Research Article

Some Results for Nonlinear $(n + 1)$ -Term Fractional Integrodifferential Inclusions with Multipoint Boundary Conditions

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We are concerned with the nonlinear $(n + 1)$ -term fractional integrodifferential inclusions $L(D)u \in F(t, u, Tu, Su)$, a.e. $t \in [0, 1]$, where $L(D) = D^\alpha - b_n D^{\beta_n} - b_{n-1} D^{\beta_{n-1}} - \dots - b_1 D^{\beta_1}$, $0 < \beta_1 < \beta_2 < \dots < \beta_{n-1} < \beta_n < 1 < \alpha < 2$, $b_1, b_2, \dots, b_n$ are constant coefficients, and $(Tu)(t) = \int_0^1 k(t, s)u(s)ds$, $(Su)(t) = \int_0^t l(t, s)u(s)ds$, subject to the nonlocal conditions $u(0) = 0$, $u(1) = \sum_{i=1}^m \gamma_i u(\eta_i)$. The existence results are obtained by using two fixed-point theorems due to Bohnenblust-Karlin and Covitz-Nadler, respectively. Our results partly generalize and improve the known ones.

1. Introduction

Fractional differential equations (FDEs) have received increasing interest for the last three decades. It is benefited by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in the modeling of many phenomena in various fields of science, engineering economics, and other fields, see for instance [1–12] and references therein.

Babakhani and Gejji [7] considered the existence of positive solutions for the nonlinear fractional differential equations

$$L(D)u = f(t, u), \quad u(0) = 0, \quad 0 < t < 1, \quad (1.1)$$

where $L(D) = D^{s_n} - a_{n-1} D^{s_{n-1}} - \dots - a_1 D^{s_1}$, $0 < s_1 < s_2 < \dots < s_n < 1$, $a_j > 0$, $j = 1, 2, \dots, n-1$, and D^{s_j} is the standard Riemann-Liouville fractional derivative. Some existence results of positive solutions are obtained by using some fixed-point theorems on a cone.

Stojanović [11] considered the existenceuniqueness of solutions for a nonlinear n -term fractional differential equation

$$b_0 D^{\beta_0} u(t) + \sum_{i=1}^{m-1} b_i D^{\beta_i} u(t) + \sum_{i=m}^{n-1} b_i D^{\alpha_i} u(t) + b_n D^{\alpha_n} u(t) = f(t, u(t)), \quad t \in (0, 1), \quad (1.2)$$

where $0 < \beta_1 < \beta_2 < \dots < \beta_{m-1} < 1 < \alpha_m < \alpha_{m+1} < \dots < \alpha_n < 2$, with initial data $u(0) = f(0)$, $u_t(0) = g(0)$.

On the other hand, realistic problems arising from economics, optimal control, and so on can be modeled as differential inclusions. Recently, El-Sayed and Ibrahim [13] initiated the study of fractional differential equations inclusions. Differential inclusions have been widely investigated by many authors, see [14–30] and references therein.

Very recently, in the survey paper [16], Agarwal et al. establish sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problems for fractional differential equations and inclusions involving the Caputo fractional derivative.

Ouahab [26] studied the following boundary value problem of fractional differential inclusions:

$$\begin{aligned} -D^\alpha y(t) &\in F(t, y(t)), \quad \text{a.e. } t \in J = [0, 1], \quad 1 < \alpha \leq 2, \\ y(0) &= y(1) = 0, \end{aligned} \quad (1.3)$$

where D^α is the standard Riemann-Liouville fractional derivative, $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values.

Chang and Nieto [27] studied boundary value problem of fractional differential inclusions

$$\begin{aligned} {}_0^C D^\delta y(t) &\in F(t, y(t)), \quad t \in J := [0, 1], \quad \delta \in (1, 2), \\ y(0) &= \alpha, \quad y(1) = \beta, \quad \alpha, \beta \neq 0, \end{aligned} \quad (1.4)$$

where ${}_0^C D^\delta y(t)$ is the Caputo's derivative, $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$.

However, to the best of our knowledge, the existence of solutions for fractional integro-differential inclusions with multipoint boundary conditions has not been paid much attention. Our goal is to fill this gap in literature.

In the present work, we consider more general fractional integro-differential inclusions

$$L(D)u \in F(t, u, Tu, Su), \quad \text{a.e. } t \in [0, 1] \quad (1.5)$$

with multipoint boundary conditions

$$u(0) = 0, \quad u(1) = \sum_{i=1}^m \gamma_i u(\eta_i), \quad (1.6)$$

where $L(D) = D^\alpha - b_n D^{\beta_n} - b_{n-1} D^{\beta_{n-1}} - \dots - b_1 D^{\beta_1}$, $0 < \beta_1 < \beta_2 < \dots < \beta_{n-1} < \beta_n < 1 < \alpha < 2$, D^α, D^{β_j} , $j = 1, 2, \dots, n$, is the standard Riemann-Liouville fractional derivative, $0 < \eta_1 < \eta_2 < \dots < \eta_m < 1$, $0 \leq \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1} < 1$, $(Tu)(t) = \int_0^1 k(t, s)u(s)ds$, $(Su)(t) = \int_0^t l(t, s)u(s)ds$,

k, l are two continuous functions on $[0, 1] \times [0, 1]$, and $F : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a given multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$). We shall consider both the cases of convex and nonconvex valued right hand side and establish some sufficient conditions which admit that the integro-differential inclusions problem has at least one solution. These results obtained by applying two fixed point theorems due to Bohnenblust-Karlin and Covitz-Nadler are complement of previously known results.

