Research Article

Spectrum of Quasi-Class \((A, k)\) Operators

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1. Introduction

Throughout this paper let \(\mathcal{H}\) be a separable complex Hilbert space with inner product \(\langle \cdot, \cdot \rangle\). Let \(B(\mathcal{H})\) denote the \(C^*\)-algebra of all bounded linear operators on \(\mathcal{H}\). The spectrum of an operator \(T \in B(\mathcal{H})\) is denoted by \(\sigma(T)\).

Here an operator \(T \in B(\mathcal{H})\) is called \(p\)-hyponormal for \(0 < p \leq 1\) if \((T^*T)^p - (TT^*)^p \geq 0\); when \(p = 1\), \(T\) is called hyponormal; when \(p = 1/2\), \(T\) is called semihyponormal. \(T\) is called log-hyponormal if \(T\) is invertible and \(\log T^*T \geq \log TT^*\). And an operator \(T\) is called paranormal if \(\|Tx\|^2 \leq \|T^2x\|\|x\|\) for all \(x \in \mathcal{H}\). By the celebrated L"owner-Heinz theorem “\(A \geq B \geq 0\) ensures \(A^p \geq B^p\) for any \(a \in [0,1]\),” every \(p\)-hyponormal operator is \(q\)-hyponormal for \(p \geq q \geq 0\). And every invertible \(p\)-hyponormal operator is log-hyponormal since \(\log t\) is an operator monotone function. We remark that \((A^p - I) / p \to \log A\) as \(p \to +0\) for positive invertible operator \(A > 0\), so that \(p\)-hyponormality of \(T\) approaches log-hyponormality of \(T\) as \(p \to +0\). In this sense, log-hyponormal can be considered as 0-hyponormal. \(p\)-hyponormal,
log-hyponormal, and paranormal operators were introduced by Aluthge [1], Tanahashi [2], and Furuta [3, 4], respectively.

In order to discuss the relations between paranormal and $p$-hyponormal and log-hyponormal operators, Furuta, et al. [5] introduced a very interesting class of bounded linear Hilbert space operators: class A defined by $|T^2| - |T|^2 \geq 0$, where $|T| = (T^*T)^{1/2}$ which is called the absolute value of $T$, and they showed that class A is a subclass of paranormal and contains $p$-hyponormal and log-hyponormal operators. Class A operators have been studied by many researchers, for example, [6–12].

Aluthge [1] introduced $\tilde{T} = |T|^{1/2}U|T|^{1/2}$, which is called Aluthge transformation of $T$. The operator $\tilde{T}$ plays an important role in the study of spectral properties of the $p$-hyponormal or log-hyponormal operator $T$. Aluthge-Wang [13] introduced $w$-hyponormal operators defined by

$$|\tilde{T}| \geq |T| \geq |\tilde{T}^*|,$$

where the polar decomposition of $T$ is $T = U|T|$ and $\tilde{T}$ is the Aluthge transformation of $T$. As a generalization of $w$-hyponormal, Ito [14] introduction class $wA(s,t)$ is defined by

$$\left(|T|^s|T^2s|T^t|t\right)^{1/(s+t)} \geq |T^2|^t,$$

$$|T|^2s \geq \left(|T|^s|T^2s|T^t|t\right)^{s/(s+t)}$$

for $s > 0$ and $t > 0$. Ito and Yamazaki [15] showed that $w$-hyponormal equals $wA(1/2,1/2)$; class A equals $wA(1,1)$. Inclusion relations among these classes are known as follows:

$$\{\text{hyponormal operators}\} \subseteq \{p\text{-hyponormal operators, } 0 < p \leq 1\}$$

$$\subseteq \{\text{class } A(s,t) \text{ operators, } s, t \in (0,1]\}$$

$$\subseteq \{\text{class } A \text{ operators}\}$$

$$\subseteq \{\text{paranormal operators}\}.$$  \hfill (1.3)

Jeon and Kim [16] introduced quasi-class A (i.e., $T^*(|T^2| - |T|^2)T \geq 0$) operators as an extension of the notion of class A operators.

Recently Tanahashi et al. [9] considered an extension of quasi-class A operators, similar with respect to the extension of the notion of $p$-quasihyponormality to $(p,k)$-quasihyponormality.

