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Research Article

Spectrum of Quasi-Class (A, k) Operators

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An operator $T \in B(\mathcal{H})$ is called quasi-class (A, k) if $T^{*k}(|T^2| - |T|^2)T^k \geq 0$ for a positive integer k, which is a common generalization of class A. In this paper, firstly we consider some spectral properties of quasi-class (A, k) operators; it is shown that if T is a quasi-class (A, k) operator, then the nonzero points of its point spectrum and joint point spectrum are identical, the eigenspaces corresponding to distinct eigenvalues of T are mutually orthogonal, and the nonzero points of its approximate point spectrum and joint approximate point spectrum are identical. Secondly, we show that Putnam's theorems hold for class A operators. Particularly, we show that if T is a class A operator and either $\sigma(|T|)$ or $\sigma(|T^*|)$ is not connected, then T has a nontrivial invariant subspace.

1. Introduction

Throughout this paper let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $B(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. The spectrum of an operator $T \in B(\mathcal{H})$ is denoted by $\sigma(T)$.

Here an operator $T \in B(\mathcal{H})$ is called p-hyponormal for $0 < p \leq 1$ if $(T^*T)^p - (TT^*)^p \geq 0$; when $p = 1$, T is called hyponormal; when $p = 1/2$, T is called semihyponormal. T is called log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. And an operator T is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$. By the celebrated Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$," every p-hyponormal operator is q-hyponormal for $p \geq q \geq 0$. And every invertible p-hyponormal operator is log-hyponormal since logt is an operator monotone function. We remark that $(A^p - I)/p \to \log \mathrm{A}$ as $p \to +0$ for positive invertible operator $A > 0$, so that p-hyponormality of T approaches log-hyponormality of T as $p \to +0$. In this sense, log-hyponormal can be considered as 0-hyponormal. p-hyponormal, log-hyponormal, and paranormal operators were introduced by Aluthge [1], Tanahashi [2], and Furuta [3, 4], respectively.

In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators, Furuta, et al. [5] introduced a very interesting class of bounded linear Hilbert space operators: class A defined by $|T^2| - |T|^2 \geq 0$, where $|T| = (T^*T)^{1/2}$ which is called the absolute value of T, and they showed that class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators. Class A operators have been studied by many researchers, for example, [6–12].

Aluthge [1] introduced $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, which is called Aluthge transformation of T. The operator $\widetilde{T}$ plays an important role in the study of spectral properties of the p-hyponormal or log-hyponormal operator T. Aluthge-Wang [13] introduced w-hyponormal operators defined by

$$\left|\widetilde{T}\right| \geq |T| \geq \left|\widetilde{T}^*\right|, \tag{1.1}$$

where the polar decomposition of T is $T = U|T|$ and $\widetilde{T}$ is the Aluthge transformation of T. As a generalization of w-hyponormal, Ito [14] introduction class $wA(s,t)$ is defined by

$$\begin{aligned} \left(|T^*|^t|T|^{2s}|T^*|^t\right)^{t/(s+t)} &\geq |T^*|^{2t}, \\ |T|^{2s} &\geq \left(|T|^s|T^*|^{2t}|T|^s\right)^{s/(s+t)} \end{aligned} \tag{1.2}$$

for $s > 0$ and $t > 0$. Ito and Yamazaki [15] showed that w-hyponormal equals $wA(1/2, 1/2)$; class A equals $wA(1,1)$. Inclusion relations among these classes are known as follows:

$$\begin{aligned} \{\text{hyponormal operators}\} &\subseteq \{p\text{-hyponormal operators},\ \ 0 < p \leq 1\} \\ &\subseteq \{\text{class A}(s,t) \text{ operators},\ \ s,t \in (0,1]\} \\ &\subseteq \{\text{class A operators}\} \\ &\subseteq \{\text{paranormal operators}\}. \end{aligned} \tag{1.3}$$

Jeon and Kim [16] introduced quasi-class A (i.e., $T^*(|T^2| - |T|^2)T \geq 0$) operators as an extension of the notion of class A operators.

Recently Tanahashi et al. [9] considered an extension of quasi-class A operators, similar with respect to the extension of the notion of p-quasihyponormality to (p,k)-quasihyponormality.