The organization of this paper is as follows. In Section 2, we present some necessary definitions and preliminaries which are used throughout this paper. Main results and their proofs are given in Section 3.

2. Preliminaries and Several Lemmas

In this section, we recall some basic definitions and notations and give several lemmas which are useful in our discussion.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) dt, \quad (2.1)$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Definition 2.2. The Riemann-Liouville fractional derivative of function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is given by

$$D^\alpha f(t) = D^n I^{n-\alpha} f(t), \quad (2.2)$$

where $n = [\alpha] + 1$, $D^n = d^n/dt^n$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.3. Let $\alpha > 0$, then

$$I^\alpha D^\alpha x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad (2.3)$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = -[-\alpha]$.

Remark 2.4. If the fractional derivative $D^\alpha x(t)$ is integrable, then

$$I^\alpha D^\beta x(t) = I^{\alpha-\beta} x(t) - \left[I^{1-\beta} x(t) \right]_{t=0} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \beta \leq \alpha < 1. \quad (2.4)$$

If x is continuous on $[0, 1]$, then $[I^{1-\beta} x(t)]_{t=0} = 0$, and (2.4) reduces to

$$I^\alpha D^\beta x(t) = I^{\alpha-\beta} x(t), \quad 0 < \beta \leq \alpha < 1. \quad (2.5)$$

The reader is referred to [4, 8, 9] for more details on fractional integrals and fractional derivatives.

Lemma 2.5. *Let $g \in L([0, 1], \mathbb{R})$, $0 < \beta_1 < \beta_2 < \dots < \beta_{n-1} < \beta_n < 1 < \alpha < 2$, and $0 < \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1} < 1$. Then $u(t)$ is a solution of the BVP*

$$\left(D^\alpha - b_n D^{\beta_n} - b_{n-1} D^{\beta_{n-1}} - \dots - b_1 D^{\beta_1}\right) u(t) = g(t), \quad a.e. \ t \in [0, 1], \quad (2.6)$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^m \gamma_i u(\eta_i). \quad (2.7)$$

if and only if $u(t)$ satisfies the integral equation

$$\begin{aligned} u(t) = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} t^{\alpha-1} u(s) ds \\ & - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} t^{\alpha-1} u(s) ds \\ & + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(s) ds \\ & + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} g(s) ds \\ & - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} g(s) ds. \end{aligned} \quad (2.8)$$

Proof. In view of Lemma 2.3 and Remark 2.4, (2.6) is equivalent to the integral equation

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \sum_{i=1}^n b_i I^{\alpha - \beta_i} u(t) + I^\alpha g(t), \quad (2.9)$$

for some $c_1, c_2 \in \mathbb{R}$, that is,

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(s) ds. \quad (2.10)$$

The boundary condition $u(0) = 0$ implies $c_2 = 0$. Thus,

$$u(t) = c_1 t^{\alpha-1} + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(s) ds. \quad (2.11)$$

In view of the boundary condition $u(1) = \sum_{i=1}^m \gamma_i u(\eta_i)$, we conclude that

$$\begin{aligned} c_1 = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} u(s) ds \\ & - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} u(s) ds \\ & + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} g(s) ds \\ & - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} g(s) ds. \end{aligned} \quad (2.12)$$

Therefore, the solution $u(t)$ of (2.6) and (2.7) satisfies (2.8).

Conversely, if $u(t)$ is a solution of (2.8), it is easy to verify that $u(t)$ satisfies (2.6) and (2.7). The proof is complete. $\square$

Now, we recall some facts from multivalued analysis.

Let (X, d) be a metric space, and $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$, $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$, and so forth.

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\}, \quad (2.13)$$

where $d(x, B) = \inf_{y \in B} d(x, y)$, $d(A, y) = \inf_{x \in A} d(x, y)$. Then $(\mathcal{P}_{bd,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [31]).

A multivalued map $N : [0, 1] \rightarrow \mathcal{P}_{cl}(X)$ is said to be measurable if for each $u \in X$, the function $Y : [0, 1] \rightarrow \mathbb{R}$, defined by

$$Y(t) = d(u, N(t)) = \inf \{d(u, z) : z \in N(t)\}, \quad (2.14)$$

is measurable.

A multivalued map $F : X \rightarrow \mathcal{P}(X)$ is convex (closed) valued if $F(u)$ is convex (closed) for all $u \in X$. F is bounded on bounded sets if $F(B) = \cup_{u \in B} F(u)$ is bounded in X for all $B \in \mathcal{P}(X)$. That is, $\sup_{u \in B} \{\sup\{|y| : y \in F(u)\}\} < \infty$. F is called upper semicontinuous (u.s.c for short) on X if for each $u_0 \in X$, the set $F(u_0)$ is nonempty closed subset of X , and if for each open set $\mathcal{U}$ of X containing $F(u_0)$, there exists an open neighborhood $\mathcal{V}$ of u_0 such that $F(\mathcal{V}) \subset \mathcal{U}$. G is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_{bd}(X)$. If the multivalued map F is completely continuous with nonempty compact valued, then G is u.s.c. if and only if F has closed graph, that is, $x_n \rightarrow x_*$, $y_n \rightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

More details on multivalued maps can be found in the books of Deimling [3], Górniewicz [32], Hu and Papageorgiou [33], and Tolstonogov [34].

