Research Article

Linearization of Two Second-Order Ordinary Differential Equations via Fiber Preserving Point Transformations

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The necessary form of a linearizable system of two second-order ordinary differential equations $y_1'' = f_1(x, y_1, y_2, y_1', y_2')$, $y_2'' = f_2(x, y_1, y_2, y_1', y_2')$ is obtained. Some other necessary conditions were also found. The main problem studied in the paper is to obtain criteria for a system to be equivalent to a linear system with constant coefficients under fiber preserving transformations. A linear system with constant coefficients is chosen because of its simplicity in finding the general solution. Examples demonstrating the procedure of using the linearization theorems are presented.

1. Introduction

Almost all physical applications of differential equations are based on nonlinear equations, which in general are very difficult to solve explicitly. Ordinary differential equations play a significant role in the theory of differential equations. In the 19th century, one of the most important problems in analysis was the problem of classification of ordinary differential equations [1–4].

One type of the classification problem is the equivalence problem. Two systems of differential equations are said to be equivalent if there exists an invertible transformation which transforms any solution of one system to a solution of the other system. The linearization problem is a particular case of the equivalence problem, where one of the systems is a linear system. It is one of essential parts in the studies of nonlinear equations.

1.1. Linearization Problem

The analysis of the linearization problem for a single ordinary differential equation was started by Lie [1]. He gave criteria for linearization of a second-order ordinary differential equation by an invertible change of the independent and dependent variables (point transformation). Cartan [4] developed another approach for solving the linearization problem of a second-order ordinary differential equation.

Later the linearization problem was also considered with respect to other types of transformations, for example, contact and generalized Sundman transformations. These transformations have been applied to third-order and fourth-order ordinary differential equations, see [5–16] and references therein. It is worth to note that fiber preserving transformations, where the change of variables only depends on the independent variable, have played a special role: either only such transformations were studied [6], or one needs to study them separately during compatibility analysis [10, 11, 13, 17].

The linearization problem for a system of second-order ordinary differential equations was studied in [18–24]. In [18], necessary and sufficient conditions for a system of $n \geq 2$ second-order ordinary differential equations to be equivalent to the free particle equations were given. Particular class of systems of two ($n = 2$) second-order ordinary differential equations were considered in [20]. In [19], criteria for linearization of a system of two second-order ordinary differential equations are related with the existence of an admitted four-dimensional Lie algebra. Some first-order and second-order relative invariants with respect to point transformations for a system of two ordinary differential equations were obtained in [21]. A new method of linearizing a system of equations is proposed in [22], where a given system of equations is reduced to a single equation to which the Lie theorem on linearization is applied.

1.2. Canonical Forms of a Linear System of Two Second-Order ODEs

The general form of a normal linear system of n second-order ordinary differential equations is

$$\ddot{v} + C\dot{v} + Dv + E = 0, \quad (1.1)$$

where $v = v(t)$ and $E = E(t)$ are vectors, $C = C(t)$ and $D = D(t)$ are $n \times n$ square matrices. It can be shown [19] that there exists a change $u = Uv$ such that system (1.1) is reduced to one of the following forms: either

$$\ddot{u} + K_1\dot{u} = 0 \quad (1.2)$$

or

$$\ddot{u} + Ku = 0. \quad (1.3)$$

Here $U(t)$, $K = K(t)$, and $K_1 = K_1(t)$ are $n \times n$ square matrices. Thus the problem of linearization via point transformations consists of solving the problem of reducibility of a system of second-order ordinary differential equations to one of these forms. In the present paper, the second canonical form (1.3) is used.

It is also worth to note that system (1.3) can be further simplified to one of three forms:

$$K = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad K = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad K = \begin{pmatrix} a & c \\ -c & a \end{pmatrix}, \quad (1.4)$$

where $a = a(t)$, $b = b(t)$, and $c = c(t) \neq 0$ are real valued (not complex) functions.

One of the main motivations for studying the linearization problem is the possibility of finding the general solution. Notice that even after finding the linearizing transformation one has to solve a linear system of second-order ordinary differential equations. The simplest case is where $K = 0$. More general and also not complicated is the case where the matrix K is constant. For example, for $n = 2$, this case leads to solving either a simple linear fourth-order ordinary differential equation with constant coefficients or two simple linear second-order ordinary differential equations. Indeed, for $n = 2$, system (1.3) is

$$\ddot{u}_1 + k_1 u_1 + k_3 u_2 = 0, \quad \ddot{u}_2 + k_4 u_1 + k_2 u_2 = 0, \quad (1.5)$$

where k_i , ($i = 1, 2, 3, 4$) are constant. If $k_3 \neq 0$, then finding u_2 from the first equation of (1.5) and substituting it into the second equation of (1.5), one obtains a fourth-order ordinary differential equation

$$u_1^{(4)} + (k_1 + k_2)\ddot{u}_1 + (k_1 k_2 - k_3 k_4)u_1 = 0. \quad (1.6)$$

Here,

$$K = \begin{pmatrix} k_1 & k_3 \\ k_4 & k_2 \end{pmatrix}. \quad (1.7)$$

The general solution of the last equation depends on the roots λ of the characteristic equation

$$\lambda^4 + (k_1 + k_2)\lambda^2 + (k_1 k_2 - k_3 k_4) = 0. \quad (1.8)$$

The solution is similar for $k_4 \neq 0$. On the other hand, if $k_3 = 0$ and $k_4 = 0$, then system (1.1) is decoupled:

$$\ddot{u}_1 + k_1 u_1 = 0, \quad \ddot{u}_2 + k_2 u_2 = 0. \quad (1.9)$$

Notice also that if in this case $k_1 = k_2$, then the last system of equations is equivalent to the system of two trivial equations $z'' = 0$.

1.3. The Problem Solved in the Present Paper

The present paper deals with a system of two second-order ordinary differential equations

$$y_1'' = F_1(x, y_1, y_2, y_1', y_2'), \quad y_2'' = F_2(x, y_1, y_2, y_1', y_2'). \quad (1.10)$$

In the next section, the form of a linearizable system (1.10) is obtained. This form coincides with that obtained in [18]. Some invariants of this form with respect to the general set of point transformations

$$t = \varphi(x, y_1, y_2), \quad u_1 = \varphi_1(x, y_1, y_2), \quad u_2 = \varphi_2(x, y_1, y_2) \quad (1.11)$$

related with a linearizable systems (1.10) were obtained in [21]. The main result of the paper consists of necessary and sufficient conditions for system (1.10) to be equivalent with respect to a fiber preserving point transformation

$$t = \varphi(x), \quad u_1 = \varphi_1(x, y_1, y_2), \quad u_2 = \varphi_2(x, y_1, y_2) \quad (1.12)$$

to system (1.5).

The paper is organized as follows. In Section 2, the necessary form of a linearizable system of two second-order ordinary differential equations is presented. In Section 3, the main results of the paper are given: necessary and sufficient criteria for a system of two second-order ordinary differential equations to be equivalent via fiber preserving point transformations (1.12) to a system of the form (1.5). During the study presented in Section 3, we also obtained some necessary conditions for linearizability for the general case of point transformations (1.11) and for the general case of the matrix $K(t)$. These conditions are shown in Section 4. Examples demonstrating the procedure of using the linearization theorems are presented in Section 5.

