Research Article

On the Maximal Eccentric Distance Sums of Graphs

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If \( G \) is a simple connected graph with vertex set \( V(G) \) and edge set \( E(G) \), let \( d_G(u,v) \) be the distance (or length of a shortest path) between vertices \( u \) and \( v \) in \( G \). For a vertex \( v \in V(G) \), the eccentricity \( ec_G(v) \) is the maximum distance from \( v \) to any other vertex, and \( DG(v) = \sum_{u \in V(G)} d_G(u,v) \) is the sum of all distances from the vertex \( v \). The eccentric distance sum of \( G \) (EDS) is defined as

\[
\xi^{d}(G) = \sum_{v \in V(G)} ec_G(v)DG(v). \tag{1.1}
\]

This graph invariant was proposed by Gupta et al. in [1]. Like the Wiener index [2–4] and eccentric connectivity index [5–9], it turned to have high discriminating power and excellent predictability both with regard to biological and physical properties and to provide valuable leads for the development of safe and potent therapeutic agents of diverse nature. So, it is of interest to study the mathematical properties of this invariant.

Yu et al. [10] considered the \( n \)-vertex trees and unicyclic graphs with minimal eccentric distance sums, respectively. Ilić et al. [11] proved that path \( P_n \) is the unique extremal trees
with \( n \) vertices having maximum eccentric distance sum, and provided various lower and upper bounds for the eccentric distance sum. In this paper, we determine the \( n \)-vertex trees with, respectively, the maximum, second-maximum, third-maximum, and fourth-maximum eccentric distance sums for \( n \geq 8 \). We also characterize the extremal unicyclic graphs with the maximal, second maximal, and third maximal eccentric distance sums.

2. The Trees with Maximal Eccentric Distance Sums

Lemma 2.1. Let \( u \) be a vertex of a connected graph \( G \) with at least two vertices. Let \( G' \) be the graph obtained by identifying \( u \) and a vertex \( v_i \) of a path \( P_{n+1} = v_0v_1 \cdots v_{n+1} \), where \( 1 \leq i \leq [n/2] \). Then, \( \xi^d(G^{-1}) > \xi^d(G') \).

Proof. By the definition of EDS, we have

\[
\xi^d(G') = \sum_{v \in V(G') \setminus \{u\}} ec_{G'}(v)D_{G'}(v) + \sum_{j=0}^{i-1} \left[ ec_{G'}(v_j)D_{G'}(v_j) + ec_{G'}(v_{n-j})D_{G'}(v_{n-j}) \right] + \sum_{k=i}^{n-i} ec_{G'}(v_k)D_{G'}(v_k).
\]

Denoted by \( A_i, B_i, \) and \( C_i \) respectively, the three sums of right equality above, we only need to prove the following three inequalities: \( A_i > A_i, B_i > B_i, C_i > C_i. \)

Note that \( ec_{G^{-1}}(v) \geq ec_{G}(v) \) for any \( v \in V(G) \), and

\[
D_{G^{-1}}(v) - D_{G}(v) = \left[ \sum_{u \in V(G)} d(v, u) + \sum_{j=0}^{i-1} (d(u, v) + j) \times 2 + \sum_{j=i}^{n-i} (d(u, v) + j) \right]
- \left[ \sum_{u \in V(G)} d(v, u) + \sum_{j=0}^{i} (d(u, v) + j) \times 2 + \sum_{j=i+1}^{n-i} (d(u, v) + j) \right]
= n - 2i + 1 > 0,
\]

and thus, the inequality \( A_i > A_i \) holds.

Note that for any \( i \leq k \leq n - i \), \( ec_{G^{-1}}(v_k) \geq ec_{G}(v_k) \), \( D_{G^{-1}}(v_k) > D_{G}(v_k) \), and thus \( C_i > C_i \). So, it suffices to prove \( B_i > B_i \).