Definition 1.1. $T \in B(\mathscr{H})$ is called a quasi-class (A,k) operator for a positive integer k if

$$T^{*k}\left(\left|T^2\right| - |T|^2\right)T^k \geq 0. \tag{1.4}$$

Remark 1.2. In [17], this class of operators is called k-quasi-class A. It is clear that

$$\begin{aligned}\{p\text{-hyponormal operators}\} &\subseteq \{\text{class A operators}\}\\ &\subseteq \{\text{quasi-class A operators}\}\\ &\subseteq \{\text{quasi-class } (A,k) \text{operators}\},\end{aligned} \tag{1.5}$$

$$\{\text{quasi-class } (A,k) \text{ operators}\} \subseteq \{\text{quasi-class } (A,k+1) \text{ operators}\}. \tag{1.6}$$

In [17], we show that the inclusion relation (1.6) is strict by an example.

In this paper, firstly we consider some spectral properties of quasi-class (A,k) operators; it is shown that if T is a quasi-class (A,k) operator for a positive integer k, then the nonzero points of its point spectrum and joint point spectrum are identical; furthermore, the eigenspaces corresponding to distinct eigenvalues of T are mutually orthogonal; the nonzero points of its approximate point spectrum and joint approximate point spectrum are identical. Secondly, we show that Putnam's theorems hold for class A operators. Particularly, we show that if T is a class A operator and either $\sigma(|T|)$ or $\sigma(|T^*|)$ is not connected, then T has a nontrivial invariant subspace.

2. Main Results

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in \mathscr{H}$ such that $(T-\lambda)x = 0$. If in addition, $(T^* - \overline{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{\mathrm{jp}}(T)$ of T. Clearly, $\sigma_{\mathrm{jp}}(T) \subseteq \sigma_p(T)$. In general, $\sigma_{\mathrm{jp}}(T) \neq \sigma_p(T)$.

In [18], Xia showed that if T is a semihyponormal operator, then $\sigma_{\mathrm{jp}}(T) = \sigma_p(T)$; Tanahashi extended this result to log-hyponormal operators in [2]. Aluthge [13] showed that if T is w-hyponormal, then nonzero points of $\sigma_{\mathrm{jp}}(T)$ and $\sigma_p(T)$ are identical; Uchiyama extended this result to class A operators in [10]. In the following, we will point out that if T is a quasi-class (A,k) operator for a positive integer k, then nonzero points of $\sigma_{\mathrm{jp}}(T)$ and $\sigma_p(T)$ are also identical and the eigenspaces corresponding to distinct eigenvalues of T are mutually orthogonal.

Lemma 2.1 (see [9, 17]). *Let $T \in B(\mathscr{H})$ be a quasi-class (A,k) operator for a positive integer k. If $\lambda \neq 0$ and $(T-\lambda)x = 0$ for some $x \in \mathscr{H}$, then $(T-\lambda)^* x = 0$.*

Theorem 2.2. *Let $T \in B(\mathscr{H})$ be a quasi-class (A,k) operator for a positive integer k. Then the following assertions hold:*

(1) $\sigma_{jp}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$,

(2) If $(T-\lambda)x = 0$, $(T-\mu)y = 0$ and $\lambda \neq \mu$, then $\langle x, y\rangle = 0$.

Proof. (1) Clearly by Lemma 2.1.

(2) Without loss of generality, we assume $\mu \neq 0$. Then we have $(T-\mu)^* y = 0$ by Lemma 2.1. Thus we have $\mu\langle x, y\rangle = \langle x, T^* y\rangle = \langle Tx, y\rangle = \lambda\langle x, y\rangle$. Since $\lambda \neq \mu$, $\langle x, y\rangle = 0$. □

A complex number λ is said to be in the approximate point spectrum $\sigma_a(T)$ of T if there is a sequence $\{x_n\}$ of unit vectors in $\mathscr{H}$ such that $(T-\lambda)x_n \to 0$. If in addition, $(T^*-\overline{\lambda})x_n \to 0$, then λ is said to be in the joint approximate point spectrum $\sigma_{\mathrm{ja}}(T)$ of T. Clearly, $\sigma_{\mathrm{ja}}(T) \subseteq \sigma_a(T)$. In general, $\sigma_{\mathrm{ja}}(T) \neq \sigma_a(T)$. In [18], Xia showed that if T is a semihyponormal operator, then $\sigma_{\mathrm{ja}}(T) = \sigma_a(T)$; Tanahashi [2] extended this result to log-hyponormal operators. Aluthge and Wang [19] showed that if T is w-hyponormal, then nonzero points of $\sigma_{\mathrm{ja}}(T)$ and $\sigma_a(T)$ are identical; Chō and Yamazaki extended this result to class A operators in [7]. In the following, we will show that if T is a quasi-class (A,k) operator for a positive integer k, then nonzero points of $\sigma_{\mathrm{ja}}(T)$ and $\sigma_a(T)$ are also identical.

