^T (\tilde{A}^T R_{11} + R_{11} \tilde{A}) x = -x^T Z_1 x < 0. \quad (2.8)$$

Which implies that

$$\lambda_i x^T (A_i^T R_{11} + R_{11} A_i) x < 0. \quad (2.9)$$

for at least one i . So for a positive definite matrix Z , construct N region of R^n as follows:

$$\Omega_i = \left\{ x \in R^n : x^T (A_i^T R_{11} + R_{11} A_i) x < -x^T Z x \right\}, \quad i \in I. \quad (2.10)$$

It is clear that $\bigcup_{i=1}^N \Omega_i = R^n \setminus \{0\}$.

Denote

$$\overline{\Omega}_1 = \Omega_1, \quad \overline{\Omega}_i = \Omega_i \setminus \bigcup_{j=1}^{i-1} \Omega_j, \quad i = 2, \dots, N. \quad (2.11)$$

It follows that $\bigcup_{i=1}^N \overline{\Omega}_i = R^n \setminus \{0\}$, $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$, $i \neq j$. The switching rule is chosen as follows:

$[S_R]$: Step 0: let $x(t) = \phi(t)$.

Step 1: set $\sigma(x) = \arg \min \{x^T (A_i^T R_{11} + R_{11} A_i) x\}$.

Step 2: stay in the i th mode as long as $x(t)$ in $\overline{\Omega}_i$.

Step 3: if $x(t)$ hits the boundary of $\overline{\Omega}_i$, go to Step 1 to determine the next mode.

Definition 2.1. Given $\alpha > 0$, the system (2.1) is α -exponentially stable if there exist a switching rule σ and a constant $\beta \geq 1$ such that every solution $x(t, \phi)$ of the system satisfies the following inequality:

$$\|x(t, \phi)\| \leq \beta e^{-\alpha t} \|\phi\|, \quad t \geq 0. \quad (2.12)$$

The following lemmas will be useful.

Lemma 2.2. For any $X, Y \in R^n$, matrices $M > 0$, $F^T F \leq I$, one has

$$2X^T Y \leq X^T M^{-1} X + Y^T M Y. \quad (2.13)$$

Lemma 2.3. For any constant matrix $M = M^T \in R^{n \times n}$, $M > 0$, scalar $\gamma \geq \eta(t) > 0$, vector function $\omega : [0, \gamma] \rightarrow R^n$ such that the integrations in the following are well defined, then

$$\eta(t) \int_0^{\eta(t)} \omega^T(\beta) M \omega(\beta) d\beta \geq \left[\int_0^{\eta(t)} \omega(\beta) d\beta \right]^T M \left[\int_0^{\eta(t)} \omega(\beta) d\beta \right]. \quad (2.14)$$

3. Main Results

Let,

$$\beta_1 = \lambda_{\min}(R_{11}), \quad (3.1)$$

$$\beta_2 = \left(1 + \bar{h}^2\right) \lambda_{\max}(R) + \frac{1 - e^{-2\alpha\bar{h}}}{2\alpha} \lambda_{\max}(P) + \frac{e^{-2\alpha\bar{h}} + 2\alpha\bar{h} - 1}{4\alpha^2} \lambda_{\max}(Q) + \frac{e^{-2\alpha\bar{\tau}} + 2\alpha\bar{\tau} - 1}{4\alpha^2} \lambda_{\max}(S), \quad (3.2)$$

$$\beta = \sqrt{\frac{\beta_2}{\beta_1}},$$

$$L_i = R_{11} A_i + A_i^T R_{11} + Z,$$

$$\bar{Z} = \begin{pmatrix} \mu_h R_{12} & 0 \\ 0 & \mu_h R_{22} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{Z} = \text{diag}\{\mu_h Z_1, \mu_h Z_2\}, \quad (3.3)$$