Definition 2.6. The multivalued map $F : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is L^1 -Carathéodory if

- (i) $t \mapsto F(t, u, v, w)$ is measurable for each $(u, v, w) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$;
- (ii) $(u, v, w) \mapsto F(t, u, v, w)$ is upper semicontinuous for almost all $t \in [0, 1]$;
- (iii) for each $r > 0$, there exists $\varphi_r \in L^1([0, 1], \mathbb{R}_+)$ such that for all $|u|, |v|, |w| \leq r$ and for almost all $t \in [0, 1]$,

$$\|F(t, u, v, w)\|_{\mathcal{P}} = \sup\{|f| : f \in F(t, u, v, w)\} \leq \varphi_r(t). \quad (2.15)$$

For any $u \in C([0, 1], \mathbb{R})$, we define the set

$$S_{F,u} = \left\{ f \in L^1([0, 1], \mathbb{R}) : f(t) \in F(t, u(t), (Tu)(t), (Su)(t)) \text{ for a.e. } t \in [0, 1] \right\}, \quad (2.16)$$

which is known as the set of selection functions.

Lemma 2.7 (see [35]). *Let X be a Banach space. Let $F : [0, 1] \times X \rightarrow \mathcal{P}_{cp,cv}(X)$ be an L^1 -Carathéodory multivalued map with $S_{F,u} \neq \emptyset$, and let Γ be a linear continuous mapping from $L^1([0, 1], X)$ into $C([0, 1], X)$. Then the operator*

$$\begin{aligned} \Gamma \circ S_F : C([0, 1], X) &\longrightarrow \mathcal{P}_{cp,cv}(C([0, 1], X)), \\ u &\longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,u}) \end{aligned} \quad (2.17)$$

is a closed graph operator in $C([0, 1], X) \times C([0, 1], X)$.

The following Bohnenblust-Karlin fixed-point lemma and the fixed-point theorem for contractive multivalued operators given by Covitz and Nadler are of great importance in the proofs of our main results. The proofs of these results can be found in Bohnenblust and Karlin [30] and in Covitz and Nadler [36].

Lemma 2.8 (see [30]). *Let X be a Banach space, D be a nonempty subset of X , which is bounded, closed, and convex. Suppose $G : D \rightarrow \mathcal{P}(X)$ is u.s.c. with closed, convex values, and such that $G(D) \subset D$ and $G(D)$ is relatively compact. Then G has a fixed point.*

Lemma 2.9 (see [36]). *Let (X, d) be a complete metric space. If N is a contraction operator, then $\text{Fix } N \neq \emptyset$.*

For convenience, let us list some conditions.

$$(H1) \quad \sum_{i=1}^m \sum_{j=1}^n (\gamma_i b_j \eta_i^{\alpha-\beta_j} / (1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j + 1)) + \sum_{i=1}^n (b_i / (1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i + 1)) + \sum_{i=1}^n (b_i / \Gamma(\alpha - \beta_i + 1)) < 1;$$

$$(H2) \quad F : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}_{cp,cv}(\mathbb{R}) \text{ is Carathéodory multivalued map.}$$

Lemma 2.10. Assume that hypothesis (H1) is satisfied. For any $g \in C([0, 1], \mathbb{R})$, the integral equation

$$\begin{aligned} u(t) = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-\beta_j-1} t^{\alpha-1} u(s) ds \\ & - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1-s)^{\alpha-\beta_i-1} t^{\alpha-1} u(s) ds \\ & + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} u(s) ds + g(t) \end{aligned} \quad (2.18)$$

has a unique solution in $C([0, 1], \mathbb{R})$.

Proof. We define the operator A as follows:

$$\begin{aligned} (Au)(t) := & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-\beta_j-1} t^{\alpha-1} u(s) ds \\ & - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1-s)^{\alpha-\beta_i-1} t^{\alpha-1} u(s) ds \\ & + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} u(s) ds + g(t) \end{aligned} \quad (2.19)$$

Obviously, A is a map from $C([0, 1], \mathbb{R})$ into $C([0, 1], \mathbb{R})$. Also, we have

$$\begin{aligned} |(Au_1)(t) - (Au_2)(t)| \leq & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-\beta_j-1} t^{\alpha-1} |u_1(s) - u_2(s)| ds \\ & + \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1-s)^{\alpha-\beta_i-1} t^{\alpha-1} |u_1(s) - u_2(s)| ds \\ & + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} |u_1(s) - u_2(s)| ds \\ \leq & \left(\sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j \eta_i^{\alpha-\beta_j}}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j + 1)} \right. \\ & \left. + \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i + 1)} + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i + 1)} \right) \|u_1 - u_2\|. \end{aligned} \quad (2.20)$$

Therefore, $\|Au_1 - Au_2\| \leq \lambda \|u_1 - u_2\|$, where $\lambda = \sum_{i=1}^m \sum_{j=1}^n (\gamma_i b_j \eta_i^{\alpha-\beta_j} / (1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j + 1)) + \sum_{i=1}^n (b_i / (1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i + 1)) + \sum_{i=1}^n (b_i / \Gamma(\alpha - \beta_i + 1))$. By (H1) and Banach contraction principle, the conclusion of lemma is true. $\square$