Theorem 2.3. *Let $T \in B(\mathscr{H})$ be a quasi-class (A,k) operator for a positive integer k. Then $\sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$.*

To prove Theorem 2.3, we need the following auxiliary results.

Lemma 2.4 (see [20]). *Let $\mathscr{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathscr{K}$ such that $\mathscr{H} \subset \mathscr{K}$ and a map $\varphi : B(\mathscr{H}) \to B(\mathscr{K})$ such that*

(1) *φ is a faithful $*$-representation of the algebra $B(\mathscr{H})$ on $\mathscr{K}$;*

(2) *$\varphi(A) \geq 0$ for any $A \geq 0$ in $B(\mathscr{H})$;*

(3) *$\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in B(\mathscr{H})$.*

Lemma 2.5 (see [18]). *Let $\varphi : B(\mathscr{H}) \to B(\mathscr{K})$ be Berberian's faithful $*$-representation. Then $\sigma_{ja}(T) = \sigma_{jp}(\varphi(T))$.*

Proof of Theorem 2.3. Let $\varphi : B(\mathscr{H}) \to B(\mathscr{K})$ be Berberian's faithful $*$-representation of Lemma 2.4. In the following, we shall show that $\varphi(T)$ is also a quasi-class (A,k) operator for a positive integer k.

In fact, since T is a quasi-class(A,k) operator, we have

$$\begin{aligned}
&(\varphi(T))^{*k}\left(\left|(\varphi(T))^2\right| - |\varphi(T)|^2\right)(\varphi(T))^k \\
&\quad = \varphi\left(T^{*k}\left(\left|T^2\right| - |T|^2\right)T^k\right) \quad \text{by Lemma 2.4(1)} \\
&\quad \geq 0 \quad \text{by Lemma 2.4(2)}.
\end{aligned} \tag{2.1}$$

Hence, we have

$$\begin{aligned}
\sigma_a(T) \setminus \{0\} &= \sigma_a(\varphi(T)) \setminus \{0\} \quad \text{by Lemma 2.4(3)} \\
&= \sigma_p(\varphi(T)) \setminus \{0\} \quad \text{by Lemma 2.4(3)} \\
&= \sigma_{\mathrm{jp}}(\varphi(T)) \setminus \{0\} \quad \text{by Theorem 2.2(1)} \\
&= \sigma_{\mathrm{ja}}(T) \setminus \{0\} \quad \text{by Lemma2.5}.
\end{aligned} \tag{2.2}$$

The proof is complete. □

Theorem 2.6. *Let $T \in B(\mathscr{H})$ be a quasi-class (A,k) operator for a positive integer k. Then $\sigma(T) \setminus \{0\} = (\sigma_a(T^*) \setminus \{0\})^*$ (i.e., $\{\lambda : \overline{\lambda} \in \sigma_a(T^*) \setminus \{0\}\}$).*

Proof. It suffices to prove $\sigma(T) \setminus \{0\} \subset (\sigma_a(T^*) \setminus \{0\})^* = \{\lambda : \overline{\lambda} \in \sigma_a(T^*) \setminus \{0\}\}$.

Xia [18] showed that $\sigma(T) = \sigma_a(T) \cup (\sigma_p(T^*))^*$ holds for any $T \in B(\mathscr{H})$. Hence we have

$$\sigma_a(T) \setminus \{0\} = \sigma_{\mathrm{ja}}(T) \setminus \{0\} \subset (\sigma_a(T^*) \setminus \{0\})^* \tag{2.3}$$

by Theorem 2.3. The proof is complete. □

Putnam [21] proved three theorems concerning spectral properties of hyponormal operators. These theorems were generalized to p-hyponormal operators by Chō et al. in [22, 23], to w-hyponormal operators by Aluthge and Wang in [24], and to $wF(p,r,q)$ operators by Yang and Yuan in [25]. In the following, we extend these theorems to quasi-class (A,k) operators.

We show the first generalization concerning points in the approximate point spectrum of a quasi-class (A,k) operator for a positive integer k as follows.