$$G_i = \begin{pmatrix} G_{11}^i & G_{12}^i & G_{13}^i & R_{11} E_i + A_i^T F_3^T & -F_0 + A_i^T F_4^T \\ * & G_{22}^i & G_{23}^i & D_i^T F_3^T & -F_1 + D_i^T F_4^T \\ * & * & G_{33}^i & R_{12}^T E_i + E_i^T F_3^T & -F_2 + E_i^T F_4^T \\ * & * & * & -\frac{1 - \mu_\tau}{\bar{\tau}} e^{-2\alpha\bar{\tau}} & -F_3 \\ * & * & * & * & -F_4 - F_4^T \end{pmatrix},$$

where “*” denotes the symmetric part in a symmetric matrix and,

$$\begin{aligned}
 G_{11}^i &= R_{12} + R_{12}^T + P + 2\alpha R_{11} + \bar{h}Q + \bar{\tau}S + F_0 A_i + A_i^T F_0^T - Z, \\
 G_{12}^i &= -R_{12} + R_{11} D_i + F_0 D_i + A_i^T F_1^T, \\
 G_{13}^i &= 2\alpha R_{12} + R_{22} + A_i^T F_2^T + F_0 E_i + A_i^T R_{12}, \\
 G_{22}^i &= -(1 - \mu_h) e^{-2\alpha \bar{h}} P + F_1 D_i + D_i^T F_1^T + \mu_h Z_1, \\
 G_{23}^i &= -R_{22} + D_i^T F_2^T + D_i R_{12} + F_1 E_i, \\
 G_{33}^i &= -\frac{1}{h} e^{-2\alpha \bar{h}} Q + 2\alpha R_{22} + E_i^T F_2^T + F_2 E_i + \mu_h Z_2.
 \end{aligned} \tag{3.4}$$

Theorem 3.1. For given $\alpha > 0$, switched linear system (2.1) is α -exponentially stable if there exist symmetric positive definite matrices $Z, Z_1, Z_2, S, P, Q, R_{11}, R_{22}, Y$, matrices $R_{12}, F_0, F_1, F_2, F_3$, and F_4 satisfying.

(i) There exist $0 < \lambda_i < 1, \quad i = 1, 2, \dots, N$ such that $\sum_{i=1}^N \lambda_i = 1$ and

$$\sum_{i=1}^N \lambda_i L_i < -Y. \tag{3.5}$$

(ii)

$$\begin{pmatrix} G_i & \bar{Z} \\ \bar{Z}^T & -\hat{Z} \end{pmatrix} < 0, \quad i = 1, \dots, N. \tag{3.6}$$

The switching rule is given by $[S_R]$, and the solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \leq \beta e^{-\alpha t} \|\phi\|, \quad t \geq 0. \tag{3.7}$$

Furthermore, the cost function in (2.5) satisfies

$$\begin{aligned}
 J \leq J^* &= \begin{pmatrix} \phi(0) \\ \int_{0-h(0)}^0 \phi(s) ds \end{pmatrix}^T R \begin{pmatrix} \phi(0) \\ \int_{0-h(0)}^0 \phi(s) ds \end{pmatrix} + \int_{-\bar{h}}^0 e^{2\alpha\theta} \phi^T(\theta) P \phi(\theta) d\theta \\
 &+ \int_{-\bar{h}}^0 \int_s^0 e^{2\alpha\theta} \phi^T(\theta) Q \phi(\theta) d\theta ds + \int_{-\tau(0)}^0 \int_s^0 e^{2\alpha\theta} \phi^T(\theta) S \phi(\theta) d\theta ds.
 \end{aligned} \tag{3.8}$$

Proof. Let $x_t := \{x(t+s), s \in [-r, 0]\}$ and consider the following Lyapunov Krasovskii functional:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(x_t). \tag{3.9}$$

