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Research Article

Minimum Group Congruences on Eventually Regular Semigroups

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An eventually regular semigroup is a semigroup in which some power of any element is regular. The minimum group congruence on an eventually regular semigroup is investigated by means of weak inverse. Furthermore, some properties of the minimum group congruence on an eventually regular semigroup are characterized.

1. Introduction

Throughout this paper, we follow the notation and conventions of Howie [1].

Recall that a semigroup is said to be eventually regular if each of its elements which has some power is regular. From the definition we conclude that eventually regular semigroups generalize both regular and finite semigroups. Edwards [2] was successful in showing that many results for regular semigroups can be obtained for eventually regular semigroups. The strategy to study eventually regular semigroups is to generalize known results for regular semigroups to eventually regular semigroups. Group congruences on regular semigroups have been investigated by many algebraists. Latorre [3] explored group congruences on regular semigroups extensively and gave the representation of group congruences on regular semigroups. Hanumantha [4] generalized the results in [3] for regular semigroups to eventually regular semigroups. Moreover, group congruences on E-inversive semigroups were studied in [5, 6].

In this paper, the author explores the minimum group congruences on eventually regular semigroups by means of weak inverses. A new representation of the minimum group congruence on an eventually regular semigroup is given. Furthermore, group congruences on eventually regular semigroups are described in the same technique.

2. Preliminaries

Let S be a semigroup and $a \in S$. As usual, E_S is the set of all idempotents of S, $\langle E_S \rangle$ is the subsemigroup of S generated by E_S and N the positive integers. An element x of S is called a *weak inverse* of a if $xax = x$. We denote by $W(a)$ the set of all weak inverses of a in S.

Let ρ be a congruence on a semigroup S. Then ρ is called *group congruence* if the quotient S/ρ is a group. In particular, a congruence ρ is said to be *the minimum group congruence* if S/ρ is the maximum group morphic image of S. For a congruence ρ of S, the subset $\{a \in S \mid a\rho \in E(S/\rho)\}$ of S is called *the kernel* of ρ denoted by $\ker \rho$.

Let S be a semigroup and H a subset of S. Then the subset $H\omega$ is called *closure* of H if $H\omega = \{x \in S \mid \exists h \in H, hx \in H\}$. In this case, H is said to be *closed* if $H\omega = H$. Moreover, a subset H of S is called full if $E_S \subseteq H$. A subsemigroup K of an eventually regular semigroup S is called *weak self-conjugate* if for any $a \in S$, $a' \in W(a)$, there exist $a'Ka \subseteq K$, $aKa' \subseteq K$. For a subset H of S, we define a binary relation named σ_H on H as

$$\sigma_H = \left\{(a,b) \in S \times S : \exists b' \in W(b),\ ab' \in H\right\}. \tag{2.1}$$

We give some lemmas which will be used in the sequel.

Lemma 2.1 (see [2, 7]). *Let S be an eventually regular semigroup and ρ a congruence on S. If $a\rho$ is an idempotent of S/ρ, then an idempotent e can be found in S such that $a\rho e$.*

Remark 2.2. Since S is an eventually regular semigroup and ρ is a group congruence on S, $x\rho$ is an idempotent of S/ρ for all $x \in \langle E_S \rangle$.

Lemma 2.3. *Let S be a regular semigroup with a unique idempotent, then S is a group.*

Lemma 2.4 (see [5, 6]). *Let S be an eventually regular semigroup. Then $W(a) \neq \emptyset$ and $aa', a'a \in E_S$ for all $a \in S$, $a' \in W(a)$.*

Lemma 2.5. *Let H be a subsemigroup of an eventually regular semigroup S and $ab \in H$ for $a, b \in S$. If H is weak self-conjugate, closed, and full, then $axb \in H$ for $x \in \langle E_S \rangle$.*

Proof. Suppose that there exist $a, b \in S$ such that $ab \in H$ and $x \in \langle E_S \rangle$. Since H is full and weak self-conjugate, we obtain $b'a'axb \in H$, $abb'a' \in H$ for $a' \in W(a)$, $b' \in W(b)$. It follows from $ab \in H$ that $(ab)b'a'axb \in H$. Since H is closed, we claim $axb \in H$. □

3. Main Results

We begin the section with the main result of this paper.