Theorem 2.7. *Let $T \in B(\mathscr{H})$ be a quasi-class (A,k) operator for a positive integer k. If $\lambda \neq 0$ such that $\lambda \in \sigma_a(T)$, then $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.*

To prove Theorem 2.7, we need the following auxiliary results.

Lemma 2.8 (see [26]). *Let $T = U|T|$ be the polar decomposition of T, $\lambda \neq 0$, and $\{x_n\}$ a sequence of vectors. Then the following assertions are equivalent:*

(1) $(T - \lambda)x_n \to 0$ *and* $(T^* - \overline{\lambda})x_n \to 0$,

(2) $(|T| - |\lambda|)x_n \to 0$ *and* $(U - e^{i\theta})x_n \to 0$,

(3) $(|T^*| - |\lambda|)x_n \to 0$ *and* $(U^* - e^{-i\theta})x_n \to 0$.

Proof of Theorem 2.7. If $\lambda \neq 0$ and $\lambda \in \sigma_a(T)$, a sequence of unit vectors exists such that $(T - \lambda)x_n \to 0$ and $(T^* - \overline{\lambda})x_n \to 0$ by Theorem 2.3. Hence Theorem 2.7 holds by Lemma 2.8. □

Corollary 2.9. *Let $T \in B(\mathscr{H})$ be a class A operator. If $\lambda \neq 0$ such that $\lambda \in \sigma_a(T)$, then $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.*

Let $T = U|T|$ be a p-hyponormal operator. Does it follow that if $\lambda \in \sigma(T)$, then $|\lambda| \in \sigma(|T|)$? The answer is affirmative if $\lambda \in \sigma_a(T)$ by Corollary 2.9. In general, the answer is negative even if T is hyponormal and the polar factor U is unitary; see details in [21]. However, the converse is true for many classes of operators; see the following results.

Theorem 2.10 (see [18, 21, 23]). *Let $T = U|T|$ be p-hyponormal for $p > 0$, then $\sigma(|T|) \subset \rho(\sigma(T))$, where $\rho : \mathbb{C} \to \mathbb{R}$ is defined by $\rho(z) = |z|$.*

Indeed, the above Theorem 2.10 that was shown for the case T is hyponormal by Putnam in [21], for the case T is semihyponormal by Xia in [18], and the general case by Chō et al. in [23].

Theorem 2.11 (see [24]). *Let $T = U|T|$ be w-hyponormal and $\sigma(T)$ is connected, then $\sigma(|T|) \subset \rho(\sigma(T))$, where $\rho : \mathbb{C} \to \mathbb{R}$ is defined by $\rho(z) = |z|$.*

Here we show the second generalization concerning the relation between the spectrum of T and $|T|$ to class A operators with connected spectrum.

Theorem 2.12. *Let T be a class A operator and $\sigma(T)$ is connected, then $\sigma(|T|) \subset \rho(\sigma(T))$, where $\rho : \mathbb{C} \to \mathbb{R}$ is defined by $\rho(z) = |z|$.*

The numerical range $W(T)$ of an operator T is defined by

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}. \tag{2.4}$$

Let $\overline{W}(T)$ denote the closure of $W(T)$. It is well known that for any $T \in B(\mathscr{H})$, $W(T)$ is a convex set and $\sigma(T) \subseteq \overline{W}(T)$. Moreover, if T is normal, then $\overline{W}(T) = \operatorname{conv}\sigma(T)$, the convex hull of $\sigma(T)$.

We need the following auxiliary results.

Lemma 2.13 (see [7]). *Let $T = U|T|$ be the polar decomposition of a class A operator and $\widetilde{T}_{1,1} = |T|U|T|$. Then $\widetilde{T}_{1,1}$ is semihyponormal and*

$$\sigma\left(\widetilde{T}_{1,1}\right) = \left\{r^2 e^{i\theta} : re^{i\theta} \in \sigma(T)\right\}. \tag{2.5}$$

Lemma 2.14. *Let $T = U|T|$ be the polar decomposition of a class A operator and $\widetilde{T}_{1,1} = |T|U|T|$. Then $\overline{W}(|\widetilde{T}_{1,1}|) \subseteq \overline{W}(|(\widetilde{T}_{1,1})^*|)$.*