With

$$\begin{aligned}
 V_1(x_t) &= \begin{pmatrix} x(t) \\ \int_{t-h(t)}^t x(s) ds \end{pmatrix}^T R \begin{pmatrix} x(t) \\ \int_{t-h(t)}^t x(s) ds \end{pmatrix}, \\
 V_2(x_t) &= \int_{-\bar{h}}^0 e^{2\alpha s} x^T(t+s) P x(t+s) ds, \\
 V_3(x_t) &= \int_{-\bar{h}}^0 \int_s^0 e^{2\alpha\theta} x^T(t+\theta) Q x(t+\theta) d\theta ds, \\
 V_4(x_t) &= \int_{-\tau(t)}^0 \int_s^0 e^{2\alpha\theta} x^T(t+\theta) S x(t+\theta) d\theta ds.
 \end{aligned} \tag{3.10}$$

It is easy to verify that

$$0 < \beta_1 \|x(t)\|^2 \leq V(x_t) \leq \beta_2 \|x_t\|^2, \tag{3.11}$$

where β_1 and β_2 are defined by (3.1) and (3.2). Computing the first time derivative of $V(x_t)$, we obtain

$$\begin{aligned}
 \dot{V}_1(x_t) &= 2 \begin{pmatrix} x(t) \\ \int_{t-h(t)}^t x(s) ds \end{pmatrix}^T R \begin{pmatrix} \dot{x}(t) \\ x(t) - (1 - \dot{h}(t))x(t-h(t)) \end{pmatrix} \\
 &= 2 \begin{pmatrix} x(t) \\ \int_{t-h(t)}^t x(s) ds \end{pmatrix}^T R \begin{pmatrix} A_i x(t) + D_i x(t-h(t)) + E_i \int_{t-\tau(t)}^t x(s) ds, \\ x(t) - (1 - \dot{h}(t))x(t-h(t)) \end{pmatrix}.
 \end{aligned} \tag{3.12}$$

Letting $\xi(t) = [x^T(t) \ x^T(t-h(t)) \ \int_{t-h(t)}^t x^T(s) ds \ \int_{t-\tau(t)}^t x^T(s) ds]^T$, taking account of (2.3), and using the fact that for some positive definite matrices Z_1 and Z_2 , the following inequalities hold

$$\begin{aligned}
 2\dot{h}(t)x^T(t)R_{12}x(t-h(t)) &\leq \mu_h x^T(t)R_{12}Z_1^{-1}R_{12}^T x(t) + \mu_h x^T(t-h(t))Z_1 x(t-h(t)), \\
 2\dot{h}(t)x^T(t-h(t))R_{22} \int_{t-h(t)}^t x(s) ds &\leq \mu_h x^T(t-h(t))R_{22}Z_2^{-1}R_{22}x(t-h(t)) \\
 &\quad + \mu_h \left[\int_{t-h(t)}^t x(s) ds \right]^T Z_2 \left[\int_{t-h(t)}^t x(s) ds \right].
 \end{aligned} \tag{3.13}$$

We can write

$$\dot{V}_1(x_t) \leq x^T(t) \left[R_{11}A_i + A_i^T R_{11} \right] x(t) + \xi^T(t) T_1 \xi(t) - 2\alpha V_1(x_t), \tag{3.14}$$

where

$$T_1 = \begin{pmatrix} R_{12} + R_{12}^T + 2\alpha R_{11} + \mu_h R_{12} Z_1^{-1} R_{12}^T & * & * & * \\ D_i^T R_{11} - R_{12}^T & \mu_h Z_1 + \mu_h R_{22} Z_2^{-1} R_{22}^T & * & * \\ R_{22} + 2\alpha R_{12}^T + R_{12}^T A_i & R_{12}^T D_i - R_{22} & 2\alpha R_{22} & * \\ E_i^T R_{11} & 0 & E_i^T R_{12} & 0 \end{pmatrix}. \quad (3.15)$$