Definition 1.1. $T \in B(\mathscr{H})$ is called a quasi-class $(A,k)$ operator for a positive integer $k$ if

$$T^k\left(|T^2| - |T|^2\right)T^k \geq 0.$$  \hfill (1.4)
Remark 1.2. In [17], this class of operators is called \( k \)-quasi-class \( A \). It is clear that

\[
\{ p \text{-hyponormal operators} \} \subseteq \{ \text{class } A \text{ operators} \} \subseteq \{ \text{quasi-class } A \text{ operators} \} \subseteq \{ \text{quasi-class } (A,k)\text{operators} \},
\]

(1.5)

\[
\{ \text{quasi-class } (A,k) \text{ operators} \} \subseteq \{ \text{quasi-class } (A,k+1) \text{ operators} \}.
\]

(1.6)

In [17], we show that the inclusion relation (1.6) is strict by an example.

In this paper, firstly we consider some spectral properties of quasi-class \( (A,k) \) operators; it is shown that if \( T \) is a quasi-class \( (A,k) \) operator for a positive integer \( k \), then the nonzero points of its point spectrum and joint point spectrum are identical; furthermore, the eigenspaces corresponding to distinct eigenvalues of \( T \) are mutually orthogonal; the nonzero points of its approximate point spectrum and joint approximate point spectrum are identical. Secondly, we show that Putnam’s theorems hold for class \( A \) operators. Particularly, we show that if \( T \) is a class \( A \) operator and either \( \sigma(|T|) \) or \( \sigma(|T^*|) \) is not connected, then \( T \) has a nontrivial invariant subspace.

2. Main Results

A complex number \( \lambda \) is said to be in the point spectrum \( \sigma_p(T) \) of \( T \) if there is a nonzero \( x \in \mathcal{H} \) such that \( (T - \lambda)x = 0 \). If in addition, \( (T^* - \bar{\lambda})x = 0 \), then \( \lambda \) is said to be in the joint point spectrum \( \sigma_{jp}(T) \) of \( T \). Clearly, \( \sigma_{jp}(T) \subseteq \sigma_p(T) \). In general, \( \sigma_{jp}(T) \neq \sigma_p(T) \).

In [18], Xia showed that if \( T \) is a semi-hyponormal operator, then \( \sigma_{jp}(T) = \sigma_p(T) \); Tanahashi extended this result to log-hyponormal operators in [2]. Aluthge [13] showed that if \( T \) is \( \omega \)-hyponormal, then nonzero points of \( \sigma_{jp}(T) \) and \( \sigma_p(T) \) are identical; Uchiyama extended this result to class \( A \) operators in [10]. In the following, we will point out that if \( T \) is a quasi-class \( (A,k) \) operator for a positive integer \( k \), then nonzero points of \( \sigma_{jp}(T) \) and \( \sigma_p(T) \) are also identical and the eigenspaces corresponding to distinct eigenvalues of \( T \) are mutually orthogonal.

Lemma 2.1 (see [9, 17]). Let \( T \in B(\mathcal{H}) \) be a quasi-class \( (A,k) \) operator for a positive integer \( k \). If \( \lambda \neq 0 \) and \( (T - \lambda)x = 0 \) for some \( x \in \mathcal{H} \), then \( (T - \lambda)^*x = 0 \).

Theorem 2.2. Let \( T \in B(\mathcal{H}) \) be a quasi-class \( (A,k) \) operator for a positive integer \( k \). Then the following assertions hold:

(1) \( \sigma_{jp}(T) \setminus \{ 0 \} = \sigma_p(T) \setminus \{ 0 \} \),

(2) If \( (T - \lambda)x = 0 \), \( (T - \mu)y = 0 \) and \( \lambda \neq \mu \), then \( \langle x, y \rangle = 0 \).

Proof. (1) Clearly by Lemma 2.1.

(2) Without loss of generality, we assume \( \mu \neq 0 \). Then we have \( (T - \mu)^*y = 0 \) by Lemma 2.1. Thus we have \( \mu \langle x, y \rangle = \langle x, T^*y \rangle = \langle Tx, y \rangle = \lambda \langle x, y \rangle \). Since \( \lambda \neq \mu \), \( \langle x, y \rangle = 0 \). \qed
A complex number \( \lambda \) is said to be in the approximate point spectrum \( \sigma_a(T) \) of \( T \) if there is a sequence \( \{x_n\} \) of unit vectors in \( \mathcal{H} \) such that \((T - \lambda)x_n \to 0\). If in addition, \((T^* - \overline{\lambda})x_n \to 0\), then \( \lambda \) is said to be in the joint approximate point spectrum \( \sigma_{ja}(T) \). Clearly, \( \sigma_{ja}(T) \subseteq \sigma_a(T) \).