2. Necessary Form of a Linearizable System (1.10)

For obtaining necessary conditions for system (1.10) to be linearizable via point transformations (1.11), one assumes that system (1.10) is obtained from the linear system of differential equations (1.3) by an invertible transformation (1.11). The derivatives are changed by the formulae

$$\begin{aligned} u'_1 &= g_1(x, y_1, y_2, y'_1, y'_2) = \frac{D_x \varphi_1}{D_x \varphi}, & u''_1 &= \frac{D_x g_1}{D_x \varphi}, \\ u'_2 &= g_2(x, y_1, y_2, y'_1, y'_2) = \frac{D_x \varphi_2}{D_x \varphi}, & u''_2 &= \frac{D_x g_2}{D_x \varphi}, \end{aligned} \quad (2.1)$$

where

$$D_x = \frac{\partial}{\partial x} + y'_1 \frac{\partial}{\partial y_1} + y'_2 \frac{\partial}{\partial y_2} + y''_1 \frac{\partial}{\partial y'_1} + y''_2 \frac{\partial}{\partial y'_2}. \quad (2.2)$$

Replacing u'_1 , u''_1 , u'_2 , and u''_2 in system (1.3), it becomes

$$\begin{aligned} y''_1 &= y'_1 \left(a_{11} y_1'^2 + a_{12} y_1' y_2' + a_{13} y_2'^2 \right) + a_{14} y_1'^2 + a_{15} y_1' y_2' + a_{16} y_2'^2 + a_{17} y_1' + a_{18} y_2' + a_{19}, \\ y''_2 &= y'_2 \left(a_{11} y_1'^2 + a_{12} y_1' y_2' + a_{13} y_2'^2 \right) + a_{24} y_1'^2 + a_{25} y_1' y_2' + a_{26} y_2'^2 + a_{27} y_1' + a_{28} y_2' + a_{29}, \end{aligned} \quad (2.3)$$

where the coefficients a_{ij} are expressed through the functions φ , φ_1 , φ_2 , their partial derivatives, and the entries of the matrix $K = (k_{ij}(t))$ as follows:

$$\begin{aligned}
a_{11} &= \frac{(h_1\varphi_{1y_1y_1} + h_2\varphi_{2y_1y_1} - v\varphi_{y_1y_1} + f_1\varphi_{y_1}^3 - f_2\varphi_{y_1}^2\varphi_{y_2})}{\Delta}, \\
a_{12} &= \frac{2(h_1\varphi_{1y_1y_2} + h_2\varphi_{2y_1y_2} - v\varphi_{y_1y_2} + f_1\varphi_{y_1}^2\varphi_{y_2} - f_2\varphi_{y_1}\varphi_{y_2}^2)}{\Delta}, \\
a_{13} &= \frac{(h_1\varphi_{1y_2y_2} + h_2\varphi_{2y_2y_2} - v\varphi_{y_2y_2} + f_1\varphi_{y_1}\varphi_{y_2}^2 - f_2\varphi_{y_2}^3)}{\Delta}, \\
a_{14} &= \frac{(2h_1\varphi_{1xy_1} + 2h_2\varphi_{2xy_1} + h_3\varphi_{1y_1y_1} + h_4\varphi_{2y_1y_1} + h_5\varphi_{y_1y_1} - 2v\varphi_{xy_1} + 3f_1\varphi_x\varphi_{y_1}^2 - 2f_2\varphi_x\varphi_{y_1}\varphi_{y_2} - \mathfrak{J})}{\Delta}, \\
a_{15} &= \frac{2(h_1\varphi_{1xy_2} + h_2\varphi_{2xy_2} + h_3\varphi_{1y_1y_2} + h_4\varphi_{2y_1y_2} + h_5\varphi_{y_1y_2} - 2v\varphi_{xy_2} + 2f_1\varphi_x\varphi_{y_1}\varphi_{y_2} - f_2\varphi_x\varphi_{y_2}^2 - \mathfrak{R})}{\Delta}, \\
a_{16} &= \frac{(h_3\varphi_{1y_2y_2} + h_4\varphi_{2y_2y_2} + h_5\varphi_{y_2y_2} + f_1\varphi_x\varphi_{y_2}^2 - f_3\varphi_{y_2}^3)}{\Delta}, \\
a_{17} &= \frac{(h_1\varphi_{1xx} + h_2\varphi_{2xx} + 2h_3\varphi_{1xy_1} + 2h_4\varphi_{2xy_1} + 2h_5\varphi_{xy_1} - v\varphi_{xx} + 3f_1\varphi_x^2\varphi_{y_1} - f_2\varphi_x^2\varphi_{y_2} - 2f_3\varphi_x\varphi_{y_1})}{\Delta}, \\
a_{18} &= \frac{2(h_3\varphi_{1xy_2} + h_4\varphi_{2xy_2} + 2h_5\varphi_{xy_2} + f_1\varphi_x^2\varphi_{y_2} - f_3\varphi_x\varphi_{y_2}^2)}{\Delta}, \\
a_{19} &= \frac{(h_3\varphi_{1xx} + h_4\varphi_{2xx} + h_5\varphi_{xx} + f_1\varphi_x^3 - f_3\varphi_x^2\varphi_{y_2})}{\Delta}, \\
a_{24} &= \frac{(h_6\varphi_{1y_1y_1} + h_7\varphi_{2y_1y_1} + h_8\varphi_{y_1y_1} - f_2\varphi_x\varphi_{y_1}^2 + f_3\varphi_{y_1}^3)}{\Delta}, \\
a_{25} &= \frac{2(h_1\varphi_{1xy_1} + h_2\varphi_{2xy_1} - v\varphi_{xy_1} + h_6\varphi_{1y_1y_2} + h_7\varphi_{2y_1y_2} + f_1\varphi_x\varphi_x\varphi_{y_1}^2 - 2f_2\varphi_x\varphi_{y_1}\varphi_{y_2} - f_3\varphi_{y_1}^2\varphi_{y_2})}{\Delta}, \\
a_{26} &= \frac{(2h_1\varphi_{1xy_2} + 2h_2\varphi_{2xy_2} - 2v\varphi_{xy_2} + h_6\varphi_{1y_2y_2} + h_7\varphi_{2y_2y_2} + h_8\varphi_{y_2y_2} + 2f_1\varphi_x\varphi_{y_1}\varphi_{y_2} - 3f_2\varphi_x\varphi_{y_2}^2 - \mathfrak{R})}{\Delta}, \\
a_{27} &= \frac{2(h_6\varphi_{1xy_1} + h_7\varphi_{2xy_1} + h_8\varphi_{xy_1} - f_2\varphi_x^2\varphi_{y_1} + f_3\varphi_x\varphi_{y_1}^2)}{\Delta}, \\
a_{28} &= \frac{(h_1\varphi_{1xx} + h_2\varphi_{2xx} - v\varphi_{xx} + 2h_6\varphi_{1xy_2} + 2h_7\varphi_{2xy_2} + 2h_8\varphi_{xy_2} + f_1\varphi_x^2\varphi_{y_1} - 3f_2\varphi_x^2\varphi_{y_2} + \mathfrak{L})}{\Delta}, \\
a_{29} &= \frac{(h_6\varphi_{1xx} + h_7\varphi_{2xx} + h_8\varphi_{xx} - f_2\varphi_x^3 + f_3\varphi_x^2\varphi_{y_1})}{\Delta},
\end{aligned} \tag{2.4}$$

where $\mathfrak{J}$ denotes $f_3\varphi_{y_1}^2\varphi_{y_2}$, $\mathfrak{R}$ denotes $f_3\varphi_{y_1}\varphi_{y_2}^2$, $\mathfrak{U}$ denotes $2f_3\varphi_x\varphi_{y_1}\varphi_{y_2}$, and $\Delta \neq 0$ is the Jacobian of the change of variables (1.11)

$$\begin{aligned}\Delta &= \left(\varphi_x \varphi_{1y_1} \varphi_{2y_2} - \varphi_x \varphi_{1y_2} \varphi_{2y_1} - \varphi_{y_1} \varphi_{1x} \varphi_{2y_2} + \varphi_{y_1} \varphi_{1y_2} \varphi_{2x} + \varphi_{y_2} \varphi_{1x} \varphi_{2y_1} - \varphi_{y_2} \varphi_{1y_1} \varphi_{2x} \right), \\ f_1 &= \varphi_{1y_2} (k_{22}\varphi_2 + k_{21}\varphi_1) - \varphi_{2y_2} (k_{11}\varphi_1 + k_{12}\varphi_2), \\ f_2 &= \varphi_{1y_1} (k_{22}\varphi_2 + k_{21}\varphi_1) - \varphi_{2y_1} (k_{11}\varphi_1 + k_{12}\varphi_2), \\ f_3 &= \varphi_{1x} (k_{22}\varphi_2 + k_{21}\varphi_1) - \varphi_{2x} (k_{11}\varphi_1 + k_{12}\varphi_2), \\ v &= \varphi_{1y_2} \varphi_{2y_1} - \varphi_{1y_1} \varphi_{2y_2}, \quad h_1 = \varphi_{y_2} \varphi_{2y_1} - \varphi_{y_1} \varphi_{2y_2}, \quad h_2 = \varphi_{y_1} \varphi_{1y_2} - \varphi_{y_2} \varphi_{1y_1}, \\ h_3 &= \varphi_{y_2} \varphi_{2x} - \varphi_x \varphi_{2y_2}, \quad h_4 = \varphi_x \varphi_{1y_2} - \varphi_{y_2} \varphi_{1x}, \quad h_5 = \varphi_{1x} \varphi_{2y_2} - \varphi_{1y_2} \varphi_{2x}, \\ h_6 &= \varphi_x \varphi_{2y_1} - \varphi_{y_1} \varphi_{2x}, \quad h_7 = \varphi_{y_1} \varphi_{1x} - \varphi_x \varphi_{1y_1}, \quad h_8 = \varphi_{1y_1} \varphi_{2x} - \varphi_{1x} \varphi_{2y_1}.\end{aligned}\tag{2.5}$$

Equation (2.3) presents the necessary form of a system of two second-order ordinary differential equations which can be mapped into a linear equation (1.3) via point transformations.