Let \( ec_G(u) = s \) and \( n_1 = |V(G)| \), and we distinguish the following two cases:

Case 1 (\( s \geq n - i + 1 \)). By direct calculation, it follows that

\[
B_i = \sum_{j=0}^{i-1} ec_{G^{-1}}(v_j)D_{G^{-1}}(v_j) + ec_{G^{-1}}(v_{n-j})D_{G^{-1}}(v_{n-j})
\]

\[
= \sum_{j=0}^{i-1} \left[ ec_{G}(v_j) - 1 \right] \left[ \sum_{v \in V(P_{n+1})} d(v_j, v) + \sum_{v \in V(G) \setminus \{u\}} (d_G(v_j, u) - 1 + d_G(u, v)) \right]
\]
\[ \begin{aligned}
& + \left[ \text{ec}_G(v_{n-j}) + 1 \right] \left\{ \sum_{v \in V(P_{n-1})} d(v_{n-j}, v) + \sum_{v \in V(G) \setminus \{u\}} (d_G(v_{n-j}, u) + 1 + d_G(u, v)) \right\} \\
& = B_i + \sum_{j=0}^{i-1} \left( (n_1 - 1) \left[ \text{ec}_G(v_{n-j}) - \text{ec}_G(v_j) \right] \\
& \quad + \sum_{v \in V(P_{n-1})} [d(v_{n-j}, v) - d(v_j, v)] + (n_1 - 1) [d(v_{n-j}, u) - d(v_j, u)] + 2(n_1 - 1) \right).
\end{aligned} \]

(2.3)

For \( 0 \leq j \leq i - 1 \), it is easily seen that \( \text{ec}_G(v_{n-j}) > \text{ec}_G(v_j) \) and \( d(v_{n-j}, u) \geq d(v_j, u) \). Furthermore, \( \sum_{v \in V(P_{n-1})} [d(v_{n-j}, v) - d(v_j, v)] = 0 \), and thus, \( B^{i-1} > B^i \).

\textbf{Case 2} \((s \leq n - i)\). For \( 0 \leq j \leq i - 1 \), it is easily seen that \( \text{ec}_{G^{-1}}(v_j) = \text{ec}_G(v_j), \text{ec}_{G^{-1}}(v_{n-j}) \geq \text{ec}_G(v_{n-j}) \). Then,

\[ B^{i-1} = \sum_{j=0}^{i-1} \left[ \text{ec}_{G^{-1}}(v_j) D_{G^{-1}}(v_j) + \text{ec}_{G^{-1}}(v_{n-j}) D_{G^{-1}}(v_{n-j}) \right] \]

\[ \geq \sum_{j=0}^{i-1} \left[ \text{ec}_G(v_j) D_{G^{-1}}(v_j) + \text{ec}_G(v_{n-j}) D_{G^{-1}}(v_{n-j}) \right] \]

\[ = \sum_{j=0}^{i-1} \left\{ \text{ec}_G(v_j) \left[ \sum_{v \in V(P_{n-1})} d(v_j, v) + \sum_{v \in V(G) \setminus \{u\}} (d_G(v_j, u) - 1 + d_G(u, v)) \right] \\
+ \text{ec}_G(v_{n-j}) \left[ \sum_{v \in V(P_{n-1})} d(v_{n-j}, v) + \sum_{v \in V(G) \setminus \{u\}} (d_G(v_{n-j}, u) + 1 + d_G(u, v)) \right] \right\} \]

\[ = B_i + \sum_{j=0}^{i-1} \left[ \text{ec}_G(v_{n-j}) - \text{ec}_G(v_j) \right] \]

\[ > B^i, \]

where the last inequality follows as \( \text{ec}_G(v_{n-j}) > \text{ec}_G(v_j) \) for \( 0 \leq j \leq i - 1 \).

By Lemma 2.1 the inequality \( \xi^d(G^i) < \xi^d(G^0) \) follows easily. Note that \( G^i \) has the number of pendent vertices greater than that of \( G^0 \). Let \( T \) be a tree with at least \( i \) pendent vertices, where \( 3 \leq i \leq n - 1 \), by applying the above transformation to \( T \) repeatedly, then we can obtain a new tree with exactly \( i - 1 \) pendent vertices, which has larger eccentric distance sum. So it is easy to prove the following result, which is also obtained by Ilić et al. in [11].

\textbf{Theorem 2.2}. Among all trees with \( n \) vertices, \( P_n \) has the maximal eccentric distance sum.

Let \( T_{n,i} \) be the tree obtained from \( P_{n-1} = v_0 v_1 \cdots v_{n-2} \) by attaching a pendent vertex \( v_{n-1} \) to \( v_i \), where \( 1 \leq i \leq \lfloor (n - 2)/2 \rfloor \).
Lemma 2.3. Let $n \geq 8$. Then,

$$
\xi^d(T_{n,1}) - \xi^d(T_{n,2}) > 2n - 4, \quad \xi^d(T_{n,2}) - \xi^d(T_{n,3}) > 4n - 10. \tag{2.5}
$$