And “*” denotes the symmetric part in a symmetric matrix

$$\begin{aligned} \dot{V}_2(x_t) &= x^T(t) P x(t) - (1 - \dot{h}(t)) e^{-2\alpha h(t)} x^T(t - h(t)) P x(t - h(t)) - 2\alpha V_2(x_t) \\ &\leq x^T(t) P x(t) - (1 - \mu_h) e^{-2\alpha \bar{h}} x^T(t - h(t)) P x(t - h(t)) - 2\alpha V_2(x_t), \\ \dot{V}_3(x_t) &= \bar{h} x^T(t) Q x(t) - \int_{t-\bar{h}}^t e^{2\alpha(\theta-t)} x^T(\theta) Q x(\theta) d\theta - 2\alpha V_3(x_t) \\ &\leq \bar{h} x^T(t) Q x(t) - \frac{e^{-2\alpha \bar{h}}}{\bar{h}} \left[\int_{t-h(t)}^t x(\theta) d\theta \right]^T Q \left[\int_{t-h(t)}^t x(\theta) d\theta \right] - 2\alpha V_3(x_t). \end{aligned} \quad (3.16)$$

Taking account of (2.4) and applying Lemma 2.3, we obtain

$$\begin{aligned} \dot{V}_4(x_t) &= \tau(t) x^T(t) Q x(t) - (1 - \dot{\tau}(t)) \int_{t-\tau(t)}^t e^{2\alpha(\theta-t)} x^T(\theta) S x(\theta) d\theta - 2\alpha V_4(x_t) \\ &\leq \bar{\tau} x^T(t) Q x(t) - \frac{(1 - \mu_\tau)}{\bar{\tau}} e^{-2\alpha \bar{\tau}} \left[\int_{t-\tau(t)}^t x(\theta) d\theta \right]^T S \left[\int_{t-\tau(t)}^t x(\theta) d\theta \right] - 2\alpha V_4(x_t). \end{aligned} \quad (3.17)$$

Now let $B_i = [A_i \ D_i \ E_i \ 0 \ -I]$ and $F = [F_0^T \ F_1^T \ F_2^T \ F_3^T \ F_4^T]^T$. We can easily verify that $B_i \chi = 0$, where $\chi(t) = [\xi^T(t) \ x^T(t)]^T$, and

$$\chi^T(t) \left[F B_i + B_i^T F^T \right] \chi(t) = 0, \quad i = 1, 2, \dots, N. \quad (3.18)$$

Taking account of (3.18), and adding and subtracting the term $x^T(t) Z x(t)$ with Z a positive definite matrix, we get

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq x^T(t) \left(R_{11} A_i + A_i^T R_{11} + Z \right) x(t) + \chi^T(t) \left(G_i + \bar{Z} \hat{Z}^{-1} \bar{Z}^T \right) \chi(t). \quad (3.19)$$

Since the condition (3.6) holds we have $\chi^T(t) (G_i + \bar{Z} \hat{Z}^{-1} \bar{Z}^T) \chi(t) < 0$, it follows that

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq x^T(t) L_i x(t). \quad (3.20)$$

From condition (i), we have $\sum_{i=1}^N \lambda_i L_i < -Y$, where $1 > \lambda_i > 0$, $i = 1, 2, \dots, N$ and $\sum_{i=1}^N \lambda_i = 1$. So

$$\sum_{i=1}^N \lambda_i \min_{i=1, \dots, N} x^T(t) L_i x(t) \leq \sum_{i=1}^N \lambda_i x^T(t) L_i x(t) < -x^T(t) Y x(t) < 0. \quad (3.21)$$

By choosing the switching rule as

$$\sigma(x) = \arg \min_{i=1, \dots, N} x^T(t) L_i x(t). \quad (3.22)$$

We have

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq x^T(t) L_i x(t) \leq \sum_{i=1}^N \lambda_i x^T(t) L_i x(t) < -x^T(t) Y x(t) < 0. \quad (3.23)$$

This implies that $V(x_t) \leq V(\phi) e^{-2\alpha t}$, $t \geq 0$. Taking account of (3.11), we obtain

$$\beta_1 \|x(t)\|^2 \leq V(x_t) \leq V(\phi) e^{-2\alpha t} \leq \beta_2 e^{-2\alpha t} \|\phi\|^2. \quad (3.24)$$