3. Sufficient Conditions for Equivalency to (1.5) via Fiber Preserving Transformations

For obtaining sufficient conditions of linearizability of system (2.3), one has to solve the compatibility problem of the system of (2.4), considering it as an overdetermined system of partial differential equations for the functions φ_1 , φ_2 , and φ with given coefficients a_{ij} of system (2.3). The compatibility analysis depends on the value of φ_{1y_1} .

The next part of the present paper deals with a fiber preserving set of point transformations (1.12):

$$t = \varphi(x), \quad u_1 = \varphi_1(x, y_1, y_2), \quad u_2 = \varphi_2(x, y_1, y_2),\tag{3.1}$$

and constant matrix

$$K = \begin{pmatrix} k_1 & k_3 \\ k_4 & k_2 \end{pmatrix}.\tag{3.2}$$

3.1. Case $\varphi_{1y_1} \neq 0$

Substitution of $\varphi_{y_1} = 0$ and $\varphi_{y_2} = 0$ into (2.4) gives

$$a_{11} = 0, \quad a_{12} = 0, \quad a_{13} = 0,\tag{3.3}$$

$$v_x = \frac{(2\varphi_{xx}v - \varphi_x v(a_{17} + a_{28}))}{(2\varphi_x)},$$

$$v_{y_1} = -\frac{v(2a_{14} + a_{25})}{2},$$

$$v_{y_2} = -\frac{v(a_{15} + 2a_{26})}{2},$$

$$\psi_{1xx} = \frac{(\varphi_{xx}\psi_{1x} - \varphi_x^3(k_1\psi_1 + k_3\psi_2) - \varphi_x\psi_{1y_1}a_{19} - \varphi_x\psi_{1y_2}a_{29})}{\varphi_x},$$

$$\psi_{1xy_1} = \frac{(\varphi_{xx}\psi_{1y_1} - \varphi_x\psi_{1y_1}a_{17} - \varphi_x\psi_{1y_2}a_{27})}{(2\varphi_x)},$$

$$\psi_{1xy_2} = \frac{(\varphi_{xx}\psi_{1y_2} - \varphi_x\psi_{1y_1}a_{18} - \varphi_x\psi_{1y_2}a_{28})}{(2\varphi_x)},$$

$$\psi_{1y_1y_1} = -(\psi_{1y_1}a_{14} + \psi_{1y_2}a_{24}),$$

$$\psi_{1y_1y_2} = -\frac{(\psi_{1y_1}a_{15} + \psi_{1y_2}a_{25})}{2},$$

$$\psi_{1y_2y_2} = -(\psi_{1y_1}a_{16} + \psi_{1y_2}a_{26}),$$

$$\psi_{2y_2} = \frac{(\psi_{1y_2}\psi_{2y_1} - v)}{\psi_{1y_1}},$$

$$\psi_{2y_1y_1} = \frac{(a_{24}v - \psi_{1y_1}\psi_{2y_1}a_{14} - \psi_{1y_2}\psi_{2y_1}a_{24})}{\psi_{1y_1}},$$

$$\psi_{2xy_1} = \frac{(\varphi_{xx}\psi_{1y_1}\psi_{2y_1} - \varphi_x\psi_{1y_1}\psi_{2y_1}a_{17} - \varphi_x\psi_{1y_2}\psi_{2y_1}a_{27} + \varphi_xa_{27}v)}{(2\varphi_x\psi_{1y_1})},$$

$$\psi_{2xx} = \frac{(\varphi_{xx}\psi_{1y_1}\psi_{2x} - \varphi_x^3\psi_{1y_1}(k_2\psi_2 + k_4\psi_1) - \varphi_x\psi_{1y_1}\psi_{2y_1}a_{19} - \varphi_x\psi_{1y_2}\psi_{2y_1}a_{29} + \varphi_xa_{29}v)}{(\varphi_x\psi_{1y_1})},$$

where $\Delta = -\varphi_x v \neq 0$. Comparing the mixed derivatives $(v_x)_{y_1} = (v_{y_1})_x$, $(v_x)_{y_2} = (v_{y_2})_x$ and $(v_{y_2})_{y_1} = (v_{y_1})_{y_2}$, one obtains

$$2a_{14x} - a_{17y_1} + a_{25x} - a_{28y_1} = 0, \quad a_{15x} - a_{17y_2} + 2a_{26x} - a_{28y_2} = 0,$$

$$a_{15y_1} - 2a_{14y_2} - a_{25y_2} + 2a_{26y_1} = 0.$$

Considering the conditions $(\psi_{i_{xx}})_{y_j} = (\psi_{i_{xy_j}})_x$ ($i, j = 1, 2$), one has

$$\varphi_{xxx} = \frac{(3\varphi_{xx}^2\psi_{1y_1} - 4\varphi_x^4\psi_{1y_1}k_1 - 4\varphi_x^4\psi_{2y_1}k_3 + \varphi_x^2\psi_{1y_1}(\lambda_{20} - \lambda_{16}) + \varphi_x^2\psi_{1y_2}\lambda_{12})}{(2\varphi_x\psi_{1y_1})}, \quad (3.16)$$

$$k_4 = \frac{(4\varphi_x^2\psi_{1y_1}\psi_{2y_1}(k_1 - k_2) + 4\varphi_x^2\psi_{2y_1}^2k_3 - v\lambda_{12})}{(4\varphi_x^2\psi_{1y_1}^2)}, \quad (3.17)$$

$$k_3 = \frac{(\psi_{1y_2}^2\lambda_{12} - \psi_{1y_1}^2\lambda_{15} - \psi_{1y_1}\psi_{1y_2}\lambda_{16})}{(4\varphi_x^2v)},$$

$$k_2 = \frac{(4\varphi_x^2\psi_{1y_1}k_1v - 2\psi_{1y_1}^2\psi_{2y_1}\lambda_{15} - 2\psi_{1y_1}\psi_{1y_2}\psi_{2y_1}\lambda_{16} + \psi_{1y_1}v\lambda_{16} + 2\psi_{1y_2}^2\psi_{2y_1}\lambda_{12} - 2\psi_{1y_2}v\lambda_{12})}{(4\varphi_x^2\psi_{1y_1}v)}. \quad (3.18)$$

Here, the functions $\lambda_n(x, y_1, y_2)$ are defined through $a_{ij}(x, y_1, y_2)$ and their derivatives (presented in the appendix).

Equating the mixed derivatives $(\psi_{i_{xy_1}})_{y_2} = (\psi_{i_{xy_2}})_{y_1}$, $(\psi_{i_{xy_1}})_{y_1} = (\psi_{i_{y_1y_1}})_x$, $(\psi_{1_{xy_2}})_{y_i} = (\psi_{1_{y_iy_2}})_x$, $(\psi_{1_{y_1y_2}})_{y_2} = (\psi_{1_{y_1y_2}})_{y_1}$, $(\psi_{2_{y_1y_1}})_{y_2} = (\psi_{2_{y_2}})_{y_1y_1}$ and using the conditions $(k_{i+1})_{y_2} = 0$, ($i = 1, 2$), one obtains

$$\lambda_n = 0 \quad (n = 1, 2, \dots, 11). \quad (3.19)$$

Note that $(\psi_{1_{xy_2}})_{y_1} - (\psi_{1_{y_1y_2}})_x = 0$ is satisfied. Differentiating (3.16) and (3.17) with respect to y_1 and y_2 , one has

$$2\lambda_{15y_1} - 2a_{14}\lambda_{15} - a_{15}\lambda_{16} + a_{25}\lambda_{15} = 0, \quad \lambda_{27+j} = 0 \quad (j = 0, 1, 2). \quad (3.20)$$

Equations $(k_i)_x = 0$ ($i = 2, 3, 4$), become

$$4\varphi_{xx}\lambda_{12} + \varphi_x\lambda_{14} = 0, \quad (3.21)$$

$$2\varphi_{xx}\lambda_{16} - \varphi_x(\lambda_{16x} + a_{18}\lambda_{12} - a_{27}\lambda_{15}) = 0, \quad (3.22)$$

$$4\varphi_{xx}\lambda_{15} + \varphi_x(a_{17}\lambda_{15} - 2\lambda_{15x} + a_{18}\lambda_{16} - a_{28}\lambda_{15}) = 0. \quad (3.23)$$

Further analysis of the compatibility depends on the values of the coefficients λ_{12} , λ_{16} and λ_{15} of the last three equations (3.21)–(3.23).