Definition 2.11. A function $u \in C[0, 1]$ is said to be a solution of the BVP (1.5) and (1.6), if there exists a function $f \in L^1([0, 1], \mathbb{R})$ such that $f(t) \in F(t, u(t), (Tu)(t), (Su)(t))$ a.e. on $[0, 1]$ and

$$\begin{aligned} u(t) = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-\beta_j-1} t^{\alpha-1} u(s) ds \\ & - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha-\beta_i-1} t^{\alpha-1} u(s) ds \\ & + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha-\beta_i-1} u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds \\ & + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} f(s) ds \\ & - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} f(s) ds. \end{aligned} \quad (2.21)$$

3. Main Results

In this section, we present our main results and prove them. Firstly, under convexity condition on the multivalued right-hand side, we are to establish the existence theorem of solutions for fractional differential inclusions (1.5) and (1.6), by employing the Bohnenblust-Karlin fixed-point theorem. Then, under nonconvexity condition on the multivalued right-hand side, the existence theorem of solutions are gotten, by employing the Covitz-Nadler fixed-point theorem.

Theorem 3.1. Assume that hypotheses (H1) and (H2) are satisfied. Then BVP (1.5) and (1.6) has at least one solution in $C([0, 1], \mathbb{R})$, provided that

$$(H3) \quad \tau / \Gamma(\alpha) + \sum_{i=1}^m (\gamma_i \mu_i / (1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)) + (\omega / (1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)) < 1 - \lambda,$$

where

$$\begin{aligned} \tau &= \liminf_{r \rightarrow \infty} \int_0^1 \frac{\max\{\varphi_r(s), \varphi_{\tilde{k}r}(s), \varphi_{\tilde{l}r}(s)\}}{r} ds, \\ \mu_i &= \liminf_{r \rightarrow \infty} \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1} \max\{\varphi_r(s), \varphi_{\tilde{k}r}(s), \varphi_{\tilde{l}r}(s)\}}{r} ds, \\ \omega &= \liminf_{r \rightarrow \infty} \int_0^1 \frac{(1 - s)^{\alpha-1} \max\{\varphi_r(s), \varphi_{\tilde{k}r}(s), \varphi_{\tilde{l}r}(s)\}}{r} ds, \\ \tilde{k} &= \sup_{t \in [0, 1]} \int_0^1 k(t, s) ds, \quad \tilde{l} = \sup_{t \in [0, 1]} \int_0^t l(t, s) ds, \end{aligned} \quad (3.1)$$

and λ is defined in the proof of Lemma 2.10.

Proof. To transform the problem into a fixed-point problem, we consider the multivalued operator, $N : C([0, 1], \mathbb{R}) \rightarrow \mathcal{P}(C([0, 1], \mathbb{R}))$, where for any $u \in C([0, 1], \mathbb{R})$, $N(u)$ is defined by

$$\left\{ h \in C([0, 1], \mathbb{R}) : h(t) = \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} t^{\alpha-1} h(s) ds \right. \\ - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} t^{\alpha-1} h(s) ds \\ + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds \\ + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} f(s) ds \\ \left. - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} f(s) ds, f \in S_{F,u} \right\}. \quad (3.2)$$

For any $f \in L^1([0, 1], \mathbb{R})$, we have $\int_0^t (t - s)^{\alpha-1} f(s) ds, \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} f(s) ds, \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} f(s) ds \in C([0, 1], \mathbb{R})$. In view of Lemma 2.10, N is well defined. Moreover, it follows from the convexity of $S_{F,u}$ (because F has convex values) that $N(u)$ is convex for each $u \in C([0, 1], \mathbb{R})$. Clearly, the fixed points of N are solutions of (1.5) and (1.6).

We shall show that N has a fixed point in three steps.

Step 1. we claim that there exists a $r^* > 0$, such that $N(B_{r^*}) \subset B_{r^*}$, where $B_{r^*} = \{u \in C([0, 1], \mathbb{R}) : \|u\| \leq r^*\}$.

In fact, if it is not true, then for any $r > 0$, there exists a function $u_r \in B_r$, $h_r \in N(u_r)$ but $h_r \notin B_r$, that is $\|h_r\| > r$ and for some $f_r \in S_{F,u_r}$

$$h_r(t) = \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} t^{\alpha-1} h_r(s) ds \\ - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} t^{\alpha-1} h_r(s) ds \\ + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} h_r(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f_r(s) ds \quad (3.3) \\ + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} f_r(s) ds \\ - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} f_r(s) ds.$$