And then, $\|x(t)\| \leq \beta e^{-\alpha t} \|\phi\|$, $t \geq 0$. Furthermore, since (3.11) holds, we have

$$\dot{V}(x_t) \leq -x^T(t) Y x(t). \quad (3.25)$$

Integrating both sides of (3.25) from 0 to T and using the initial conditions, we obtain

$$\begin{aligned} & - \int_0^T x^T(t) Y x(t) dt \\ & \geq \begin{pmatrix} x(T) \\ \int_{T-h(T)}^T x(s) ds \end{pmatrix}^T R \begin{pmatrix} x(T) \\ \int_{T-h(T)}^T x(s) ds \end{pmatrix} - \begin{pmatrix} x(0) \\ \int_{0-h(0)}^0 x(s) ds \end{pmatrix}^T R \begin{pmatrix} x(0) \\ \int_{0-h(0)}^0 x(s) ds \end{pmatrix} \\ & \quad + \int_{T-h}^T e^{-2\alpha T} e^{2\alpha\theta} x^T(\theta) P x(\theta) d\theta - \int_{-h}^0 e^{2\alpha\theta} x^T(\theta) P x(\theta) d\theta \\ & \quad + \int_{-h}^0 \int_{T+s}^T e^{-2\alpha T} e^{2\alpha\theta} x^T(\theta) Q x(\theta) d\theta ds - \int_{-h}^0 \int_s^0 e^{2\alpha\theta} x^T(\theta) Q x(\theta) d\theta ds \\ & \quad + \int_{-\tau(T)}^0 \int_{T+s}^T e^{-2\alpha T} e^{2\alpha\theta} x^T(\theta) S x(\theta) d\theta ds - \int_{-\tau(0)}^0 \int_s^0 e^{2\alpha\theta} x^T(\theta) S x(\theta) d\theta ds. \end{aligned} \quad (3.26)$$

As the system is asymptotically stable, when $T \rightarrow \infty$ we have

$$\begin{aligned} \left(\int_{T-h(T)}^T x(s) ds \right)^T R \left(\int_{T-h(T)}^T x(s) ds \right) &\rightarrow 0, \quad \int_{-\bar{h}}^0 \int_{T+s}^T e^{-2\alpha T} e^{2\alpha\theta} x^T(\theta) Q x(\theta) d\theta ds \rightarrow 0, \\ \int_{T-\bar{h}}^T e^{-2\alpha T} e^{2\alpha\theta} x^T(\theta) P x(\theta) d\theta &\rightarrow 0, \quad \int_{-\tau(T)}^0 \int_{T+s}^T e^{-2\alpha T} e^{2\alpha\theta} x^T(\theta) S x(\theta) d\theta ds \rightarrow 0. \end{aligned} \quad (3.27)$$

Hence, we get

$$\begin{aligned} \int_0^T x^T(t) Y x(t) dt &\leq \left(\int_{0-h(0)}^0 \phi(s) ds \right)^T R \left(\int_{0-h(0)}^0 \phi(s) ds \right) + \int_{-\bar{h}}^0 e^{2\alpha\theta} \phi^T(\theta) P \phi(\theta) d\theta \\ &\quad + \int_{-\bar{h}}^0 \int_s^0 e^{2\alpha\theta} \phi^T(\theta) Q \phi(\theta) d\theta ds + \int_{-\tau(0)}^0 \int_s^0 e^{2\alpha\theta} \phi^T(\theta) S \phi(\theta) d\theta ds \end{aligned} \quad (3.28)$$

which concludes the proof. $\square$

Remark 3.2. In order to improve the results, we can use instead of (3.18) the relation

$$\chi^T(t) \left[F_i B_i + B_i^T F_i^T \right] \chi(t) = 0, \quad i = 1, 2, \dots, N, \quad (3.29)$$

where F_i is given by $F_i = [F_{0i}^T \ F_{1i}^T \ F_{2i}^T \ F_{3i}^T \ F_{4i}^T]^T$. Then we can state the following result.