Proof. Let $\widetilde{T}_{1,1} = V|\widetilde{T}_{1,1}|$ be the polar decomposition of $\widetilde{T}_{1,1}$. The nonzero points of $\sigma(|\widetilde{T}_{1,1}|)$ and $\sigma(|(\widetilde{T}_{1,1})^*|)$ are identical. Since T is a class A operator, $\widetilde{T}_{1,1} = |T|U|T|$ is semihyponormal by Lemma 2.13, that is, $|\widetilde{T}_{1,1}| \geq |(\widetilde{T}_{1,1})^*|$. It follows that $0 \in \sigma(|(\widetilde{T}_{1,1})^*|)$ if $0 \in \sigma(|\widetilde{T}_{1,1}|)$. Therefore $\sigma(|\widetilde{T}_{1,1}|) \subseteq \sigma(|(\widetilde{T}_{1,1})^*|)$. Hence

$$\overline{W}\left(\left|\widetilde{T}_{1,1}\right|\right) = \operatorname{conv}\sigma\left(\left|\widetilde{T}_{1,1}\right|\right) \subseteq \operatorname{conv}\sigma\left(\left|\left(\widetilde{T}_{1,1}\right)^*\right|\right) = \overline{W}\left(\left|\left(\widetilde{T}_{1,1}\right)^*\right|\right). \tag{2.6}$$

□

Lemma 2.15. *Let $T = U|T|$ be the polar decomposition of a class A operator and $\widetilde{T}_{1,1} = |T|U|T|$. Then $\sigma(|T|^2) \subseteq \overline{W}(|(\widetilde{T}_{1,1})^*|)$.*

Proof. Since T is a class A operator, we have

$$\left|\widetilde{T}_{1,1}\right| \geq |T|^2 \geq \left|\left(\widetilde{T}_{1,1}\right)^*\right| \tag{2.7}$$

by the proof of Theorem 2.1 in [7]. So we have

$$\left\langle\left|\widetilde{T}_{1,1}\right|x, x\right\rangle \geq \left\langle|T|^2 x, x\right\rangle \geq \left\langle\left|\left(\widetilde{T}_{1,1}\right)^*\right|x, x\right\rangle \tag{2.8}$$

for any unit vector x. By Lemma 2.14, $\langle |\widetilde{T}_{1,1}|x, x\rangle \in \overline{W}(|(\widetilde{T}_{1,1})^*|)$. The convexity of $\overline{W}(|(\widetilde{T}_{1,1})^*|)$ and the above inequalities imply

$$\left\langle |T|^2 x, x \right\rangle \in \overline{W}\left(\left|\left(\widetilde{T}_{1,1}\right)^*\right|\right). \tag{2.9}$$

Hence

$$\sigma\left(|T|^2\right) \subseteq \operatorname{conv} \sigma\left(|T|^2\right) = \overline{W}\left(|T|^2\right) \subseteq \overline{W}\left(\left|\left(\widetilde{T}_{1,1}\right)^*\right|\right). \tag{2.10}$$

□

Proof of Theorem 2.12. Since T is a class A operator, $\widetilde{T}_{1,1} = |T|U|T|$ is semihyponormal by Lemma 2.13. It follows from Theorem 2.10 that

$$\sigma\left(\left|\widetilde{T}_{1,1}\right|\right) \subseteq \rho\left(\sigma\left(\widetilde{T}_{1,1}\right)\right). \tag{2.11}$$

Since the nonzero points of $\sigma(|\widetilde{T}_{1,1}|)$ and $\sigma(|(\widetilde{T}_{1,1})^*|)$ are identical, and $0 \in \sigma(|(\widetilde{T}_{1,1})^*|)$ implies that $(\widetilde{T}_{1,1})^*$ is not invertible, and hence $0 \in \sigma(\widetilde{T}_{1,1})$, the above containment may be modified to become

$$\sigma\left(\left|\left(\widetilde{T}_{1,1}\right)^*\right|\right) \subseteq \rho\left(\sigma\left(\widetilde{T}_{1,1}\right)\right). \tag{2.12}$$

By Lemma 2.13, we have

$$\sigma\left(\widetilde{T}_{1,1}\right) = \left\{r^2 e^{i\theta} : re^{i\theta} \in \sigma(T)\right\}. \tag{2.13}$$

So

$$\rho\left(\sigma\left(\widetilde{T}_{1,1}\right)\right) = (\rho(\sigma(T)))^2. \tag{2.14}$$