3.1.1. Case $\lambda_{12} \neq 0$

Substituting φ_{xx} , found from (3.21), into (3.23) and (3.22), one obtains

$$\begin{aligned} a_{17}\lambda_{12}\lambda_{15} - 2\lambda_{15x}\lambda_{12} + a_{18}\lambda_{12}\lambda_{16} - a_{28}\lambda_{12}\lambda_{15} - \lambda_{14}\lambda_{15} &= 0, \\ 2\lambda_{16x}\lambda_{12} + 2a_{18}\lambda_{12}^2 - 2a_{27}\lambda_{12}\lambda_{15} + \lambda_{14}\lambda_{16} &= 0. \end{aligned} \quad (3.24)$$

Differentiating (3.21) with respect to y_1 and y_2 , one has

$$\lambda_{12y_1}\lambda_{14} - \lambda_{14y_1}\lambda_{12} = 0, \quad \lambda_{12y_2}\lambda_{14} - \lambda_{14y_2}\lambda_{12} = 0. \quad (3.25)$$

Equation (3.16) becomes

$$\begin{aligned} k_1 = & \left(16\varphi_{1y_1}^2\varphi_{2y_1}\lambda_{12}^2\lambda_{15} + 16\varphi_{1y_1}\varphi_{1y_2}\varphi_{2y_1}\lambda_{12}^2\lambda_{16} \right. \\ & + 16\varphi_{1y_1}\lambda_{12}^2v(\lambda_{20} - \lambda_{16}) + 4\varphi_{1y_1}\lambda_{12}v(2\lambda_{14x} + a_{17}\lambda_{14} - a_{28}\lambda_{14}) \\ & \left. + \varphi_{1y_1}\lambda_{14}v(4a_{27}\lambda_{16} + 5\lambda_{14}) - 16\varphi_{1y_2}^2\varphi_{2y_1}\lambda_{12}^3 + 16\varphi_{1y_2}\lambda_{12}^3v \right) / \left(64\varphi_x^2\varphi_{1y_1}\lambda_{12}^2v \right). \end{aligned} \quad (3.26)$$

Differentiating (3.26) with respect to x , one gets the condition

$$32\lambda_{12}^3\lambda_{17} + 8\lambda_{12}^2\lambda_{18} + 2\lambda_{12}\lambda_{19} + \lambda_{14}(8a_{27}^2\lambda_{16}^2 + 18a_{27}\lambda_{14}\lambda_{16} + 15\lambda_{14}^2) = 0. \quad (3.27)$$

Notice that $(k_1)_{y_1} = 0$ and $(k_1)_{y_2} = 0$ are satisfied. Hence, there are no new conditions for the functions $\varphi(x)$, $\varphi_1(x, y_1, y_2)$, and $\varphi_2(x, y_1, y_2)$. In summary, the criteria for linearization are conditions (3.3), (3.15), (3.19), (3.20), (3.24), (3.25), and (3.27).

3.1.2. Case $\lambda_{12} = 0$ and $\lambda_{16} \neq 0$

Since $\lambda_{12} = 0$ and $\varphi_x \neq 0$, (3.21) leads to the condition $\lambda_{14} = 0$, and (3.22) becomes

$$\varphi_{xx} = \frac{\varphi_x\lambda_{16x}}{(2\lambda_{16})}. \quad (3.28)$$

Substituting φ_{xx} into (3.23) and (3.16), one gets

$$2(\lambda_{16x}\lambda_{15} - \lambda_{15x}\lambda_{16}) + \lambda_{15}\lambda_{16}(a_{17} - a_{28}) + a_{18}\lambda_{16}^2 = 0, \quad (3.29)$$

$$k_1 = \frac{\left(4\varphi_{1y_1}\varphi_{2y_1}\lambda_{16}^2\lambda_{15} + 4\varphi_{1y_2}\varphi_{2y_1}\lambda_{16}^3 - 4\lambda_{16}^3v + 4\lambda_{16}^2\lambda_{23}v - 4\lambda_{16}\lambda_{16xx}v + 5\lambda_{16x}^2v \right)}{\left(16\varphi_x^2\lambda_{16}^2v \right)}. \quad (3.30)$$

Equation $(k_1)_x = 0$ leads to the condition

$$8\lambda_{16}^3\lambda_{21} + 4\lambda_{16}^2\lambda_{22} + 18\lambda_{16}\lambda_{16xx}\lambda_{16x} - 15\lambda_{16x}^3 = 0. \quad (3.31)$$

Note that $(\varphi_{xx})_{y_i} = 0$ and $(k_1)_{y_i} = 0$, $(i = 1, 2)$ are satisfied. Hence, there are no other conditions for the functions $\varphi(x)$, $\varphi_1(x, y_1, y_2)$, and $\varphi_2(x, y_1, y_2)$. Summarizing, the linearization criteria in the case $\lambda_{12} = 0$ and $\lambda_{16} \neq 0$ are conditions (3.3), (3.15), (3.19), (3.20), (3.29), and (3.31).

3.1.3. Case $\lambda_{12} = 0$, $\lambda_{16} = 0$, and $\lambda_{15} \neq 0$

Substituting φ_{xx} , found from (3.23), into (3.16), one has

$$k_1 = \frac{(16\varphi_{1y_1}\varphi_{2y_1}\lambda_{15}^3 + \lambda_{26}v)}{(64\varphi_x^2\lambda_{15}^2v)}. \quad (3.32)$$

Differentiating (3.32) with respect to x , one gets

$$\lambda_{15}^3\lambda_{24} + 2\lambda_{15}^2\lambda_{25} - 120\lambda_{15x}^3 + 36\lambda_{15}\lambda_{15x}(4\lambda_{15xx} + \lambda_{15x}a_{17} - \lambda_{15x}a_{28}) = 0. \quad (3.33)$$

Note that the equations $(\varphi_{xx})_{y_i} = 0$ and $(k_1)_{y_i} = 0$, $(i = 1, 2)$ are satisfied. Hence, there are no more conditions for the compatibility, and the linearization criteria in the studied case are (3.3), (3.15), (3.19), (3.20) and (3.33).

Remark 3.1. In the case $\lambda_{12} = 0$, $\lambda_{16} = 0$ and $\lambda_{15} = 0$, one has $k_2 = k_1$ and $k_3 = k_4 = 0$. This case corresponds to (1.9).

Combining all derived results in the case $\varphi_{1y_1} \neq 0$, the following theorem is proven.

Theorem 3.2. *Necessary and sufficient conditions for system (2.3) to be equivalent to a linear system (1.3) with constant matrix K via fiber preserving transformations are as follows.*

(I) *The conditions are (3.3), (3.15), (3.19), and (3.20), and the additional conditions are as*

(I.1) *If $\lambda_{12} \neq 0$, then the additional conditions are (3.24), (3.25), and (3.27).*

(I.2) *If $\lambda_{12} = 0$ and $\lambda_{16} \neq 0$, then the additional conditions are (3.29) and (3.31).*

(I.3) *If $\lambda_{12} = 0$, $\lambda_{16} = 0$, and $\lambda_{15} \neq 0$, then the additional condition is (3.33).*

(I.4) *If $\lambda_{12} = 0$, $\lambda_{16} = 0$, and $\lambda_{15} = 0$, then there are no additional conditions.*