In general, \( \sigma_{ja}(T) \neq \sigma_a(T) \). In [18], Xia showed that if \( T \) is a semi-hyponormal operator, then \( \sigma_{ja}(T) = \sigma_a(T) \); Tanahashi [2] extended this result to log-hyponormal operators. Aluthge and Wang [19] showed that if \( T \) is \( \omega \)-hyponormal, then nonzero points of \( \sigma_{ja}(T) \) and \( \sigma_a(T) \) are identical; Chô and Yamazaki extended this result to class \( A \) operators in [7]. In the following, we will show that if \( T \) is a quasi-class \((A, k)\) operator for a positive integer \( k \), then nonzero points of \( \sigma_{ja}(T) \) and \( \sigma_a(T) \) are also identical.

**Theorem 2.3.** Let \( T \in B(\mathcal{H}) \) be a quasi-class \((A, k)\) operator for a positive integer \( k \). Then \( \sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\} \).

To prove Theorem 2.3, we need the following auxiliary results.

**Lemma 2.4** (see [20]). Let \( \mathcal{H} \) be a complex Hilbert space. Then there exists a Hilbert space \( \mathcal{K} \) such that \( \mathcal{H} \subset \mathcal{K} \) and a map \( \varphi : B(\mathcal{H}) \to B(\mathcal{K}) \) such that

1. \( \varphi \) is a faithful \(*\)-representation of the algebra \( B(\mathcal{H}) \) on \( \mathcal{K} \);
2. \( \varphi(A) \geq 0 \) for any \( A \geq 0 \) in \( B(\mathcal{H}) \);
3. \( \sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_{ja}(\varphi(T)) \) for any \( T \in B(\mathcal{H}) \).

**Lemma 2.5** (see [18]). Let \( \varphi : B(\mathcal{H}) \to B(\mathcal{K}) \) be Berberian’s faithful \(*\)-representation. Then \( \sigma_{ja}(T) = \sigma_{ja}(\varphi(T)) \).

**Proof of Theorem 2.3.** Let \( \varphi : B(\mathcal{H}) \to B(\mathcal{K}) \) be Berberian’s faithful \(*\)-representation of Lemma 2.4. In the following, we shall show that \( \varphi(T) \) is also a quasi-class \((A, k)\) operator for a positive integer \( k \).

In fact, since \( T \) is a quasi-class \((A, k)\) operator, we have

\[
(\varphi(T))^k \left( \left| (\varphi(T))^2 \right| - |\varphi(T)|^2 \right) (\varphi(T))^k \\
= \varphi \left( T^k \left( |T|^2 - |T|^2 \right) T^k \right) \\
\geq 0 \quad \text{by Lemma 2.4(1)}
\]

(2.1)

Hence, we have

\[
\sigma_a(T) \setminus \{0\} = \sigma_a(\varphi(T)) \setminus \{0\} \quad \text{by Lemma 2.4(3)}
\]

\[
= \sigma_{ja}(\varphi(T)) \setminus \{0\} \quad \text{by Lemma 2.4(3)}
\]

\[
= \sigma_{ja}(T) \setminus \{0\} \quad \text{by Theorem 2.2(1)}
\]

\[
= \sigma_a(T) \setminus \{0\} \quad \text{by Lemma 2.5.}
\]

The proof is complete. \( \square \)
Theorem 2.6. Let $T \in B(\mathcal{K})$ be a quasi-class $(A,k)$ operator for a positive integer $k$. Then $\sigma(T) \setminus \{0\} = (\sigma_a(T^*) \setminus \{0\})^*$ (i.e., $\{ \lambda : \overline{\lambda} \in \sigma_a(T^*) \setminus \{0\} \}$).

Proof. It suffices to prove $\sigma(T) \setminus \{0\} \subset (\sigma_a(T^*) \setminus \{0\})^* = \{ \lambda : \overline{\lambda} \in \sigma_a(T^*) \setminus \{0\} \}$.

Xia [18] showed that $\sigma(T) = \sigma_a(T) \cup (\sigma_p(T^*))^*$ holds for any $T \in B(\mathcal{K})$. Hence we have

$$\sigma_a(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\} \subset (\sigma_a(T^*) \setminus \{0\})^*$$

(2.3)

by Theorem 2.3. The proof is complete. \qed

Putnam [21] proved three theorems concerning spectral properties of hyponormal operators. These theorems were generalized to $p$-hyponormal operators by Chô et al. in [22, 23], to $w$-hyponormal operators by Aluthge and Wang in [24], and to $wF(p,r,q)$ operators by Yang and Yuan in [25]. In the following, we extend these theorems to quasi-class $(A,k)$ operators.