Theorem 3.3. For given $\alpha > 0$, switched linear system (2.1) is α -exponentially stable if there exist symmetric positive definite matrices $Z, Z_1, Z_2, S, P, Q, R_{11}, R_{22}, Y$, matrices $R_{12}, F_{0i}, F_{1i}, F_{2i}, F_{3i}$ and F_{4i} satisfying the conditions (3.5) and (3.6), where the matrices $F_k, k = 0, 1, \dots, 4$ are replaced with $F_{ki}, k = 0, 1, \dots, 4, i = 1, \dots, N$. The switching rule is given by $[S_R]$, the solution $x(t, \phi)$ of the system satisfies (3.7), and the cost function in (2.5) satisfies (3.8).

If $h(t) \neq 0$ and $\tau(t) = 0$, the system (2.2) is reduced to the system as follows:

$$\dot{x}(t) = A_i x(t) + D_i x(t - h(t)), \quad t > 0. \quad (3.30)$$

In this case, we have the following corollary.

Corollary 3.4. For given $\alpha > 0$, switched linear system (3.30) is α -exponentially stable if there exist symmetric positive definite matrices $Z, Z_1, Z_2, S, P, Q, R_{11}, R_{22}, Y$, matrices $R_{12}, F_{ki}, k = 0, \dots, 4, i = 1, \dots, N$ satisfying condition (3.5) and the following LMI:

$$\begin{pmatrix} \mathcal{J}_i & \hat{\phi} \\ \hat{\phi}^T & -\hat{Z} \end{pmatrix} < 0, \quad i = 1, \dots, N, \quad (3.31)$$

where,

$$\mathfrak{J}_i = \begin{pmatrix} \mathfrak{J}_{11}^i & \mathfrak{J}_{12}^i & 2\alpha R_{12} + R_{22} + A_i^T F_{2i}^T + A_i^T R_{12} & -F_{0i} + A_i^T F_{3i}^T \\ * & \mathfrak{J}_{22}^i & -R_{22} + D_i^T F_{2i}^T + D_i R_{12} & -F_{1i} + D_i^T F_{3i}^T \\ * & * & -\frac{1}{h} e^{-2\alpha\bar{h}} Q + 2\alpha R_{22} + \mu_h Z_2 & -F_{2i} \\ * & * & * & -F_{3i} - F_{3i}^T \end{pmatrix} < 0, \quad \wp = \begin{pmatrix} \mu_h R_{12} & 0 \\ 0 & \mu_h R_{22} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.32)$$

And where “*” denotes the symmetric part and

$$\begin{aligned} \mathfrak{J}_{11}^i &= R_{12} + R_{12}^T + P + 2\alpha R_{11} + \bar{h}Q + F_{0i}A_i + A_i^T F_{0i}^T - Z, \\ \mathfrak{J}_{12}^i &= -R_{12} + R_{11}D_i + F_{0i}D_i + A_i^T F_{1i}^T, \\ \mathfrak{J}_{22}^i &= -(1 - \mu_h)e^{-2\alpha\bar{h}}P + F_{1i}D_i + D_i^T F_{1i}^T + \mu_h Z_1. \end{aligned} \quad (3.33)$$

Remark 3.5. In [16], the results are given in terms of a set of generalized Lyapunov equations type, and in [18], the results are expressed in terms of generalized algebraic Riccati equation type, while in this paper, the results are expressed in terms of linear matrix inequalities.

4. Examples

In this section, two examples will be presented for illustration and comparison.

Example 4.1 (see [18]). Consider the uncertain switched linear systems (2.1), where $N = 2$, $h = 1$, $\tau = 1$ and

$$\begin{aligned} A_1 &= \begin{pmatrix} -20 & 1 \\ -3 & 2 \end{pmatrix}, & D_1 &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, & E_1 &= \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 4 & -1 \\ 1 & -32 \end{pmatrix}, & D_2 &= \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}, & E_2 &= \begin{pmatrix} -1 & 1 \\ -1 & 4 \end{pmatrix}. \end{aligned} \quad (4.1)$$