3.2. Case $\psi_{1y_1} = 0$

If $\psi_{2y_2} \neq 0$, then the change $\bar{x} = x$, $\bar{y}_1 = y_2$, $\bar{y}_2 = y_1$ leads to the previous case. Hence, without loss of generality one can assume that $\psi_{2y_2} = 0$. Equations (2.4) become

$$a_{11} = 0, \quad a_{12} = 0, \quad a_{13} = 0, \quad a_{15} = 0, \quad a_{16} = 0, \quad a_{18} = 0, \quad a_{24} = 0, \quad a_{25} = 0, \quad a_{27} = 0, \quad (3.34)$$

$$\begin{aligned} \psi_{2y_1y_1} &= -\psi_{2y_1} a_{14}, & \psi_{2xy_1} &= \frac{(\varphi_{xx}\psi_{2y_1} - \varphi_x\psi_{2y_1}a_{17})}{(2\varphi_x)}, \\ \psi_{1y_2y_2} &= -\psi_{1y_2} a_{26}, & \psi_{1xy_2} &= \frac{(\varphi_{xx}\psi_{1y_2} - \varphi_x\psi_{1y_2}a_{28})}{(2\varphi_x)}, \\ \psi_{1xx} &= \frac{(\varphi_{xx}\psi_{1x} - \varphi_x^3(k_1\psi_1 + k_3\psi_2) - \varphi_x\psi_{1y_2}a_{29})}{\varphi_x}, \\ \psi_{2xx} &= \frac{(\varphi_{xx}\psi_{2x} - \varphi_x^3(k_2\psi_2 + k_4\psi_1) - \varphi_x\psi_{2y_1}a_{19})}{\varphi_x}, \end{aligned} \quad (3.35)$$

and $\Delta = -\varphi_x\psi_{1y_2}\psi_{2y_1} \neq 0$.

Comparing all mixed derivatives of third-order of the functions ψ_j , $j = 1, 2$, one finds

$$a_{28y_1} = 0, \quad a_{28y_2} = 2a_{26x}, \quad a_{26y_1} = 0, \quad a_{17y_2} = 0, \quad a_{17y_1} = 2a_{14x}, \quad a_{14y_2} = 0, \quad (3.36)$$

$$\begin{aligned} k_1 &= \frac{(3\varphi_{xx}^2 - 2\varphi_{xxx}\varphi_x + \varphi_x^2\mu_1)}{(4\varphi_x^4)}, & k_2 &= \frac{(3\varphi_{xx}^2 - 2\varphi_{xxx}\varphi_x - \varphi_x^2\mu_2)}{(4\varphi_x^4)}, \\ k_3 &= \frac{(-a_{29y_1}\psi_{1y_2})}{(\varphi_x^2\psi_{2y_1})}, & k_4 &= \frac{(-a_{19y_2}\psi_{2y_1})}{(\varphi_x^2\psi_{1y_2})}, \end{aligned} \quad (3.37)$$

where the coefficients μ_n are defined through a_{ij} and their derivatives (presented in the appendix).

Equations $(k_i)_x = 0$, $(k_i)_{y_j} = 0$ ($i = 1, 2, 3, 4$, $j = 1, 2$), become

$$\varphi_{xxxx} = \frac{(12\varphi_{xxx}\varphi_{xx}\varphi_x - 12\varphi_{xx}^3 + 2\varphi_{xx}\varphi_x^2\mu_2 - \varphi_x^3\mu_{2x})}{(2\varphi_x^2)}, \quad (3.38)$$

$$(\varphi_{xx} - \varphi_x\mu_3)a_{29y_1} = 0, \quad (\varphi_{xx} - \varphi_x\mu_4)a_{19y_2} = 0, \quad 2\varphi_{xx}\mu_5 - \varphi_x\mu_{5x} = 0, \quad (3.39)$$

$$\begin{aligned} a_{19y_2}a_{14} - a_{19y_1y_2} &= 0, & a_{29y_1}a_{26} - a_{29y_1y_2} &= 0, \\ a_{29y_1y_1} + a_{29y_1}a_{14} &= 0, & a_{19y_2y_2} + a_{19y_2}a_{26} &= 0, \\ a_{26xx} - a_{26x}a_{28} + a_{26y_2}a_{29} - a_{29y_2y_2} + a_{29y_2}a_{26} &= 0, \\ a_{14xx} - a_{14x}a_{17} + a_{14y_1}a_{19} - a_{19y_1y_1} + a_{19y_1}a_{14} &= 0. \end{aligned} \quad (3.40)$$

Notice that $(\varphi_{xxxx})_{y_j} = 0 (j = 1, 2)$ are satisfied. Equation (3.39) generates a further analysis of the compatibility depending on the values of a_{29y_1} , a_{19y_2} , and μ_5 .

3.2.1. Case $a_{29y_1} \neq 0$

From the first equation of (3.39), one obtains that

$$\varphi_{xx} = \varphi_x \mu_3. \quad (3.41)$$

Substituting φ_{xx} into the second and third equations of (3.39), one obtains

$$2\mu_5\mu_3 - \mu_{5x} = 0, \quad a_{19y_2}(\mu_3 - \mu_4) = 0. \quad (3.42)$$

Substitution of φ_{xx} into (3.38) gives

$$2\mu_{3xx} + \mu_{2x} - 6\mu_{3x}\mu_3 - 2\mu_2\mu_3 + 2\mu_3^3 = 0. \quad (3.43)$$

Note that $(\varphi_{xx})_{y_1} = 0$ and $(\varphi_{xx})_{y_2} = 0$ are satisfied. Hence, there are no new conditions. In summary, the linearization criteria are (3.34), (3.36), (3.40), (3.42), and (3.43).

3.2.2. Case $a_{29y_1} = 0$ and $a_{19y_2} \neq 0$

From the second equation of (3.39), one obtains that

$$\varphi_{xx} = \varphi_x \mu_4. \quad (3.44)$$

Substituting φ_{xx} into the third equation of (3.39), one obtains

$$\mu_{5x} - 2\mu_5\mu_4 = 0. \quad (3.45)$$

Substitution of φ_{xx} into (3.38) gives

$$6\mu_{4x}\mu_4 - \mu_{2x} - 2\mu_{4xx} + 2\mu_2\mu_4 - 2\mu_4^3 = 0. \quad (3.46)$$

Note that $(\varphi_{xx})_{y_1} = 0$ and $(\varphi_{xx})_{y_2} = 0$ are satisfied. Hence, there are no other conditions. Thus, the linearization criteria in this case are (3.34), (3.36), (3.40), (3.45), and (3.46).

3.2.3. Case $a_{29y_1} = 0$, $a_{19y_2} = 0$, and $\mu_5 \neq 0$

From the third equation of (3.39), one obtains

$$\varphi_{xx} = \frac{(\varphi_x \mu_{5x})}{(2\mu_5)}. \quad (3.47)$$

Substitution of φ_{xx} into (3.38) leads to the condition

$$\mu_5^2(4\mu_{5x}\mu_2 - 4\mu_{5xx}) + 18\mu_{5xx}\mu_{5x}\mu_5 - 15\mu_{5x}^3 - 4\mu_{2x}\mu_5^3 = 0. \quad (3.48)$$

Note that $(\varphi_{xx})_{y_1} = 0$ and $(\varphi_{xx})_{y_2} = 0$ are satisfied. Hence, there are no more conditions. In brief, the linearization criteria are conditions (3.34), (3.36), (3.40), and (3.48). Notice also that

$$k_2 = k_1 + \frac{\mu_5}{(4\varphi_x^2)}, \quad k_3 = 0, \quad k_4 = 0. \quad (3.49)$$

Remark 3.3. In the case $a_{29y_1} = 0$, $a_{19y_2} = 0$, $\mu_5 = 0$, one has $k_2 = k_1$ and $k_3 = k_4 = 0$. This case corresponds to (1.9).

Combining all obtained results in the case $\varphi_{1y_1} = 0$ and $\varphi_{2y_2} = 0$, the following theorem is proven.

Theorem 3.4. *Necessary and sufficient conditions for system (2.3) to be equivalent to a linear system (1.3) with constant matrix K by fiber preserving transformations are as follows.*

(II) *The conditions are (3.34), (3.36), and (3.40), and the additional conditions are as*

- (II.1) *If $a_{29y_1} \neq 0$, then the additional conditions are (3.42) and (3.43).*
- (II.2) *If $a_{29y_1} = 0$ and $a_{19y_2} \neq 0$, then the additional conditions are (3.45) and (3.46).*
- (II.3) *If $a_{29y_1} = 0$, $a_{19y_2} = 0$, and $\mu_5 \neq 0$, then the additional condition is (3.48).*
- (II.4) *If $a_{29y_1} = 0$, $a_{19y_2} = 0$, and $\mu_5 = 0$, then there are no additional conditions.*

Remark 3.5. If one assumes that the conditions (II) of Theorem 3.4 are valid, then the conditions (I) of Theorem 3.2 vanish. Moreover, these conditions also imply that $\lambda_{12} = -4a_{29y_1}$, $\lambda_{15} = -4a_{19y_2}$, $\lambda_{16} = -\mu_5$, and the following is valid: (a) the conditions (II.1) become a particular case of the conditions (I.1); (b) the conditions (II.3) are a particular case of the conditions (I.2); (c) the conditions (II.2) with $\mu_5 \neq 0$ and $\mu_5 = 0$ form particular cases of the conditions (I.2) and (I.3), respectively. This allows to propose the conjecture that Theorem 3.2 is valid independently of the values of φ_{1y_1} and φ_{2y_2} .