Theorem 3.1. *Let S be an eventually regular semigroup and $H = \langle E_S \rangle \omega$. Then the following statements are true.*

(1) *If H is a weak self-conjugate, closed subsemigroup, then σ_H is the minimum group congruence on S and* $\ker \sigma_H = H$.

(2) *If the relation σ is a group congruence on S and* $\ker\sigma = H$, *then σ is the minimum group congruence on S and H is weak self-conjugate, closed, and full subsemigroup with* $\sigma = \sigma_{\ker\sigma}$.

The following lemma plays an important role in the proof of Theorem 3.1.

Lemma 3.2. *Let S be an eventually regular semigroup and $a, b \in S$. If the subsemigroup H of S is weak self-conjugate, closed, and full, then the following statements are equivalent:*

(1) $a\sigma_H b$;

(2) $ab' \in H$, $ba' \in H$ *for* $a' \in W(a)$, $b' \in W(b)$;

(3) $b'a \in H$ *for* $b' \in W(b)$.

Proof. (1) $\Rightarrow$ (2) Suppose $a\sigma_H b$ for $a, b \in S$, then there exists $a'' \in W(a)$ such that $ab'' \in H$, and so $ab''bb' \in H$ for $b' \in W(b)$. For any $a' \in W(a)$, $a'a \in E_S$, it follows from Lemma 2.4 that

$$ab''b(a'a)b' = ab''ba'(ab') \in H. \tag{3.1}$$

Since H is weak self-conjugate, closed, and full, we deduce $ab''ba' \in H$, so that $ab' \in H$. In a similar way, we prove $ba' \in H$ for $a' \in W(a)$.

(2) $\Rightarrow$ (3) Using the statement (2), we conclude that there exists $b' \in W(b)$ such that $ab' \in H$. Since H is weak self-conjugate, we obtain $a'ab'a \in H$ and $a'a \in E_S \subseteq H$, so that $b'a \in H$.

(3) $\Rightarrow$ (1) For $a, b \in S$, there exists $b' \in W(b)$ such that $b'a \in H$. From the weak self-conjugate of H, we deduce $bb'ab' \in H$ and $bb' \in H$. And since H is closed, we have $ab' \in H$, which leads to $a\sigma_H b$. □

We now give the proof of Theorem 3.1.

Proof of Theorem 3.1. (1) To show that σ_H is an equivalence, let $H = \langle E_S\rangle\omega$ be a weak self-conjugate, closed subsemigroup. It is obvious that H is full and $\langle E_S\rangle \subseteq H$. For $a \in S$, there exists $a' \in W(a)$ such that $aa' \in E_S \subseteq H$, so that $a\sigma_H a$, and so σ_H is reflexive. To prove the symmetry, suppose $a\sigma_H b$ for $a, b \in S$, then there exists $b' \in W(b)$, $a' \in W(a)$ such that $ab' \in H$. And since H is weak self-conjugate, full, we obtain $ab'ba' \in H$, so that $b\sigma_H a$, and so σ_H is symmetry. To prove the transitivity, let $a\sigma_H b$, $b\sigma_H c$ for $a, b, c \in S$. Then there exist $b' \in W(b)$, $c' \in W(c)$ such that $ab' \in H$, $bc' \in H$, hence $ab'bc' \in H$. And there exists $a' \in W(a)$ such that $a'a \in E_S$, and it follows from Lemma 2.4 that $ab'b(a'a)c' = (ab'ba')ac' \in H$. Since H is weak self-conjugate and full, we deduce $ab'ba' \in H$, $ac' \in H$, and so $a\sigma_H c$, which says that σ_H is transitivity. Therefore σ_H is an equivalence, as required.