Proof. Suppose that $T_{n,i}$ has the same vertex labeling as the above definition. We have two cases based on the parity of $n$. If $n \geq 8$ is even, then

$$
\xi^d(T_{n,1}) - \xi^d(T_{n,2}) = \sum_{i=0}^{n-1} [\text{ec}_{T_{n,i}}(v_i)D_{T_{n,i}}(v_i) - \text{ec}_{T_{n,2}}(v_i)D_{T_{n,2}}(v_i)]
$$

$$
= (n - 2) \cdot (-1) + (n - 3) \cdot (-1) + (n - 4) \cdot 1 + \cdots + \frac{n}{2} \cdot 1
$$

$$
+ \left(\frac{n}{2} - 1\right) \cdot 1 + \frac{n}{2} \cdot 1 + \cdots + (n - 3) \cdot 1
$$

$$
+ (n - 2) \cdot 1 + \frac{1}{2} \left(3n^2 - 19n + 36\right)
$$

$$
= \frac{1}{4} \left(9n^2 - 62n + 116\right)
$$

$$
= 2(n - 2) \left(\frac{9}{8}n - \frac{44}{8}\right) + 7
$$

$$
> 2(n - 2),
$$

(2.6)

$$
\xi^d(T_{n,2}) - \xi^d(T_{n,3}) = \sum_{i=0}^{n-1} [\text{ec}_{T_{n,i}}(v_i)D_{T_{n,i}}(v_i) - \text{ec}_{T_{n,3}}(v_i)D_{T_{n,3}}(v_i)]
$$

$$
= (n - 2) \cdot (-1) + (n - 3) \cdot (-1) + (n - 4) \cdot (-1) + (n - 5) \cdot 1
$$

$$
+ \cdots + \frac{n}{2} \cdot 1 + \left(\frac{n}{2} - 1\right) \cdot 1 + \frac{n}{2} \cdot 1 + \cdots + (n - 3) \cdot 1 + (n - 2) \cdot 1
$$

$$
+ \frac{1}{2} \left(3n^2 - 27n + 72\right)
$$

$$
= \frac{1}{4} \left(9n^2 - 86n + 220\right)
$$

$$
= (4n - 10) \left(\frac{9}{16}n - \frac{127}{32}\right) + \frac{227}{16}
$$

$$
> 4n - 10.
$$

If $n \geq 9$ is odd, then we make a similar calculation as above and obtain that

$$
\xi^d(T_{n,1}) - \xi^d(T_{n,2}) = 2(n - 2) \left(\frac{9}{8}n - \frac{44}{8}\right) + \frac{29}{4} > 2(n - 2),
$$

(2.7)

$$
\xi^d(T_{n,2}) - \xi^d(T_{n,3}) = (4n - 10) \left(\frac{9}{16}n - \frac{143}{32}\right) + \frac{149}{16} > 4n - 10.
$$

These complete the proof. \qed
Lemma 2.4. Among all trees on \( n \) vertices, where \( n \geq 8 \), \( T_{n,3} \) has the maximal eccentric distance sum except \( P_n, T_{n,1}, \) and \( T_{n,2} \).

Proof. Suppose that \( T \) is an \( n \)-vertex tree different from \( P_n, T_{n,1}, \) and \( T_{n,2} \). Let \( p \) be the number of pendant vertices of \( T \), then \( p \geq 3 \).

If \( p = 3 \), then \( T \) is a tree obtained by identifying three pendant vertices of three paths. Denote by \( r, s, \) and \( t \), respectively, the lengths of the three paths, where \( r \geq s \geq t \geq 1 \) and \( r + s + t + 1 = n \). Here, we denote it by \( T_n(r, s, t) \). Clearly, \( T_n(n - i - 2, i, 1) \equiv T_{n,i} \) (\( 1 \leq i \leq [(n - 2) / 2] \)).

Suppose first that \( t = 1 \). Then, \( T = T_{n,i} \) with \( i \geq 3 \). For \( 3 \leq i \leq [(n - 4) / 2] \), by Lemma 2.1, we have \( \xi^d(T_{n,i+1}) < \xi^d(T_{n,i}) \), and thus, \( \xi^d(T) \leq \xi^d(T_{n,3}) \) with equality if and only if \( T \equiv T_{n,3} \). Now suppose that \( t \geq 2 \), then \( r \geq 3 \) for \( n \geq 8 \), and by Lemma 2.1, we can obtain a new tree \( T_n(n - s + t - 1, 1) \) or \( T_n(s + t - 1, r, 1) \), which is not isomorphic to \( T_{n,1}, T_{n,2}, \) and \( T_{n,3} \) and has larger eccentric distance sum.