Notice that this conjecture is to be expected. For example, for a linearizable single second-order equation via a point transformation, the linearizable criteria combine to only two conditions, whereas during compatibility analysis, one has to study two separable cases [17].

4. Necessary Conditions of Linearization under Point Transformations

During the study presented in the previous section, several relations for linearizability for the general case of point transformations (1.11) and for the general case of the matrix $K(t)$ were noted. These relations are the necessary conditions for linearization, and they were obtained as follows. For example, assuming that $\varphi_{1y_1} \neq 0$, from (2.4), one obtains the derivatives

$v_x, v_{y_j}, \varphi_{xx}, \varphi_{xy_j}, \varphi_{y_j y_k}, \psi_{1xx}, \psi_{1xy_j}, \psi_{1y_j y_k}, (j, k, l = 1, 2)$. Comparing the mixed derivatives of the functions v, φ, ψ_1 , and ψ_2 , one can find the expressions of the quantities

$$\omega_n \quad (n = 1, 2, \dots, 15), \quad (4.1)$$

where ω_n are expressed through a_{ij} and their derivatives (shown in the appendix). Excluding the functions v, φ, ψ_1 , and ψ_2 from these expressions, one obtains the conditions

$$J_i = 0 \quad (i = 1, 2, \dots, 15), \quad (4.2)$$

where

$$\begin{aligned} J_1 &= \omega_1 \omega_{11} - 2\omega_1 \omega_9 + 2\omega_{10} \omega_2 - \omega_3 \omega_6, & J_2 &= \omega_1 \omega_5 + 2\omega_2 \omega_6, \\ J_3 &= 6\omega_1 \omega_8 - 2\omega_1 \omega_{12} + 10\omega_{11} \omega_2 - 20\omega_2 \omega_9 - 5\omega_3^2, \\ J_4 &= 2\omega_{10} \omega_2 - \omega_1 \omega_9, & J_5 &= 10\omega_1 \omega_7 + \omega_{12} \omega_2 - 3\omega_2 \omega_8, \\ J_6 &= \omega_1 \omega_8 + \omega_{11} \omega_2 - 3\omega_2 \omega_9, & J_7 &= 4\omega_1 \omega_{12} - 2\omega_1 \omega_8 + 10\omega_3^2 + 5\omega_3 \omega_5, \\ J_8 &= 10\omega_1 \omega_{13} + \omega_{12} \omega_3 - 3\omega_3 \omega_8, & J_9 &= \omega_1 \omega_{15} - \omega_{10} \omega_3, \\ J_{10} &= \omega_1 \omega_{14} + \omega_{11} \omega_3 - 3\omega_3 \omega_9, & J_{11} &= 2\omega_{12} \omega_2 - \omega_2 \omega_8 + 5\omega_3 \omega_4, \\ J_{12} &= \omega_{13} \omega_2 - \omega_3 \omega_7, & J_{13} &= 2\omega_{15} \omega_2 - \omega_3 \omega_9, \\ J_{14} &= \omega_{14} \omega_2 - \omega_3 \omega_8, & J_{15} &= 2\omega_1 \omega_4 - 2\omega_2 \omega_3 - \omega_2 \omega_5. \end{aligned} \quad (4.3)$$

After obtaining these relations, one can directly check, by substituting (2.4) into (4.2), that they are satisfied for the general case of point transformations (1.11) and for the general case of the matrix $K(t)$. Notice also that using this substitution into the conditions obtained in [18, 23], one obtains that they are not satisfied unless the matrix $K = 0$.

Thus, the following theorem can be stated.

Theorem 4.1. *The conditions (4.2) are necessary for system (2.3) to be linearizable under point transformations.*

5. Examples

In this section, examples demonstrating the procedure of using the linearization theorems are presented.

Example 5.1. Let us consider a system of two second-order quadratically semilinear ordinary differential equations

$$\begin{aligned} y_1'' &= a(y_1, y_2) y_1'^2 + 2b(y_1, y_2) y_1' y_2' + c(y_1, y_2) y_2'^2, \\ y_2'' &= d(y_1, y_2) y_1'^2 + 2e(y_1, y_2) y_1' y_2' + f(y_1, y_2) y_2'^2. \end{aligned} \quad (5.1)$$

In [18, 20], it is shown that system (5.1) is equivalent via point transformations to the simplest equations $\ddot{u}_1 = 0$, $\ddot{u}_2 = 0$ if and only if

$$S_i = 0 \quad (i = 1, 2, 3, 4), \quad (5.2)$$

where

$$\begin{aligned} S_1 &= a_{y_2} - b_{y_1} + be - cd, & S_2 &= b_{y_2} - c_{y_1} + (ac - b^2) + (bf - ce), \\ S_3 &= d_{y_2} - e_{y_1} - (ae - bd) - (df - e^2), & S_4 &= b_{y_1} + f_{y_1} - a_{y_2} - e_{y_2}. \end{aligned} \quad (5.3)$$

Application of fiber preserving transformation to system (5.1) also leads to the same conditions (5.2).

Example 5.2. Consider a nonlinear system

$$y_1'' = -y_1'^2 - y_2'^2 - q_1, \quad y_2'' = q_2 - 2y_1'y_2', \quad (5.4)$$

where q_1, q_2 are constant. Applying the linearization criteria obtained in [18, 23] to system (5.4), one obtains that system (5.4) is equivalent to the free particle equations via point transformations if and only if $q_2 = 0$. Let us consider the case $q_2 \neq 0$. Note that, for system (5.4),

$$\lambda_{12} = -4q_2, \quad \lambda_{14} = 0, \quad \lambda_{15} = -4q_2, \quad \lambda_{16} = 0. \quad (5.5)$$

Since $q_2 \neq 0$, then $\lambda_{12} \neq 0$, and (3.21) becomes $\varphi_{xx} = 0$. Taking the simplest solution $\varphi = x$ of this equation and solving the compatible system of (3.5)–(3.14) for the functions φ_1 , and φ_2 , one gets the solution $\varphi_1 = (1/2)e^{(y_1 - y_2)}$ and $\varphi_2 = (1/2)e^{(y_1 + y_2)}$. Substituting φ , φ_1 and φ_2 into (3.18) and (3.26), one obtains $k_1 = q_1 + q_2$ and $k_2 = q_1 - q_2$. Thus, Theorem 3.2 guarantees that system (5.4) can be transformed to the system of linear equations:

$$\ddot{u}_1 + k_1 u_1 = 0, \quad \ddot{u}_2 + k_2 u_2 = 0, \quad (5.6)$$

and the linearizing transformation is

$$t = x, \quad u_1 = \frac{1}{2}e^{(y_1 - y_2)}, \quad u_2 = \frac{1}{2}e^{(y_1 + y_2)}. \quad (5.7)$$

Example 5.3. A variety of applications in science and engineering such as the well-known oscillator system, the vibration of springs, and some types of conservative systems with two

degrees of freedom are of the form

$$y_1'' = g_1(x)y_1 + g_2(x)y_2, \quad y_2'' = g_3(x)y_2 + g_4(x)y_1. \quad (5.8)$$

For system (5.8),

$$\lambda_{12} = -4g_4, \quad \lambda_{16} = 4(g_1 - g_3), \quad \lambda_{15} = -4g_2, \quad \lambda_{21} = -2g_{3x}, \quad \lambda_{22} = 4g_3\lambda_{16x} - \lambda_{16xxx}. \quad (5.9)$$

Theorem 3.2 provides conditions sufficiently for system (5.8) to be reduced to a linear system with constant coefficients via a fiber preserving transformation. For example, for the oscillator system ($g_2 = 0, g_4 = 0$), the condition is

$$8\lambda_{16}^3\lambda_{21} + 4\lambda_{16}^2\lambda_{22} + 18\lambda_{16}\lambda_{16xx}\lambda_{16x} - 15\lambda_{16x}^3 = 0, \quad (5.10)$$

whereas the criteria of [18, 23] are only satisfied when $g_1 = g_3$.