We show the first generalization concerning points in the approximate point spectrum of a quasi-class $(A,k)$ operator for a positive integer $k$ as follows.

Theorem 2.7. Let $T \in B(\mathcal{K})$ be a quasi-class $(A,k)$ operator for a positive integer $k$. If $\lambda \neq 0$ such that $\lambda \in \sigma_a(T)$, then $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.

To prove Theorem 2.7, we need the following auxiliary results.

Lemma 2.8 (see [26]). Let $T = U|T|$ be the polar decomposition of $T$, $\lambda \neq 0$, and $\{x_n\}$ a sequence of vectors. Then the following assertions are equivalent:

1. $(T - \lambda)x_n \to 0$ and $(T^* - \overline{\lambda})x_n \to 0$,
2. $(|T| - |\lambda|)x_n \to 0$ and $(U - e^{i\theta})x_n \to 0$,
3. $(|T^*| - |\lambda|)x_n \to 0$ and $(U^* - e^{-i\theta})x_n \to 0$.

Proof of Theorem 2.7. If $\lambda \neq 0$ and $\lambda \in \sigma_a(T)$, a sequence of unit vectors exists such that $(T - \lambda)x_n \to 0$ and $(T^* - \overline{\lambda})x_n \to 0$ by Theorem 2.3. Hence Theorem 2.7 holds by Lemma 2.8. \qed

Corollary 2.9. Let $T \in B(\mathcal{K})$ be a class A operator. If $\lambda \neq 0$ such that $\lambda \in \sigma_a(T)$, then $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.

Let $T = U|T|$ be a $p$-hyponormal operator. Does it follow that if $\lambda \in \sigma(T)$, then $|\lambda| \in \sigma(|T|)$? The answer is affirmative if $\lambda \in \sigma_a(T)$ by Corollary 2.9. In general, the answer is negative even if $T$ is hyponormal and the polar factor $U$ is unitary; see details in [21]. However, the converse is true for many classes of operators; see the following results.

Theorem 2.10 (see [18, 21, 23]). Let $T = U|T|$ be $p$-hyponormal for $p > 0$, then $\sigma(|T|) \subset \rho(\sigma(T))$, where $\rho : \mathbb{C} \to \mathbb{R}$ is defined by $\rho(z) = |z|$.

Indeed, the above Theorem 2.10 that was shown for the case $T$ is hyponormal by Putnam in [21], for the case $T$ is semihyponormal by Xia in [18], and the general case by Chô et al. in [23].

Theorem 2.11 (see [24]). Let $T = U|T|$ be $w$-hyponormal and $\sigma(T)$ is connected, then $\sigma(|T|) \subset \rho(\sigma(T))$, where $\rho : \mathbb{C} \to \mathbb{R}$ is defined by $\rho(z) = |z|$.
Here we show the second generalization concerning the relation between the spectrum of \( T \) and \(|T|\) to class A operators with connected spectrum.

**Theorem 2.12.** Let \( T \) be a class A operator and \( \sigma(T) \) is connected, then \( \sigma(|T|) \subset \rho(\sigma(T)) \), where \( \rho : \mathbb{C} \to \mathbb{R} \) is defined by \( \rho(z) = |z| \).

The numerical range \( W(T) \) of an operator \( T \) is defined by

\[
W(T) = \{ \langle Tx, x \rangle : \|x\| = 1 \}. \tag{2.4}
\]

Let \( \overline{W}(T) \) denote the closure of \( W(T) \). It is well known that for any \( T \in B(H) \), \( W(T) \) is a convex set and \( \sigma(T) \subset \overline{W}(T) \). Moreover, if \( T \) is normal, then \( \overline{W}(T) = \text{conv} \sigma(T) \), the convex hull of \( \sigma(T) \).

We need the following auxiliary results.

**Lemma 2.13** (see [7]). Let \( T = U|T| \) be the polar decomposition of a class A operator and \( \tilde{T}_{1,1} = |T|U|T| \). Then \( \tilde{T}_{1,1} \) is semihyponormal and

\[
\sigma(\tilde{T}_{1,1}) = \{ r^2 e^{i\theta} : re^{i\theta} \in \sigma(T) \}. \tag{2.5}
\]

**Lemma 2.14.** Let \( T = U|T| \) be the polar decomposition of a class A operator and \( \tilde{T}_{1,1} = |T|U|T| \). Then \( \overline{W}(\tilde{T}_{1,1}) \subset \overline{W}(\sigma(\tilde{T}_{1,1})) \).