On the other hand, from (H2), we obtain

$$\begin{aligned}
|h_r(t)| &\leq \lambda \|h_r\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_r(s) ds \\
&\quad + \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} t^{\alpha-1} f_r(s) ds \\
&\quad + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} f_r(s) ds \\
&\leq \lambda \|h_r\| + \frac{1}{\Gamma(\alpha)} \int_0^1 (|t-s|)^{\alpha-1} \varphi_{\max\{r, \tilde{k}r, \tilde{l}r\}}(s) ds \\
&\quad + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} \varphi_{\max\{r, \tilde{k}r, \tilde{l}r\}}(s) ds \\
&\quad + \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} t^{\alpha-1} \varphi_{\max\{r, \tilde{k}r, \tilde{l}r\}}(s) ds.
\end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned}
r < \|h_r\| &\leq \frac{1}{(1-\lambda)\Gamma(\alpha)} \int_0^1 (|t-s|)^{\alpha-1} \varphi_{\max\{r, \tilde{k}r, \tilde{l}r\}}(s) ds \\
&\quad + \sum_{i=1}^m \frac{\gamma_i}{(1-\lambda)(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \varphi_{\max\{r, \tilde{k}r, \tilde{l}r\}}(s) ds \\
&\quad + \frac{1}{(1-\lambda)(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{\max\{r, \tilde{k}r, \tilde{l}r\}}(s) ds.
\end{aligned} \tag{3.5}$$

Dividing both sides by r and taking the lower limit as $r \rightarrow +\infty$, we get

$$1 \leq \frac{\tau}{(1-\lambda)\Gamma(\alpha)} + \sum_{i=1}^m \frac{\gamma_i \mu_i}{(1-\lambda)(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} + \frac{\omega}{(1-\lambda)(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)}. \tag{3.6}$$

This is a contraction to (H3). Hence there exists a $r^* > 0$ such that $N(B_{r^*}) \subset B_{r^*}$.

Step 2. $N(B_{r^*})$ is equicontinuous.

Let $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, $u \in B_{r^*}$, and $h \in N(u)$. Then there exists $f \in S_{F,u}$ such that for each $t \in [0, 1]$, we have

$$\begin{aligned}
 h(t) = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-\beta_j-1} t^{\alpha-1} h(s) ds \\
 & - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1-s)^{\alpha-\beta_i-1} t^{\alpha-1} h(s) ds \\
 & + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \\
 & + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} f(s) ds \\
 & - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} t^{\alpha-1} f(s) ds.
 \end{aligned} \tag{3.7}$$

Therefore,

$$\begin{aligned}
 |h(t_1) - h(t_2)| \leq & a \left(t_2^{\alpha-1} - t_1^{\alpha-1} \right) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \varphi_{\max\{r^*, \tilde{k}r^*, \tilde{l}r^*\}}(s) ds \\
 & + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right) \varphi_{\max\{r^*, \tilde{k}r^*, \tilde{l}r^*\}}(s) ds \\
 & + \sum_{i=1}^n \frac{b_i r^*}{\Gamma(\alpha - \beta_i + 1)} \left(t_2^{\alpha-\beta_i} - t_1^{\alpha-\beta_i} + 2(t_2 - t_1)^{\alpha-\beta_i} \right),
 \end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
 a = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j \eta_i^{\alpha-\beta_j} r^*}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j + 1)} + \sum_{i=1}^n \frac{b_i r^*}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i + 1)} \\
 & + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \varphi_{\max\{r^*, \tilde{k}r^*, \tilde{l}r^*\}}(s) ds \\
 & + \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{\max\{r^*, \tilde{k}r^*, \tilde{l}r^*\}}(s) ds.
 \end{aligned} \tag{3.9}$$

The right-hand side of the above inequality tends to zeros independently of $h \in B_{r^*}$, as $t_2 - t_1 \rightarrow 0$. This shows that $N(B_{r^*})$ is equicontinuous.

Step 3. N has a closed graph.

Let $u_n \rightarrow u_*$, $h_n \in N(u_n)$ and $h_n \rightarrow h_*$. We will prove that $h_* \in N(u_*)$. Now $h_n \in N(u_n)$ implies that there exists $f_n \in S_{F,u_n}$ such that for each $t \in [0, 1]$,

$$\begin{aligned} h_n(t) = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} t^{\alpha-1} h_n(s) ds \\ & - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} t^{\alpha-1} h_n(s) ds \\ & + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} h_n(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f_n(s) ds \\ & + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} f_n(s) ds \\ & - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} f_n(s) ds. \end{aligned} \quad (3.10)$$

We need to show that there exists $f_* \in S_{F,u}$ such that for each $t \in [0, 1]$,

$$\begin{aligned} h_*(t) = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} t^{\alpha-1} h_*(s) ds \\ & - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} t^{\alpha-1} h_*(s) ds \\ & + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} h_*(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f_*(s) ds \\ & + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} f_*(s) ds \\ & - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} f_*(s) ds. \end{aligned} \quad (3.11)$$

In (2.18), taking

$$\begin{aligned} g(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} f(s) ds \\ & - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} f(s) ds. \end{aligned} \quad (3.12)$$

In view of Lemma 2.10, we know that for any $f \in L([0, 1], \mathbb{R})$, the integral equation (2.18) has a unique solution, which we denote by h_f . Because of this, we can define the operator

$$\begin{aligned}\Phi : L([0, 1], \mathbb{R}) &\longrightarrow C([0, 1], \mathbb{R}), \\ f &\longmapsto h_f.\end{aligned}\tag{3.13}$$