6. Conclusion

The necessary form of a linearizable system of two second-order ordinary differential equations $y_1'' = f_1(x, y_1, y_2, y_1', y_2')$, $y_2'' = f_2(x, y_1, y_2, y_1', y_2')$ via point transformations is obtained. Some other necessary conditions were also found. Necessary and sufficient conditions for a system of two second-order ordinary differential equations to be transformed to the general form of linear system with constant coefficients via fiber preserving transformations are obtained. A linear system with constant coefficients is chosen because of its simplicity in finding the general solution. On the way of establishing of main theorems, we also give an explicit procedure for constructing this linearizing transformation.

Appendix

Consider the following:

$$\lambda_1 = 2a_{28y_1} - 2a_{27y_2} + a_{17}a_{25} - 2a_{18}a_{24} - a_{25}a_{28} + 2a_{26}a_{27},$$

$$\lambda_2 = 2a_{18y_1} - 2a_{17y_2} - 2a_{14}a_{18} + a_{15}a_{17} - a_{15}a_{28} + 2a_{16}a_{27},$$

$$\lambda_3 = 4a_{24x} - 2a_{27y_1} - 2a_{14}a_{27} + 2a_{17}a_{24} - 2a_{24}a_{28} + a_{25}a_{27},$$

$$\lambda_4 = 2a_{28y_1} - 2a_{25x} + a_{15}a_{27} - 2a_{18}a_{24},$$

$$\lambda_5 = 2a_{28y_2} - 4a_{26x} + 2a_{16}a_{27} - a_{18}a_{25},$$

$$\lambda_6 = 2a_{25y_1} - 4a_{24y_2} + 2a_{14}a_{25} - 2a_{15}a_{24} + 4a_{24}a_{26} - a_{25}^2,$$

$$\lambda_7 = 4a_{16x} - 2a_{18y_2} + a_{15}a_{18} - 2a_{16}a_{17} + 2a_{16}a_{28} - 2a_{18}a_{26},$$

$$\lambda_8 = 4a_{16y_1} - 2a_{15y_2} - 4a_{14}a_{16} + a_{15}^2 - 2a_{15}a_{26} + 2a_{16}a_{25},$$

$$\lambda_9 = 2a_{25y_2} - 4a_{26y_1} - a_{15}a_{25} + 4a_{16}a_{24},$$

$$\begin{aligned}\lambda_{10} = & 2a_{18x}a_{15} - 8a_{16y_1}a_{19} - 8a_{16y_2}a_{29} - 4a_{17x}a_{16} - 4a_{18xy_2} \\ & - 4a_{18x}a_{26} + 2a_{18y_1}a_{18} + 2a_{18y_2}a_{17} + 2a_{18y_2}a_{28} + 8a_{19y_1}a_{16} \\ & + 8a_{19y_2y_2} - 8a_{19y_2}a_{15} + 8a_{19y_2}a_{26} + 4a_{26x}a_{18} + 4a_{28x}a_{16} \\ & - 16a_{29y_2}a_{16} - 2a_{14}a_{18}^2 - 2a_{15}a_{18}a_{28} + 2a_{16}a_{17}^2 - 2a_{16}a_{28}^2 \\ & + 2a_{17}a_{18}a_{26} + a_{18}^2a_{25} + 2a_{18}a_{26}a_{28},\end{aligned}$$

$$\begin{aligned}\lambda_{11} = & 2a_{18y_2}a_{27} - 4a_{19y_2}a_{25} - 4a_{26xx} + 4a_{26x}a_{28} - 4a_{26y_1}a_{19} - 4a_{26y_2}a_{29} \\ & + 4a_{29y_1}a_{16} + 4a_{29y_2y_2} - 4a_{29y_2}a_{26} - a_{15}a_{18}a_{27} + 2a_{16}a_{17}a_{27} \\ & - 2a_{16}a_{27}a_{28} + 2a_{18}a_{26}a_{27},\end{aligned}$$

$$\lambda_{12} = 2a_{27x} - 4a_{29y_1} - a_{17}a_{27} + 4a_{19}a_{24} + 2a_{25}a_{29} - a_{27}a_{28},$$

$$\lambda_{14} = \lambda_{12}(a_{28} - a_{17}) - a_{27}\lambda_{16} - 2\lambda_{12x},$$

$$\lambda_{15} = 2a_{18x} - 4a_{19y_2} + 2a_{15}a_{19} + 4a_{16}a_{29} - a_{17}a_{18} - a_{18}a_{28},$$

$$\begin{aligned}\lambda_{16} = & 4a_{19y_1} - 2a_{17x} + 2a_{28x} - 4a_{29y_2} - 4a_{14}a_{19} - 2a_{15}a_{29} + a_{17}^2 \\ & + 2a_{19}a_{25} + 4a_{26}a_{29} - a_{28}^2,\end{aligned}$$

$$\begin{aligned}\lambda_{17} = & 2a_{19x}a_{25} - 2a_{19y_2}a_{27} + 2a_{25x}a_{19} + 4a_{26x}a_{29} + 2a_{28xx} - 2a_{28x}a_{28} \\ & - 4a_{29xy_2} + 4a_{29x}a_{26} - 2a_{29y_1}a_{18} + a_{15}a_{19}a_{27} + 2a_{16}a_{27}a_{29} - a_{17}a_{18}a_{27} \\ & + 2a_{18}a_{19}a_{24} + a_{18}a_{25}a_{29} - a_{18}a_{27}a_{28} - a_{27}\lambda_{15},\end{aligned}$$

$$\begin{aligned}\lambda_{18} = & 2a_{19y_1}\lambda_{14} + 4a_{28x}\lambda_{14} - 10a_{29y_2}\lambda_{14} + 2\lambda_{14xx} + 2\lambda_{14x}a_{17} - 2\lambda_{14x}a_{28} \\ & - 2a_{14}a_{19}\lambda_{14} - a_{15}a_{29}\lambda_{14} + a_{17}^2\lambda_{14} - a_{17}a_{28}\lambda_{14} - 3a_{18}a_{27}\lambda_{14} \\ & + 5a_{19}a_{25}\lambda_{14} + 10a_{26}a_{29}\lambda_{14} - 2a_{28}^2\lambda_{14},\end{aligned}$$

$$\begin{aligned}\lambda_{19} = & 8a_{29y_1}\lambda_{14}\lambda_{16} + 8\lambda_{14x}a_{27}\lambda_{16} + 18\lambda_{14x}\lambda_{14} + 8a_{17}a_{27}\lambda_{14}\lambda_{16} + 9a_{17}\lambda_{14}^2 \\ & - 8a_{19}a_{24}\lambda_{14}\lambda_{16} - 4a_{25}a_{29}\lambda_{14}\lambda_{16} + 4a_{27}^2\lambda_{14}\lambda_{15} - 4a_{27}a_{28}\lambda_{14}\lambda_{16} \\ & - 9a_{28}\lambda_{14}^2,\end{aligned}$$

$$\lambda_{20} = 2a_{28x} - 4a_{29y_2} - a_{18}a_{27} + 2a_{19}a_{25} + 4a_{26}a_{29} - a_{28}^2,$$

$$\lambda_{21} = 2a_{26x}a_{29} + a_{28xx} - a_{28x}a_{28} - 2a_{29xy_2} + 2a_{29x}a_{26},$$