**Proof.** Let \( \tilde{T}_{1,1} = V\tilde{T}_{1,1}V \) be the polar decomposition of \( \tilde{T}_{1,1} \). The nonzero points of \( \sigma(\tilde{T}_{1,1}) \) and \( \sigma(|\tilde{T}_{1,1}|) \) are identical. Since \( T \) is a class A operator, \( \tilde{T}_{1,1} = |T|U|T| \) is semihyponormal by Lemma 2.13, that is, \( |\tilde{T}_{1,1}| \geq |(\tilde{T}_{1,1})^*| \). It follows that \( 0 \in \sigma(|\tilde{T}_{1,1}|) \) if \( 0 \in \sigma((\tilde{T}_{1,1})^*) \). Therefore \( \sigma(|\tilde{T}_{1,1}|) \subset \sigma((\tilde{T}_{1,1})^*) \).

Hence

\[
\overline{W}(\tilde{T}_{1,1}) = \text{conv} \sigma(\tilde{T}_{1,1}) \subset \text{conv} \sigma((\tilde{T}_{1,1})^*) = \overline{W}((\tilde{T}_{1,1})^*). \tag{2.6}
\]

**Lemma 2.15.** Let \( T = U|T| \) be the polar decomposition of a class A operator and \( \tilde{T}_{1,1} = |T|U|T| \). Then \( \sigma(|T|^2) \subset \overline{W}((\tilde{T}_{1,1})^*) \).

**Proof.** Since \( T \) is a class A operator, we have

\[
|\tilde{T}_{1,1}| \geq |T|^2 \geq |(\tilde{T}_{1,1})^*| \tag{2.7}
\]

by the proof of Theorem 2.1 in [7]. So we have

\[
\langle |\tilde{T}_{1,1}| x, x \rangle \geq \langle |T|^2 x, x \rangle \geq \langle |(\tilde{T}_{1,1})^*| x, x \rangle \tag{2.8}
\]
Lemma 2.13. It follows from Theorem 2.10 that

\[ \langle |T|^2 x, x \rangle \in \overline{W(|T|^2)}. \tag{2.9} \]

Hence

\[ \sigma(|T|^2) \subseteq \text{conv} \sigma(|T|^2) = \overline{W(|T|^2)} \subseteq \overline{W(|\bar{T}_{1,1}|^*)}. \tag{2.10} \]

**Proof of Theorem 2.12.** Since \( T \) is a class \( A \) operator, \( \bar{T}_{1,1} = |T|U|T| \) is semihyponormal by Lemma 2.13. It follows from Theorem 2.10 that

\[ \sigma(|\bar{T}_{1,1}|) \subseteq \rho(\sigma(\bar{T}_{1,1})). \tag{2.11} \]

Since the nonzero points of \( \sigma(|\bar{T}_{1,1}|) \) and \( \sigma(|(\bar{T}_{1,1})^*|) \) are identical, and \( 0 \in \sigma(|(\bar{T}_{1,1})^*|) \) implies that \( (\bar{T}_{1,1})^* \) is not invertible, and hence \( 0 \in \sigma(\bar{T}_{1,1}) \), the above containment may be modified to become

\[ \sigma(|(\bar{T}_{1,1})^*|) \subseteq \rho(\sigma(\bar{T}_{1,1})). \tag{2.12} \]

By Lemma 2.13, we have

\[ \sigma(\bar{T}_{1,1}) = \{ r^2 e^{i\theta} : r e^{i\theta} \in \sigma(T) \}. \tag{2.13} \]

So

\[ \rho(\sigma(\bar{T}_{1,1})) = (\rho(\sigma(T)))^2. \tag{2.14} \]

Since \( \sigma(T) \) is connected, \( (\rho(\sigma(T)))^2 \) is a closed convex subset of \( \mathbb{R} \). Hence by Lemma 2.15, we have

\[ \sigma(|T|^2) \subseteq \overline{W(|T|^2)} = \text{conv} \sigma(|(\bar{T}_{1,1})^*|) \subseteq \text{conv} \rho(\sigma(\bar{T}_{1,1})) = (\rho(\sigma(T)))^2. \tag{2.15} \]