It is easy to verify that the operator Φ is linear. On the other hand, we can get

$$\begin{aligned}\|h_f\| &\leq \frac{1}{(1-\lambda)\Gamma(\alpha)} \int_0^1 (|t-s|)^{\alpha-1} |f(s)| ds \\ &\quad + \frac{1}{(1-\lambda)(1-\sum_{i=1}^m \gamma_i \eta_i^{\alpha-1})\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s)| ds \\ &\quad + \sum_{i=1}^m \frac{\gamma_i}{(1-\lambda)(1-\sum_{i=1}^m \gamma_i \eta_i^{\alpha-1})\Gamma(\alpha)} \int_0^{\eta_i} (\eta_i-s)^{\alpha-1} |f(s)| ds \\ &\leq \mu \int_0^1 |f(s)| ds,\end{aligned}\tag{3.14}$$

where

$$\mu = \frac{1}{(1-\lambda)\Gamma(\alpha)} + \sum_{i=1}^m \frac{\gamma_i \eta_i^{\alpha-1}}{(1-\lambda)(1-\sum_{i=1}^m \gamma_i \eta_i^{\alpha-1})\Gamma(\alpha)} + \frac{1}{(1-\lambda)(1-\sum_{i=1}^m \gamma_i \eta_i^{\alpha-1})\Gamma(\alpha)},\tag{3.15}$$

that is, $\|\Phi(f)\| \leq \mu \|f\|_{L^1}$, which implies Φ is continuous. From Lemma 2.7, it follows that $\Phi \circ S_{F,u}$ is a closed graph operator. Moreover, we know $h_n \in \Phi(S_{F,u_n})$. Since $u_n \rightarrow u_*$, it follows from Lemma 2.7 that (3.11) holds for some $f_* \in S_{F,u_*}$.

Therefore, N is a compact multivalued map, u.s.c. with convex closed values. As a consequence of Lemma 2.8, we immediately conclude that N has a fixed-point u which is a solution of the problem (1.5) and (1.6). The proof is complete. $\square$

As a direct corollary of Theorem 3.1, we can immediately obtain the following result when the nonlinearity F has sublinear growth in the state variable.

Corollary 3.2. *Suppose that (H1) and (H2) are satisfied. Then the problem (1.5) and (1.6) has at least one solution in $C([0, 1], \mathbb{R})$, provided that*

(H4) *there exist functions $p(t), q(t), r(t), \theta(t) \in L([0, 1], \mathbb{R})$, and $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ such that*

$$\|F(t, u, v, w)\|_p \leq p(t)|u|^{\sigma_1} + q(t)|v|^{\sigma_2} + r(t)|w|^{\sigma_3} + \theta(t)\tag{3.16}$$

for each $(t, u, v, w) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Proof. We only need to verify that (H4) implies (H3) in Theorem 3.1. Taking $\varphi(t) = p(t)|u|^{\sigma_1} + q(t)|v|^{\sigma_2} + r(t)|w|^{\sigma_3} + \theta(t)$, it is easy to prove

$$\begin{aligned}\tau &= \liminf_{r \rightarrow \infty} \int_0^1 \frac{\max\{\varphi_r(s), \varphi_{\tilde{k}r}(s), \varphi_{\tilde{l}r}(s)\}}{r} ds = 0, \\ \mu_i &= \liminf_{r \rightarrow \infty} \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1} \max\{\varphi_r(s), \varphi_{\tilde{k}r}(s), \varphi_{\tilde{l}r}(s)\}}{r} ds = 0, \\ \omega &= \liminf_{r \rightarrow \infty} \int_0^1 \frac{(1-s)^{\alpha-1} \max\{\varphi_r(s), \varphi_{\tilde{k}r}(s), \varphi_{\tilde{l}r}(s)\}}{r} ds = 0,\end{aligned}\tag{3.17}$$

because of $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$. Then (H3) is satisfied. The proof is finished. $\square$

In the next part, we are concerned with the BVP (1.5) and (1.6) with nonconvex valued right-hand side. By using Covitz and Nadler fixed-point theorem, we obtain the following result.

Theorem 3.3. *Assume that the following hypotheses hold:*

- (A1) $F : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow P_{cl}(\mathbb{R})$; $t \rightarrow F(t, u, v, w)$ is measurable for each $u, v, w \in \mathbb{R}$;
- (A2) There exist three functions $p, q, r \in L^1([0, 1], \mathbb{R})$ such that for a.e. $t \in [0, 1]$ and $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$,

$$\begin{aligned}H_d(F(t, u_1, v_1, w_1), F(t, u_2, v_2, w_2)) &\leq p(t)|u_1 - u_2| + q(t)|v_1 - v_2| + r(t)|w_1 - w_2|, \\ d(0, F(t, 0, 0, 0)) &\leq m(t).\end{aligned}\tag{3.18}$$

Then the BVP (1.5) and (1.6) has at least one solution in $C([0, 1], \mathbb{R})$ provided that

$$\tilde{a}_0 + \tilde{b}_0 + \tilde{c}_0 + \sum_{i=1}^{m+1} \frac{\tilde{a}_i + \tilde{k}\tilde{b}_i + \tilde{l}\tilde{c}_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1})} < 1 - \lambda,\tag{3.19}$$

where

$$\begin{aligned}\tilde{a}_0 &= \frac{1}{\Gamma(\alpha)} \int_0^1 p(s) ds, & \tilde{a}_i &= \frac{1}{\Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} p(s) ds, & i &= 1, 2, \dots, m+1, \\ \tilde{b}_0 &= \frac{1}{\Gamma(\alpha)} \int_0^1 q(s) ds, & \tilde{b}_i &= \frac{1}{\Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} q(s) ds, & i &= 1, 2, \dots, m+1, \\ \tilde{c}_0 &= \frac{1}{\Gamma(\alpha)} \int_0^1 r(s) ds, & \tilde{c}_i &= \frac{1}{\Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} r(s) ds, & i &= 1, 2, \dots, m+1,\end{aligned}\tag{3.20}$$

$\eta_{m+1} = 1$ and λ defined as in Lemma 2.10, $\tilde{k}, \tilde{l}$ defined by (3.1) in Theorem 3.1.