We now prove that σ_H is a congruence. Suppose $a\sigma_H b$ for $a, b, c \in S$. Then there exists $(cb)' \in W(cb)$, and so $b(cb)' \in W(c)$, $(cb)'c \in W(b)$. Put $c' = b(cb)'$, $b' = (cb)'c$. Then $b'c' \in W(cb)$, $(cb)' = b'c'$. It follows from Lemma 3.2 that $ab'' \in H$ for $b'' \in W(b)$, and so $b' = (cb)'c \in W(b)$, so that $ab' \in H$. Since H is weak self-conjugate and $(ca)(cb)' = cab'c'$, we conclude $ca(cb)' = cab'c' \in H$, so that $ca\sigma_H cb$. Therefore σ_H is left compatible. On the other hand, a similar argument will show that σ_H satisfies right compatible. Thus σ_H is a congruence on S.

We now turn to show σ_H is a group congruence on S. For any $e, f \in E_S$, there exists $f \in W(f) \subseteq \langle E_S\rangle$ such that $ef \in \langle E_S\rangle \subseteq H$, so that $e\sigma_H f$. It follows from Lemma 2.1 that

S/σ_H has a uniue idempotent. For any $a \in S$, there exists $m \in N$ such that a^m is regular element. Furthermore, there exists $(a^m)' \in W(a^m)$ such that

$$a^m(a^m)' \in E_S, \quad a^m(a^m)'a(a^m)' = a^m(a^m)' \in E_S, \tag{3.2}$$

and so $a(a^m)' \in \langle E_S \rangle \omega = H$, which leads to $a\sigma_H a^m$. Therefore, we conclude that S/σ_H is a regular semigroup. It follows from Lemma 2.3 that S/σ_H is a group, so that σ_H is a group congruence on S.

We then show that σ_H is the minimum group congruence on S. Let $a\sigma_H b$ for $a, b \in S$, and let ρ be any group congruence on S with $e\rho$ as the unique idempotent of S/ρ. It follows from Lemma 3.2 that there exists $b' \in W(b)$ such that $ab' \in H$, and so there exists $t \in \langle E_S \rangle$ such that $tab' \in \langle E_S \rangle$. Notice that

$$(tab')\rho = e\rho = (a\rho)b'\rho, \qquad (aa')\rho = e\rho = (a\rho)a'\rho, \tag{3.3}$$

for $a' \in W(a)$, so that $b'\rho$ and $a'\rho$ are the group inverse of $a\rho$. In view of the uniqueness of group inverses, we have $a'\rho = b'\rho$. Since $a'\rho$ is the group inverse of $a\rho$ and $b'\rho$ is the group inverse of $b\rho$, we claim $a\rho = b\rho$, which leads to $\sigma_H \subseteq \rho$. Thus σ_H is the minimum group congruence on S.

We finally prove $\ker \sigma_H = H$. For any $a \in \ker \sigma_H$, it follows from Lemma 2.1 that there exists $e \in E_S$ such that $a\sigma_H e$. We, by Lemma 3.2, deduce that there exists $e' \in W(e)$ such that $e'a \in H$, $ee' \in E_S \subseteq H$. Since H is closed, we have $e' \in H$, $a \in H$, and so $\ker \sigma_H \subseteq H$. To show $\ker \sigma_H \supseteq H$, let $a \in H$. Since there exists $t \in \langle E_S \rangle$, $(ta)' \in W(ta)$ such that

$$ta \in \langle E_S \rangle, \quad (ta)\sigma_H \subseteq E\left(\frac{S}{\sigma_H}\right), \quad (ta)(ta)' \in E_S, \tag{3.4}$$

and so $a(ta)' \in H$, so that $a\sigma_H(ta)$. Therefore $a\sigma_H \subseteq E(S/\sigma_H)$, and so $a \in \ker \sigma_H$. Thus $\ker \sigma_H = H$, as required.