Since

\[ \sigma(|T|^2) = (\sigma(|T|))^2, \tag{2.16} \]

so we have

\[ (\sigma(|T|))^2 \subseteq \rho(\sigma(T))^2, \tag{2.17} \]
Theorem 2.16. Let 

\[ \sigma(|T|) \subset \rho(\sigma(T)). \]  

(2.18)

The proof is complete. \( \square \)

Putnam [21] proved that if \( T \) is hyponormal and \( \sigma(|T^*|) \) is not an interval, then \( T \) has a nontrivial invariant subspace. This result has been generalized by many authors. Cho et al. generalized Putnam’s result to \( p \)-hyponormal operators in [22]. In [24], Aluthge and Wang proved that if \( T \) is \( w \)-hyponormal and either \( \sigma(|T|) \) or \( \sigma(|T^*|) \) is not an interval, then \( T \) has a nontrivial invariant subspace.

Here we shall generalize the above result to class A operators and give an application of Theorem 2.12.

A complex number \( \lambda \) is said to be in the compression spectrum \( \sigma_c(T) \) of \( T \) if \( \text{ran}(T - \lambda) \) is not dense in \( \mathcal{A} \). It is well known that \( \sigma(T) = \sigma_c(T) \cup \sigma_a(T) \) for any \( T \in B(\mathcal{A}) \). Moreover, if \( \lambda \in \sigma_c(T) \) and \( T \neq \lambda I \), then \( \text{ran}(T - \lambda) \) is a nontrivial invariant subspace of \( T \).

**Theorem 2.16.** Let \( T \in B(\mathcal{A}) \) be a quasi-class \((A,k)\) operator for a positive integer \( k \). If there is a \( \lambda \in \sigma(T), \lambda \neq 0, \) with \( |\lambda| \notin \sigma(|T|) \cap \sigma(|T^*|) \), then \( T \) has a nontrivial invariant subspace.

**Proof.** We have that \( \lambda \notin \sigma_a(T) \) by Theorem 2.7. So we have \( \lambda \in \sigma_c(T) \). By the assumption, we have that \( T \neq \lambda I \). Hence \( T \) has a nontrivial subspace. \( \square \)

**Corollary 2.17.** Let \( T \in B(\mathcal{A}) \) be a class A operator. If there is a \( \lambda \in \sigma(T), \lambda \neq 0, \) with \( |\lambda| \notin \sigma(|T|) \cap \sigma(|T^*|) \), then \( T \) has a nontrivial invariant subspace.

**Theorem 2.18.** Let \( T \in B(\mathcal{A}) \) be a class A operator for a positive integer \( k \). If either \( \sigma(|T|) \) or \( \sigma(|T^*|) \) is not connected, then \( T \) has a nontrivial invariant subspace.

**Proof.** We only give the proof for the case that \( \sigma(|T^*|) \) is not connected, for the case \( \sigma(|T|) \) is not connected can be proved similarly.

If \( \sigma(T) \) is not connected, then the Theorem is clear, so we assume that \( \sigma(T) \) is connected. By the assumption, we have that \( \sigma(|T^*|) \) is not an interval, so there exist \( s, t \in \sigma(|T^*|), 0 \leq s < t \) such that

\[ (s,t) \cap \sigma(|T^*|) = \emptyset. \]  

(2.19)

Let \( N = \{ z : s < |z| < t \} \). Since \( \sigma(|T|) \setminus \{ 0 \} = \sigma(|T^*|) \setminus \{ 0 \} \), there exists a \( v \in \sigma(T) \) for which \( |v| = t \). Similarly, if \( 0 < s \), then there exists a \( u \in \sigma(T) \) for which \( |u| = s \) by Theorem 2.12.

On the other hand, if \( s = 0 \), then \( T^* \) is not invertible and hence \( 0 \in \sigma(T) \). Hence there exists a \( u = 0 \in \sigma(T) \) such that \( |u| = s \). So both the outer and inner boundaries of the annulus \( N \) contain a point of \( \sigma(T) \). Since \( \sigma(T) \) is connected, we have that \( N \cap \sigma(T) \neq \emptyset \).

Hence there exists a \( \lambda \in N \cap \sigma(T) \), thus \( |\lambda| \in (s,t) \). So we have that \( \lambda \neq 0 \) and \( |\lambda| \notin \sigma(|T^*|) \) by (2.19). Therefore Theorem 2.18 holds by Corollary 2.17. \( \square \)
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