If \( p \geq 5 \) by applying transformation of Lemma 2.1 to \( T \) repeatedly, we can obtain a new tree with exactly four pendant vertices and larger eccentric distance sum. Thus, it suffices to consider the case \( p = 4 \).

Now, suppose that \( p = 4 \). In this case, \( T \) has at most two vertices with degree more than 2.

Case 1. If \( T \) has exactly two vertices with degree more than 2, say \( u \) and \( v \), then \( d(u) = d(v) = 3 \). Suppose that the length of path connecting \( u \) and \( v \) is \( a \), the lengths of pendant paths at \( u \) are \( b, c \), and the lengths of pendant paths at \( v \) are \( d, f \). We denote this tree by \( T_n(a; b, c; d, f) \), where \( a \geq 1, b \geq c, d \geq f \) and \( a + b + c + d + f + 1 = n \). If \( b \geq 3 \), then \( b, d + f + a \geq 3 \), and by Lemma 2.1 and above proof, we have \( \xi^d(T_n(a; b, c; d, f)) < \xi^d(T_n(b, d + f + a, c)) < \xi^d(T_{n,3}) \), where we suppose that \( b \geq d + f + a \geq c \). If \( b = c = 2 \), then by Lemma 2.1 and the result above we have \( \xi^d(T_n(a; 2, 2; d, f)) < \xi^d(T_n(a; 3, 1; d, f)) < \xi^d(T_{n,3}) \). If \( b = 2, c = 1 \) or \( b = 1, c = 1 \), applying similar proof of Lemma 2.3, it is easily proven that \( \xi^d(T_n(n - 6; 1, 1; 2, 1)) - \xi^d(T_{n,3}) > 0 \), \( \xi^d(T_n(n - 5; 1, 1; 1, 1)) - \xi^d(T_{n,3}) > 0 \), and thus, we only need to prove that \( \xi^d(T_n(n - 5; 1, 1; 1, 1)) < \xi^d(T_{n,3}) \).

Here, we write \( T_n \) instead of \( T_n(n - 5; 1, 1; 1, 1) \). Then, \( T_n \) is a tree obtained from the path \( v_0v_1 \cdots v_{n-4}v_{n-3} \) by joining an isolated vertex \( v_{n-2} \) to \( v_{n-4} \) and an isolated vertex \( v_{n-1} \) to \( v_1 \). Let \( T_{n,1} \) be a tree obtained from \( T_n \) by deleting the edge \( v_1v_{n-1} \) and adding the edge \( v_0v_{n-1} \).

If \( n \geq 8 \) is odd, then

\[
ec_{T_{n,1}}(v_0)D_{T_{n,1}}(v_0) - ec_{T_n}(v_0)D_{T_n}(v_0) = -(n - 3),
\]

\[
ec_{T_{n,1}}(v_1)D_{T_{n,1}}(v_1) - ec_{T_n}(v_1)D_{T_n}(v_1) = n - 4,
\]

\[
ec_{T_{n,1}}(v_{n-1})D_{T_{n,1}}(v_{n-1}) - ec_{T_n}(v_{n-1})D_{T_n}(v_{n-1})
= (n - 2)(D_{T_n}(v_{n-1}) + n - 3) - (n - 3)D_{T_n}(v_{n-1})
= \frac{3n^2 - 13n + 16}{2},
\]

\[
ec_{T_{n,1}}(v_{n-3})D_{T_{n,1}}(v_{n-3}) - ec_{T_n}(v_{n-3})D_{T_n}(v_{n-3})
= ec_{T_{n,1}}(v_{n-2})D_{T_{n,1}}(v_{n-2}) - ec_{T_n}(v_{n-2})D_{T_n}(v_{n-2})
= (n - 2)(D_{T_n}(v_{n-2}) + 1) - (n - 3)D_{T_n}(v_{n-2})
= \frac{n^2 - n}{2},
\]
for 2 ≤ i ≤ (n - 5)/2, ec_{T_n}(v_i)D_{T_n}(v_i) - ec_{T_n}(v_i)D_{T_n}(v_i) = n - 3 - i; for (n - 1)/2 ≤ i ≤ n - 4,

\[ ec_{T_n}(v_i)D_{T_n}(v_i) - ec_{T_n}(v_i)D_{T_n}(v_i) = (i + 1)(D_{T_n}(v_i) + 1) - iD_{T_n}(v_i) \]

\[ = i^2 - (n - 4)i + \frac{n^2 - 3n + 2}{2}. \]