We can see that each subsystem is unstable. Applying the results of [18], it is found that the system is asymptotically stable with decay rate $\alpha = 0.5$. Applying the results of [14], by choosing $Q = 8 * I$, and applying Theorem 3.1, the critical delay of asymptotic stability (without decay rate) is found as 0.5722. Therefore, with the results of [14], we can't conclude about stability of the considered system. Note that both [14, 18] study the systems with constant delays. It is well known that the results for time varying delays are more conservative than those with constant delays. Applying our results, we let $\mu_h = 0$ and $\mu_\tau = 0$,

by Theorem 3.1, with $\lambda_1 = \lambda_2 = 0.5$, we obtain a decay rate $\alpha = 0.984$ with the following solutions:

$$\begin{aligned}
 P &= \begin{pmatrix} 3.1461 & -3.6923 \\ -3.6923 & 5.1228 \end{pmatrix}, & R &= 10^{-2} \begin{pmatrix} 112.4135 & -41.2719 & -0.1118 & -3.2593 \\ -41.2719 & 73.7166 & -0.2017 & 1.6277 \\ -1.0907 & -0.2017 & 0.0337 & -0.0079 \\ -3.2593 & 1.6277 & -0.0079 & 0.1106 \end{pmatrix}, \\
 Z &= \begin{pmatrix} 17.153 & -8.754 \\ -8.754 & 22.110 \end{pmatrix}, & Q &= \begin{pmatrix} 0.25 & 0.0024 \\ 0.0024 & 0.6492 \end{pmatrix}, & F_0 &= \begin{pmatrix} 0.1546 & 0.0216 \\ -0.1153 & 0.3648 \end{pmatrix}, \\
 F_1 &= 10^{-2} \begin{pmatrix} -0.8709 & 3.3759 \\ 2.7474 & -4.8734 \end{pmatrix}, & F_2 &= 10^{-2} \begin{pmatrix} 0.1475 & 0.5318 \\ 2.2716 & 0.1949 \end{pmatrix}, & S &= \begin{pmatrix} 2.1245 & -0.4151 \\ -0.4151 & 7.8435 \end{pmatrix}, \\
 F_3 &= 10^{-2} \begin{pmatrix} 1.4000 & -0.0729 \\ -3.5071 & -0.2284 \end{pmatrix}, & F_4 &= 10^{-2} \begin{pmatrix} 0.7830 & 0.2873 \\ -0.2020 & 0.0708 \end{pmatrix}, \\
 Z_1 &= \begin{pmatrix} 2.975 & 0 \\ 0 & 2.975 \end{pmatrix}, & Z_2 &= \begin{pmatrix} 2.975 & 0 \\ 0 & 2.975 \end{pmatrix}, & Y &= 10^{-2} \begin{pmatrix} 0.4050 & -0.0579 \\ -0.0579 & 0.2367 \end{pmatrix}.
 \end{aligned} \tag{4.2}$$

And the matrices

$$L_1 = \begin{pmatrix} -25.336 & -2.413 \\ -2.413 & 24.233 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 25.320 & 2.415 \\ 2.415 & -24.243 \end{pmatrix}. \tag{4.3}$$

Satisfy $\lambda_1 L_1 + \lambda_2 L_2 = 10^{-3} \begin{pmatrix} -8 & 1 \\ 1 & -5 \end{pmatrix} < -Y$.

The sets Ω_1 and Ω_2 are given by

$$\begin{aligned}
 \Omega_1 &= \left\{ (x_1, x_2) \in R^2 : -25.336x_1^2 - 4.826x_1x_2 + 24.233x_2^2 < 0 \right\}, \\
 \Omega_2 &= \left\{ (x_1, x_2) \in R^2 : 25.32x_1^2 + 4.83x_1x_2 - 24.243x_2^2 < 0 \right\}.
 \end{aligned} \tag{4.4}$$

It can be seen that $\Omega_1 \cup \Omega_2 = R^2 \setminus \{0\}$, therefore, the switching regions are given by: $\overline{\Omega}_1 = \Omega_1$, $\overline{\Omega}_2 = \Omega_2 \setminus \overline{\Omega}_1$, and the switching rule is given by

$$\sigma = \begin{cases} 1 & \text{if } x(t) \in \overline{\Omega}_1, \\ 2 & \text{if } x(t) \in \overline{\Omega}_2. \end{cases} \tag{4.5}$$