Proof. Transform the problem into a fixed-point problem. Let $N : C([0, 1], \mathbb{R}) \rightarrow \mathcal{P}(C([0, 1], \mathbb{R}))$ be defined as in Theorem 3.1. We shall show that N satisfies the assumptions of Lemma 2.9. The proof will be given in two steps.

Step 1. $N(u) \in \mathcal{P}_{cl}(C([0, 1], \mathbb{R}))$ for each $u \in C([0, 1], \mathbb{R})$.

Indeed, let $(u_n)_{n \geq 0} \in N(u)$ such that $u_n \rightarrow \tilde{u}$ in $C([0, 1], \mathbb{R})$. Then there exists $g_n \in S_{F, u}$ such that for each $t \in [0, 1]$,

$$\begin{aligned} u_n(t) = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} t^{\alpha-1} u_n(s) ds \\ & - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} t^{\alpha-1} u_n(s) ds \\ & + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} u_n(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g_n(s) ds \\ & + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} g_n(s) ds \\ & - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} g_n(s) ds. \end{aligned} \quad (3.21)$$

For every $w \in F(t, u(t), (Tu)(t), (Su)(t))$, from (A2), we have

$$|w| \leq d(0, F(t, 0, 0, 0)) + H_d(F(t, 0, 0, 0), F(t, u(t), (Tu)(t), (Su)(t))). \quad (3.22)$$

Then,

$$|g_n(t)| \leq m(t) + p(t)|u(t)| + q(t)|(Tu)(t)| + r(t)|(Su)(t)| \quad \text{a.e. } t \in [0, 1], \quad (3.23)$$

that is, $g_n(t) \in B(0, m(t) + p(t)|u(t)| + q(t)|(Tu)(t)| + r(t)|(Su)(t)|)$, where

$$\begin{aligned} & B(0, m(t) + p(t)|u(t)| + q(t)|(Tu)(t)| + r(t)|(Su)(t)|) \\ & = \{w \in \mathbb{R} : |w| \leq m(t) + p(t)|u(t)| + q(t)|(Tu)(t)| + r(t)|(Su)(t)|\} := \Psi(t). \end{aligned} \quad (3.24)$$

It is clear that $\Psi : [0, 1] \rightarrow \mathcal{P}_{cp,cv}(\mathbb{R})$ is a multivalued integrable bounded map. Since $g_n(\cdot) \in \Psi(\cdot)$, $n \geq 1$, we may pass to a subsequence if necessary to get that g_n converges weakly to g in $L_w^1([0, 1], \mathbb{R})$. From Mauz's lemma [37], there exists $g \in \overline{\text{conv}}\{g_n(t) : n \geq 1\}$, then there

exists a subsequence $\{\bar{g}_n : n \geq 1\}$ in $\overline{\text{conv}}\{g_n : n \geq 1\}$, such that $\bar{g}_n$ converges strongly to g in $L^1([0, 1], \mathbb{R})$, which implies $g \in L^1([0, 1], \mathbb{R})$. Then for each $t \in [0, 1]$,

$$\begin{aligned}
u_n(t) \longrightarrow \tilde{u}(t) = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} t^{\alpha-1} \tilde{u}(s) ds \\
& - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} t^{\alpha-1} \tilde{u}(s) ds \\
& + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} \tilde{u}(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(s) ds \\
& + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} g(s) ds \\
& - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} g(s) ds.
\end{aligned} \tag{3.25}$$

So, $\tilde{u} \in N(u) \in \rho_{cl}(C([0, 1], \mathbb{R}))$.

Step 2. There exists $\gamma < 1$, such that

$$H_d(N(u), N(u_*)) \leq \gamma \|u - u_*\|_\infty \quad \text{for each } u, u_* \in C([0, 1], \mathbb{R}). \tag{3.26}$$

Let $u, u_* \in C([0, 1], \mathbb{R})$ and $h \in N(u)$. Then there exists $f(t) \in F(t, u(t), (Tu)(t), (Su)(t))$ such that for each $t \in [0, 1]$

$$\begin{aligned}
h(t) = & \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} t^{\alpha-1} h(s) ds \\
& - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} t^{\alpha-1} h(s) ds \\
& + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds \\
& + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} f(s) ds \\
& - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} f(s) ds.
\end{aligned} \tag{3.27}$$

It follows from (A2) that

$$\begin{aligned} & H_d(F(t, u(t), (Tu)(t), (Su)(t)), F(t, u_*(t), (Tu_*)(t), (Su_*)(t))) \\ & \leq p(t)|u(t) - u_*(t)| + q(t)|(Tu)(t) - (Tu_*)(t)| + r(t)|(Su)(t) - (Su_*)(t)|, \quad t \in [0, 1]. \end{aligned} \quad (3.28)$$