It follows that

\[
\xi^d(T_{n,1}) - \xi^d(T_{n-n/5;1,1,1,1}) = \frac{5n^2 - 15n + 14}{2} + \frac{(n-5/2)(n-3-i)}{4} + \frac{n^2 - 3n + 2}{2} \\
= \frac{25n^2 - 48n + 111}{8} + \frac{(n-5)(n+3)(2n-5)}{12} = 4n^3 + 57n^2 - 304n + 337 \]

\[ > \frac{18n^2 - 148n + 337}{4} = \xi^d(T_{n,1}) - \xi^d(T_{n,3}). \]

Thus, \( \xi^d(T_{n,3}) > \xi^d(T_{n-n/5;1,1,1,1}) \).

Similarly, for even \( n \), we have

\[
\xi^d(T_{n,1}) - \xi^d(T_{n-n/5;1,1,1,1}) = 5n^2 - 15n + 14 + \frac{n^2 - 3n + 2}{2} \\
= \frac{18n^2 - 148n + 256}{4} = \xi^d(T_{n,1}) - \xi^d(T_{n,3}), \]

and the inequality \( \xi^d(T_{n,3}) > \xi^d(T_{n-n/5;1,1,1,1}) \) also holds.

Case 2. If \( T \) has an unique vertex of degree greater than 2, say \( 2 \), then \( d(2) = 4 \), and \( T \) is a tree obtained by identifying four pendent vertices of four paths. Denote by \( T_n(r,s,t,l) \) this tree, where \( r, s, t, l \) are the lengths of the four pendent paths respectively and \( r \geq s \geq t \geq l \geq 1, r+s+t+l+1 = n \). If \( t = 2 \), by Lemma 2.1, we have \( \xi^d(T_n(r,s,t,l)) < \xi^d(T_n(r+1,s+1,l)) < \xi^d(T_{n,3}) \). Similarly, we can show the result for \( t = l = 1 \) and \( r \geq 2 \). In the following, we will prove that \( \xi^d(T_n(n-4,1,1,1)) < \xi^d(T_{n,3}) \). We label the vertices of \( T_n(n-4,1,1,1) \) and \( T_{n,3} \) such
that $T_n(n - 4, 1, 1, 1)$ can be viewed as obtained from the path $v_0v_1 \cdots v_{n-4}$ by joining three isolated vertices $v_{n-3}$, $v_{n-2}$ and $v_{n-1}$ to $v_{n-4}$, and $T_{n,3}$ can be viewed as obtained from the path $v_0v_1 \cdots v_{n-2}$ by joining an isolated vertex $v_{n-1}$ to $v_{n-5}$. Clearly, $ecr_{T_{n,3}}(v_i) \geq ecr_{T_{n}}(v_i)$ for $0 \leq i \leq n - 3$, $DT_{T_{n,3}}(v_i) = DT_{T_{n}}(v_i(n-4,1,1,1))$ for $0 \leq i \leq n - 5$ and $i = n - 3$, and $DT_{T_{n,3}}(v_{n-4}) = DT_{T_{n}}(v_{n-4}) + 2$. Note that

$$ecr_{T_{n,3}}(v_{n-2})DT_{T_{n,3}}(v_{n-2}) - ecr_{T_{n}}(v_{n-2})DT_{T_{n}}(v_{n-2})$$

$$= (n - 2) \frac{DT_{T_{n}}(v_{n-2}) + n - 2} - (n - 3)DT_{T_{n}}(v_{n-2})$$

$$= (n - 2)(n - 2) + \frac{1}{2}(n - 2)(n - 3) + 4$$

$$= \frac{3n^2 - 13n + 22}{2},$$

(2.12)

$$ecr_{T_{n,3}}(v_{n-1})DT_{T_{n,3}}(v_{n-1}) - ecr_{T_{n}}(v_{n-1})DT_{T_{n}}(v_{n-1})$$

$$= (n - 4) \frac{DT_{T_{n}}(v_{n-2}) - n + 8} - (n - 3)DT_{T_{n}}(v_{n-2})$$

$$= -(n - 2)(n - 8) + \frac{1}{2}(n - 2)(n - 3) - 4$$

$$= \frac{3n^2 - 25n + 46}{2},$$

then

$$\xi^d(T_{n,3}) - \xi^d(T_{n}(n - 4, 1, 1, 1)) > \frac{3n^2 - 13n + 22}{2} - \frac{3n^2 - 25n + 46}{2} > 0,$$

(2.13)

and this completes the proof.