$$\begin{aligned}
\lambda_{22} &= 4a_{29y_2}\lambda_{16x} - 2a_{28x}\lambda_{16x} - \lambda_{16xx} - 4\lambda_{16x}a_{26}a_{29} + \lambda_{16x}a_{28}^2, \\
\lambda_{23} &= 2a_{28x} - 4a_{29y_2} + 4a_{26}a_{29} - a_{28}^2, \\
\lambda_{24} &= 32(a_{19y_1} - a_{19x}a_{14}) - 16(a_{17y_1}a_{19} + a_{18y_1}a_{29} + a_{29x}a_{15} - a_{14}a_{18}a_{29}) \\
&\quad + 56(a_{19y_1}a_{17} - a_{14}a_{17}a_{19}) - 24(a_{19y_1}a_{28} + a_{14}a_{19}a_{28}) + 21a_{28}^3 \\
&\quad + 160(a_{26x}a_{29} - a_{29x}a_{26} + a_{29x}a_{26}) + 120(a_{17}a_{26}a_{29} - a_{29y_2}a_{17}) \\
&\quad + 64a_{28xx} + 48a_{28x}a_{17} - 112a_{28x}a_{28} + 88(a_{29y_2}a_{28} - a_{26}a_{28}a_{29}) \\
&\quad - 36a_{15}a_{17}a_{29} + 20a_{15}a_{28}a_{29} + 15a_{17}^3 - 9a_{17}^2a_{28} - 27a_{17}a_{28}^2, \\
\lambda_{25} &= 12(\lambda_{15xx}a_{28} - \lambda_{15xx}a_{17} + \lambda_{15x}a_{15}a_{29}) + 24(\lambda_{15x}a_{14}a_{19} - a_{19y_1}\lambda_{15x}) \\
&\quad - 16\lambda_{15xx} - 32a_{28x}\lambda_{15x} - 9\lambda_{15x}a_{17}^2 + 6\lambda_{15x}a_{17}a_{28} + 19\lambda_{15x}a_{28}^2 \\
&\quad + 88(a_{29y_2}\lambda_{15x} - \lambda_{15x}a_{26}a_{29}), \\
\lambda_{26} &= (\lambda_{15}^2(16a_{19y_1} + 32a_{28x} - 80a_{29y_2} - 16a_{14}a_{19} \\
&\quad - 8a_{15}a_{29} + 5a_{17}^2 - 2a_{17}a_{28} + 80a_{26}a_{29} - 19a_{28}^2) \\
&\quad + 4\lambda_{15}(\lambda_{15x}a_{28} - 4\lambda_{15xx} - \lambda_{15x}a_{17}) + 20\lambda_{15x}^2) / (32\lambda_{15}^2), \\
\lambda_{27} &= 8a_{14y_1}a_{19} + 4a_{15y_1}a_{29} + 4a_{17x}y_1 - 4a_{17y_1}a_{17} + 4a_{18x}a_{24} - 2a_{18y_1}a_{27} \\
&\quad - 8a_{19y_1y_1} + 8a_{19y_1}a_{14} - 8a_{19y_2}a_{24} - 4a_{24x}a_{18} - 2a_{27x}a_{15} + 8a_{29y_1}a_{15} \\
&\quad + 2a_{14}a_{18}a_{27} + a_{15}a_{17}a_{27} - 2a_{15}a_{25}a_{29} + a_{15}a_{27}a_{28} + 8a_{16}a_{24}a_{29} \\
&\quad - 4a_{17}a_{18}a_{24} - a_{18}a_{25}a_{27}, \\
\lambda_{28} &= 2a_{17y_1}a_{27} - 8a_{19y_1}a_{24} - 4a_{24xx} + 4a_{24x}a_{28} - 4a_{24y_1}a_{19} \\
&\quad - 4a_{24y_2}a_{29} + 4a_{29y_1y_1} + 4a_{29y_1}a_{14} - 4a_{29y_1}a_{25} + 4a_{29y_2}a_{24} \\
&\quad - a_{15}a_{27}^2 + 2a_{18}a_{24}a_{27}, \\
\lambda_{29} &= 2a_{18y_1}a_{27} - 2a_{19y_1}a_{25} - 4a_{19y_2}a_{24} - 4a_{24y_2}a_{19} - 2a_{25xx} + 2a_{25x}a_{28} \\
&\quad - 4a_{26y_1}a_{29} + 4a_{29y_1y_2} + 2a_{29y_1}a_{15} - 4a_{29y_1}a_{26} - 2a_{14}a_{18}a_{27} \\
&\quad + 2a_{14}a_{19}a_{25} + a_{15}a_{17}a_{27} - 2a_{15}a_{19}a_{24} - a_{15}a_{25}a_{29} - a_{15}a_{27}a_{28} \\
&\quad + 4a_{16}a_{24}a_{29} + a_{18}a_{25}a_{27} + 4a_{19}a_{24}a_{26} - a_{19}a_{25}^2, \\
\mu_1 &= 2a_{28x} - 4a_{29y_2} + 4a_{26}a_{29} - a_{28}^2,
\end{aligned}$$

$$\begin{aligned}
\mu_2 &= 4a_{19y_1} - 2a_{17x} - 4a_{14}a_{19} + a_{17}^2, \\
a_{29y_1}\mu_3 &= \frac{2a_{29xy_1} + a_{29y_1}a_{17} - a_{29y_1}a_{28}}{4}, \\
a_{19y_2}\mu_4 &= \frac{2a_{19xy_2} - a_{19y_2}a_{17} + a_{19y_2}a_{28}}{4}, \\
\mu_5 &= -(\mu_1 + \mu_2), \\
\omega_1 &= 2a_{12y_1} - 4a_{11y_2} - 2a_{11}a_{15} + 2a_{12}a_{14} - a_{12}a_{25} + 4a_{13}a_{24}, \\
\omega_2 &= 2a_{12y_2} - 4a_{13y_1} + 4a_{11}a_{16} - a_{12}a_{15} + 2a_{12}a_{26} - 2a_{13}a_{25}, \\
\omega_3 &= a_{15y_1} - 2a_{14y_2} - a_{25y_2} + 2a_{26y_1} - 4a_{11}a_{18} + 2a_{12}a_{17} \\
&\quad - 2a_{12}a_{28} + 4a_{13}a_{27}, \\
\omega_4 &= 2a_{15y_2} - 4a_{13x} - 4a_{16y_1} - 2a_{13}a_{17} - 2a_{13}a_{28} + 4a_{14}a_{16} \\
&\quad - a_{15}^2 + 2a_{15}a_{26} - 2a_{16}a_{25}, \\
\omega_5 &= 4a_{12x} + 4a_{25y_2} - 8a_{26y_1} + 8a_{11}a_{18} - 2a_{12}a_{17} + 6a_{12}a_{28} - 8a_{13}a_{27} \\
&\quad - 2a_{15}a_{25} + 8a_{16}a_{24} + \omega_3, \\
\omega_6 &= 4a_{11x} + 4a_{24y_2} - 2a_{25y_1} + 2a_{11}a_{17} + 2a_{11}a_{28} - 2a_{14}a_{25} + 2a_{15}a_{24} \\
&\quad - 4a_{24}a_{26} + a_{25}^2, \\
\omega_7 &= 2a_{18y_2} - 4a_{16x} - 4a_{13}a_{19} - a_{15}a_{18} + 2a_{16}a_{17} - 2a_{16}a_{28} + 2a_{18}a_{26}, \\
\omega_8 &= 5a_{15x} - a_{17y_2} - 4a_{18y_1} - 6a_{26x} + 3a_{28y_2} + 4a_{12}a_{19} - 8a_{13}a_{29} \\
&\quad + 4a_{14}a_{18} - 2a_{15}a_{17} + 2a_{15}a_{28} + 4a_{16}a_{27} - 4a_{18}a_{25}, \\
\omega_9 &= a_{17y_1} - 2a_{14x} + 3a_{25x} - 3a_{28y_1} + 4a_{12}a_{29} - 2a_{15}a_{27} + 4a_{18}a_{24}, \\
\omega_{10} &= 2a_{27y_1} - 4a_{24x} - 4a_{11}a_{29} + 2a_{14}a_{27} - 2a_{17}a_{24} + 2a_{24}a_{28} - a_{25}a_{27}, \\
\omega_{11} &= 4a_{25x} + 4a_{27y_2} - 8a_{28y_1} + 8a_{11}a_{19} + 8a_{12}a_{29} - 2a_{15}a_{27} - 2a_{17}a_{25} \\
&\quad + 8a_{18}a_{24} + 2a_{25}a_{28} - 4a_{26}a_{27}, \\
\omega_{12} &= 12a_{17y_2} - 12a_{18y_1} - 8a_{26x} + 4a_{28y_2} - 8a_{12}a_{19} - 24a_{13}a_{29} + 12a_{14}a_{18} \\
&\quad - 6a_{15}a_{17} + 6a_{15}a_{28} - 8a_{16}a_{27} - 2a_{18}a_{25}, \\
\omega_{13} &= 2\lambda_{15}, \quad \omega_{14} = 4\lambda_{16}, \quad \omega_{15} = -2\lambda_{12}.
\end{aligned}$$

(A.1)

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