Since $\sigma(T)$ is connected, $(\rho(\sigma(T)))^2$ is a closed convex subset of $\mathbb{R}$. Hence by Lemma 2.15, we have

$$\sigma\left(|T|^2\right) \subseteq \overline{W}\left(\left|\left(\widetilde{T}_{1,1}\right)^*\right|\right) = \operatorname{conv} \sigma\left(\left|\left(\widetilde{T}_{1,1}\right)^*\right|\right) \subseteq \operatorname{conv} \rho\left(\sigma\left(\widetilde{T}_{1,1}\right)\right) = (\rho(\sigma(T)))^2. \tag{2.15}$$

Since

$$\sigma\left(|T|^2\right) = (\sigma(|T|))^2, \tag{2.16}$$

so we have

$$(\sigma(|T|))^2 \subseteq (\rho(\sigma(T)))^2, \tag{2.17}$$

that is,

$$\sigma(|T|) \subset \rho(\sigma(T)). \tag{2.18}$$

The proof is complete. □

Putnam [21] proved that if T is hyponormal and $\sigma(|T^*|)$ is not an interval, then T has a nontrivial invariant subspace. This result has been generalized by many authors. Chō et al. generalized Putnam's result to p-hyponormal operators in [22]. In [24], Aluthge and Wang proved that if T is w-hyponormal and either $\sigma(|T|)$ or $\sigma(|T^*|)$ is not an interval, then T has a nontrivial invariant subspace.

Here we shall generalize the above result to class A operators and give an application of Theorem 2.12.

A complex number λ is said to be in the compression spectrum $\sigma_c(T)$ of T if $\operatorname{ran}(T-\lambda)$ is not dense in $\mathscr{H}$. It is well known that $\sigma(T) = \sigma_c(T) \cup \sigma_a(T)$ for any $T \in B(\mathscr{H})$. Moreover, if $\lambda \in \sigma_c(T)$ and $T \neq \lambda I$, then $\overline{\operatorname{ran}(T-\lambda)}$ is a nontrivial invariant subspace of T.

Theorem 2.16. *Let $T \in B(\mathscr{H})$ be a quasi-class (A,k) operator for a positive integer k. If there is a $\lambda \in \sigma(T)$, $\lambda \neq 0$, with $|\lambda| \notin \sigma(|T|) \cap \sigma(|T^*|)$, then T has a nontrivial invariant subspace.*

Proof. We have that $\lambda \notin \sigma_a(T)$ by Theorem 2.7. So we have $\lambda \in \sigma_c(T)$. By the assumption, we have that $T \neq \lambda I$. Hence T has a nontrivial subspace. □

Corollary 2.17. *Let $T \in B(\mathscr{H})$ be a class A operator. If there is a $\lambda \in \sigma(T)$, $\lambda \neq 0$, with $|\lambda| \notin \sigma(|T|) \cap \sigma(|T^*|)$, then T has a nontrivial invariant subspace.*

Theorem 2.18. *Let $T \in B(\mathscr{H})$ be a class A operator for a positive integer k. If either $\sigma(|T|)$ or $\sigma(|T^*|)$ is not connected, then T has a nontrivial invariant subspace.*

Proof. We only give the proof for the case that $\sigma(|T^*|)$ is not connected, for the case $\sigma(|T|)$ is not connected can be proved similarly.

If $\sigma(T)$ is not connected, then the Theorem is clear, so we assume that $\sigma(T)$ is connected. By the assumption, we have that $\sigma(|T^*|)$ is not an interval, so there exist s, $t \in \sigma(|T^*|)$, $0 \leq s < t$ such that

$$(s,t) \cap \sigma(|T^*|) = \emptyset. \tag{2.19}$$

Let $N = \{z : s < |z| < t\}$. Since $\sigma(|T|) \setminus \{0\} = \sigma(|T^*|) \setminus \{0\}$, there exists a $v \in \sigma(T)$ for which $|v| = t$. Similarly, if $0 < s$, then there exists a $u \in \sigma(T)$ for which $|u| = s$ by Theorem 2.12.

On the other hand, if $s = 0$, then T^* is not invertible and hence $0 \in \sigma(T)$. Hence there exists a $u = 0 \in \sigma(T)$ such that $|u| = s$. So both the outer and inner boundaries of the annulus N contain a point of $\sigma(T)$. Since $\sigma(T)$ is connected, we have that $N \cap \sigma(T) \neq \emptyset$.

Hence there exists a $\lambda \in N \cap \sigma(T)$, thus $|\lambda| \in (s,t)$. So we have that $\lambda \neq 0$ and $|\lambda| \notin \sigma(|T^*|)$ by (2.19). Therefore Theorem 2.18 holds by Corollary 2.17. □

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