From Lemmas 2.1 and 2.4, We have the following

**Theorem 2.5.** If $n \geq 8$, then $T_{n,1}$, $T_{n,2}$, and $T_{n,3}$ are the unique trees with the second-maximal, third-maximal, and fourth-maximal eccentric distance sums among the trees on $n$ vertices.

### 3. The Unicyclic Graphs with Maximal Eccentric Distance Sums

Let $U_n$ be the graph obtained from a path $P_{n-1} = v_0v_1 \cdots v_{n-2}$ by joining the vertex $v_{n-1}$ to $v_{n-3}$ and $v_{n-2}$. Let $Q_n$ be the graph obtained from a path $P_{n-1} = v_0v_1 \cdots v_{n-2}$ by joining the vertex $v_{n-1}$ to $v_{n-4}$ and $v_{n-2}$, and $B_n$ be obtained by joining $v_{n-1}$ to $v_{n-4}$ and $v_{n-3}$.

**Theorem 3.1.** Let $G$ be a graph with $n$ vertices and $n$ edges; that is, $G$ is an unicyclic graph, where $n \geq 8$. Then, $\xi^d(G) \leq \xi^d(U_n)$, with equality if and only if $G \equiv U_n$.

**Proof.** If $G \nsubseteq C_n$ and $G \nsubseteq U_n$, then we can always find an edge $e$ of $G$ such that $G - e$ is a tree with at least three pendant vertices and $G - e \nsubseteq T_{n,1}$, $P_n$. It follows from Lemma 2.3 that

$$\xi^d(G) < \xi^d(G - e) \leq \xi^d(T_{n,2}) < \xi^d(T_{n,1}) - 2(n - 2) = \xi^d(U_n).$$

(3.1)
Suppose that \( G \equiv C_n \) and \( v \) is any vertex of \( C_n \). If \( n \geq 8 \) is even, then
\[
ec(v)D(v) = \frac{n}{2} \left( \frac{1}{2} + \cdots + \frac{n}{2} - 1 \right) \times 2 + \frac{n}{2} = \frac{n^3}{8}.
\]
(3.2)

By the definition of EDS, we have
\[
\xi^d(U_n) = (ec(v_0)D(v_0) + ec(v_{n/2-1})D(v_{n/2-1}))
+ (ec(v_{n/2-2})D(v_{n/2-2}) + ec(v_{n/2})D(v_{n/2})) + \sum_{i \neq 0,n/2-1,n/2-2,n/2} ec(v_i)D(v_i).
\]
(3.3)

It is easily checked that
\[
ec(v_0)D(v_0) + ec(v_{n/2-1})D(v_{n/2-1}) > \frac{n^3}{4},
\]
\[
ec(v_{n/2-2})D(v_{n/2-2}) + ec(v_{n/2})D(v_{n/2}) = \frac{n^3}{4},
\]
(3.4)

and for \( i \neq 0, n/2-1, n/2-2, n/2, ec(v_i) \geq n/2 + 1, D(v_i) > n^2/4 \). Then,
\[
\xi^d(U_n) > \frac{n^3}{4} + \frac{n^3}{4} + (n-4)\left(\frac{n}{2} + 1\right)\frac{n^2}{4} = \frac{n^4}{8} + \frac{3}{8}n^2(n-2) > \frac{n^4}{8} = \xi^d(C_n).
\]
(3.5)

We can prove the result for odd number \( n \) similarly, and thus complete the proof.

**Theorem 3.2.** Let \( n \geq 8 \). Then, \( B_n, Q_n \) are the graph with, respectively, the second-maximal and third-maximal eccentric distance sums among all unicyclic graphs on \( n \) vertices.

**Proof.** If \( G \) has a spanning tree \( T \) such that \( T \not\equiv P_n, T_{n,1}, T_{n,2}, T_3 \), then
\[
\xi^d(G) < \xi^d(T) \leq \xi^d(T_{n,3}) < \xi^d(Q_n) - 4n + 10 = \xi^d(Q_n).
\]
(3.6)

Now suppose that any spanning tree of \( G \) is one of \( \{P_n, T_{n,1}, T_{n,2}\} \). Then, \( G \) must be isomorphic to \( C_n, Q_n \) or \( B_n \). It can be proven easily that \( \xi^d(C_n) < \xi^d(Q_n) \) by similar proof of Theorem 3.1. And by directed computation, we obtain that \( \xi^d(B_n) = \xi^d(Q_n) + 1 \). Thus, the result follows.

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