Hence, there is $g \in F(t, u_*(t), (Tu_*)(t), (Su_*)(t))$ such that

$$|f(t) - g| \leq p(t)|u(t) - u_*(t)| + q(t)|(Tu)(t) - (Tu_*)(t)| + r(t)|(Su)(t) - (Su_*)(t)|, \quad t \in [0, 1]. \quad (3.29)$$

Consider $G : [0, 1] \rightarrow \mathcal{P}(\mathbb{R})$, given by

$$\begin{aligned} & G(t) \\ & = \{g \in \mathbb{R} : |f(t) - g| \leq p(t)|u(t) - u_*(t)| + q(t)|(Tu)(t) - (Tu_*)(t)| + r(t)|(Su)(t) - (Su_*)(t)|\}. \end{aligned} \quad (3.30)$$

Since the multivalued operator $\tilde{G} : t \rightarrow G(t) \cap F(t, u_*(t), (Tu_*)(t), (Su_*)(t))$ is measurable (see in [38, Proposition III.4]), there exists a function g_* , which is a measurable selection for G . So, $g_*(t) \in F(t, u_*(t), (Tu_*)(t), (Su_*)(t))$ and

$$\begin{aligned} & |f(t) - g_*(t)| \\ & \leq p(t)|u(t) - u_*(t)| + q(t)|(Tu)(t) - (Tu_*)(t)| + r(t)|(Su)(t) - (Su_*)(t)|, \quad t \in [0, 1]. \end{aligned} \quad (3.31)$$

For each $t \in [0, 1]$, define

$$\begin{aligned} h_*(t) &= \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} t^{\alpha-1} h_*(s) ds \\ &\quad - \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} t^{\alpha-1} h_*(s) ds \\ &\quad + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} h_*(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g_*(s) ds \\ &\quad + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} g_*(s) ds \\ &\quad - \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} g_*(s) ds. \end{aligned} \quad (3.32)$$

Then, for $t \in [0, 1]$

$$\begin{aligned}
|h(t) - h_*(t)| &\leq \sum_{i=1}^m \sum_{j=1}^n \frac{\gamma_i b_j}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_j)} \int_0^{\eta_i} (\eta_i - s)^{\alpha - \beta_j - 1} t^{\alpha-1} |h(s) - h_*(s)| ds \\
&\quad + \sum_{i=1}^n \frac{b_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha - \beta_i)} \int_0^1 (1 - s)^{\alpha - \beta_i - 1} t^{\alpha-1} |h(s) - h_*(s)| ds \\
&\quad + \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha - \beta_i)} \int_0^t (t - s)^{\alpha - \beta_i - 1} |h(s) - h_*(s)| ds \\
&\quad + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} |f(s) - g_*(s)| ds \\
&\quad + \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} |f(s) - g_*(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |f(s) - g_*(s)| ds \\
&\leq \lambda \|h - h_*\| + \|u - u_*\| \\
&\quad \times \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} (p(s) + q(s)\tilde{k} + r(s)\tilde{l}) ds \right. \\
&\quad + \sum_{i=1}^m \frac{\gamma_i}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} t^{\alpha-1} (p(s) + q(s)\tilde{k} + r(s)\tilde{l}) ds \\
&\quad \left. + \frac{1}{(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1}) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} t^{\alpha-1} (p(s) + q(s)\tilde{k} + r(s)\tilde{l}) ds \right). \tag{3.33}
\end{aligned}$$

So,

$$\|h - h_*\| \leq \left(\frac{\tilde{a}_0 + \tilde{b}_0 + \tilde{c}_0}{1 - \lambda} + \sum_{i=1}^{m+1} \frac{\tilde{a}_i + \tilde{k}\tilde{b}_i + \tilde{l}\tilde{c}_i}{(1 - \lambda)(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1})} \right) \|u - u_*\|. \tag{3.34}$$

By an analogous relation, obtained by interchanging the roles of u, u_* , it follows that

$$H_d(N(u), N(u_*)) \leq C \|u - u_*\|. \tag{3.35}$$

where $C = (\tilde{a}_0 + \tilde{b}_0 + \tilde{c}_0) / (1 - \lambda) + \sum_{i=1}^{m+1} (\tilde{a}_i + \tilde{k}\tilde{b}_i + \tilde{l}\tilde{c}_i) / ((1 - \lambda)(1 - \sum_{i=1}^m \gamma_i \eta_i^{\alpha-1})) < 1$. Therefore, N is a contraction operator. With the help of Lemma 2.9, the problem (1.5) and (1.6) has at least one solution. $\square$

Remark 3.4. In (1.5) and (1.6), if $b_i = 0$, $\gamma_j = 0$, $i = 1, \dots, n$, $j = 1, \dots, m$, $k(t, s) = l(t, s) \equiv 0$. Then we obtain the following fractional boundary value problem of differential inclusions

$$\begin{aligned} -D^\alpha y(t) &\in F(t, y(t)), \quad \text{a.e. } t \in J = [0, 1], \quad 0 < \alpha \leq 2, \\ y(0) &= y(1) = 0 \end{aligned} \quad (3.36)$$

which has been studied in [26]. It is easy to see that our results (Theorems 3.1 and 3.3) partly generalize and improve the results obtained in it.

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