Moreover the solution of the system satisfies $\|x(t, \phi)\| \leq 3.6297e^{-0.984t}\|\phi\|$, $t \geq 0$. For the initial conditions $x_1(t) = e^{t+1}$ and $x_2(t) = 0$ for $t \in [-1, 0]$, we obtain a guaranteed cost $J^* = 14.5632$. When we apply Theorem 3.3, we obtain the stability with decay rate $\alpha = 1.186$, stability factor $\beta = 2.3345$, and the guaranteed cost $J^* = 19.244$.

Example 4.2. Consider the following system:

$$A_{01} = \begin{pmatrix} -30 & 2 \\ -5 & 4 \end{pmatrix}, \quad A_{02} = \begin{pmatrix} 2 & -2 \\ 2 & -34 \end{pmatrix}, \quad A_{d1} = \begin{pmatrix} 2 & -2 \\ -2 & -3.9 \end{pmatrix}, \quad A_{d2} = \begin{pmatrix} 6 & -2 \\ 5 & 15.31 \end{pmatrix}. \quad (4.6)$$

Applying the results of [20], when the time delay is constant, that is, $h(t) = \bar{h}$, we obtain the stability bound $\bar{h} = 0.063$ with the parameters $\beta_1 = 15$ and $\beta_2 = 0.8$. Applying our results, for comparison, we set $\mu_h = 0$ and $\alpha = 0$, then we apply Corollary 3.4 with $\lambda_1 = \lambda_2 = 0.5$, it is found that the system is asymptotically stable independent of delay. This shows the improvements of our approach. For $\bar{h} = 250$ s, we have the following results:

$$Y = \begin{pmatrix} 51.5734 & 8.3215 \\ 8.3215 & 3.6640 \end{pmatrix}, \quad L_1 = 10^3 \begin{pmatrix} -1.0045 & 0.0040 \\ 0.0040 & 0.6358 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 798.1735 & -37.3230 \\ -37.3230 & -650.4835 \end{pmatrix},$$

$$\lambda_1 L_1 + \lambda_2 L_2 = \begin{pmatrix} -103.1467 & -16.6431 \\ -16.6431 & -7.3280 \end{pmatrix} < -Y. \quad (4.7)$$

The switching regions $\bar{\Omega}_1$ and $\bar{\Omega}_2$ are given by

$$\bar{\Omega}_1 = \{(x_1, x_2) \in R^2 : -1004.5x_1^2 + 8x_1x_2 + 635.8x_2^2 < 0\},$$

$$\bar{\Omega}_2 = \{(x_1, x_2) \in R^2 : -1004.5x_1^2 + 8x_1x_2 + 635.8x_2^2 \geq 0\}, \quad (x_1, x_2) \neq (0, 0). \quad (4.8)$$

The state trajectories are depicted in Figure 1 for $h(t) = 0.5$ s and in Figure 2 for $h(t) = 250$ s. It is clear that as the delay increases as the states require much time for convergence to zeros.

Now, letting $\bar{h} = 0.5$ s, $\alpha = 0.4$, $\lambda_1 = 0.3$, $\lambda_2 = 0.7$, and the initial conditions $x_1(t) = e^{t+1}$ and $x_2(t) = 0$ for $t \in [-0.5, 0]$, we obtain a guaranteed cost $J^* = 80.004$.

5. Conclusion

By using Lyapunov Krasovskii approach, the problems of exponential stability and guaranteed cost are investigated for switched linear systems with mixed time delays. Via a designed switching rule, the results allow to the bounds that characterise the exponential stability, that is, the stability factor and the decay rate for the solution. The results are delay dependent and are expressed in terms of linear matrix inequalities. Some numerical examples are given to illustrate these results presented in this paper that have significant improvement over existing ones.

Figure 1: State trajectories for $h = 0.5$ s.

Figure 2: State trajectories for $h = 250$ s.

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