(2) Let σ be a group congruence on S and $e\sigma$ the identity of S/σ. Suppose $a\sigma b$ for $a, b \in S$, then there exist $a' \in W(a)$, $b' \in W(b)$ such that $a'\sigma$ is the group inverse of $a\sigma$ and $b'\sigma$ is the group inverse of $b\sigma$. By the uniqueness of group inverses, we have $a'\sigma b'$ and $aa'\sigma bb'\sigma ba'\sigma e$, so that $ba' \in \ker \sigma = H$, and so there exists $t \in \langle E_S \rangle$ such that $tba' \in \langle E_S \rangle$. Suppose that ρ is any group congruence on S, then

$$(tba')\rho = (ba')\rho = (b\rho)a'\rho = e\rho, \tag{3.5}$$

and so $b\rho$ is the group inverse of $a'\rho$. On the other hand, $a\rho$ is the group inverse of $a'\rho$. By the uniqueness of group inverses, we have $a\rho b$, so that $\sigma \subseteq \rho$. Therefore σ is the minimum group congruence on S.

We now prove H is weak self-conjugate, closed, and full. It is obvious that $\ker \sigma = H$ is a full subsemigroup. For any $a \in S$, $a' \in W(a)$, $x \in \ker \sigma$, then

$$(axa')\sigma = (a\sigma)e\sigma(a'\sigma) = (aa')\sigma = e\sigma, \tag{3.6}$$

which leads to $axa' \in \ker\sigma$. A similar argument shows that $a'xa \in \ker\sigma$. Therefore H is weak self-conjugate. For $x \in H\omega = (\ker\sigma)\omega$, then there exists $t \in \ker\sigma$ such that $tx \in \ker\sigma$. Hence

$$e\sigma = (tx)\sigma = t\sigma(x\sigma) = e\sigma(x\sigma) = x\sigma, \tag{3.7}$$

and so $x \in \ker\sigma$, so that $(\ker\sigma)\omega \subseteq \ker\sigma$. On the other hand, it is obvious that $(\ker\sigma)\omega \supseteq \ker\sigma$. Thus $(\ker\sigma)\omega = \ker\sigma$, and so H is weak self-conjugate, closed, and full subsemigroup of S, as required.

We finally prove $\sigma = \sigma_{\ker\sigma}$. To show $\sigma \subseteq \sigma_{\ker\sigma}$, let $a\sigma b$ for $a, b \in S$. Then there exists $b' \in W(b)$ such that $ab'\sigma bb'\sigma e$, and so $ab' \in \ker\sigma$, $a\sigma_{\ker\sigma}b$, which yields to $\sigma \subseteq \sigma_{\ker\sigma}$. We now turn to proving that the converse holds. Let $a\sigma_{\ker\sigma}b$ for $a, b \in S$. Then there exists $b' \in W(b)$ such that $ab' \in \ker\sigma = H$, and so there exists $t \in \langle E_S\rangle$ such that $tab' \in \langle E_S\rangle$. Put ρ is any group congruence on S. Notice

$$(tab')\rho = (t\rho)a\rho(b'\rho) = e\rho, \tag{3.8}$$

so that $b\rho$ and $a\rho$ are the group inverse of $b'\rho$. By the uniqueness of group inverses, we claim that $a\rho b$. Since σ is the minimum group congruence on S, σ is the intersection of all group congruence on S. Hence $a\sigma b$, so that $\sigma \supseteq \sigma_{\ker\sigma}$, and so $\sigma = \sigma_{\ker\sigma}$. The proof is then completed. □

As a specialization of Theorem 3.1, the following corollary is immediate.

Corollary 3.3. *Let S be an eventually regular semigroup. Then the following statements are true.*

(1) *If H is a weak self-conjugate, closed, and full subsemigroup, then σ_H is a group congruence on S and* $\ker\sigma_H = H$.

(2) *If the relation σ is a group congruence on S, then* $\ker\sigma$ *is a weak self-conjugate, closed, and full subsemigroup with $\sigma = \sigma_{\ker\sigma}$.*

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