1. Introduction

Throughout this paper, we follow the notation and conventions of Howie [1].

Recall that a semigroup is said to be eventually regular if each of its elements which has some power is regular. From the definition we conclude that eventually regular semigroups generalize both regular and finite semigroups. Edwards [2] was successful in showing that many results for regular semigroups can be obtained for eventually regular semigroups. The strategy to study eventually regular semigroups is to generalize known results for regular semigroups to eventually regular semigroups. Group congruences on regular semigroups have been investigated by many algebraists. Latorre [3] explored group congruences on regular semigroups extensively and gave the representation of group congruences on regular semigroups. Hanumantha [4] generalized the results in [3] for regular semigroups to eventually regular semigroups. Moreover, group congruences on $E$-inversive semigroups were studied in [5, 6].

In this paper, the author explores the minimum group congruences on eventually regular semigroups by means of weak inverses. A new representation of the minimum group congruence on an eventually regular semigroup is given. Furthermore, group congruences on eventually regular semigroups are described in the same technique.
2. Preliminaries

Let $S$ be a semigroup and $a \in S$. As usual, $E_S$ is the set of all idempotents of $S$, $\langle E_S \rangle$ is the subsemigroup of $S$ generated by $E_S$ and $N$ the positive integers. An element $x$ of $S$ is called a weak inverse of $a$ if $xax = x$. We denote by $W(a)$ the set of all weak inverses of $a$ in $S$.

Let $\rho$ be a congruence on a semigroup $S$. Then $\rho$ is called group congruence if the quotient $S/\rho$ is a group. In particular, a congruence $\rho$ is said to be the minimum group congruence if $S/\rho$ is the maximum group morphic image of $S$. For a congruence $\rho$ of $S$, the subset $\{a \in S \mid a\rho \in E(S/\rho)\}$ of $S$ is called the kernel of $\rho$ denoted by $\ker \rho$.

Let $S$ be a semigroup and $H$ a subset of $S$. Then the subset $H\omega$ is called closure of $H$ if $H\omega = \{x \in S \mid \exists h \in H, hx \in H\}$. In this case, $H$ is said to be closed if $H\omega = H$. Moreover, a subset $H$ of $S$ is called full if $E_S \subseteq H$. A subsemigroup $K$ of an eventually regular semigroup $S$ is called weak self-conjugate if for any $a \in S$, $a' \in W(a)$, there exist $a'K a \subseteq K$, $aK a' \subseteq K$. For a subset $H$ of $S$, we define a binary relation named $\sigma_H$ on $H$ as

$$\sigma_H = \{(a, b) \in S \times S : \exists b' \in W(b), \ a b' \in H\}. \quad (2.1)$$

We give some lemmas which will be used in the sequel.

**Lemma 2.1** (see [2, 7]). Let $S$ be an eventually regular semigroup and $\rho$ a congruence on $S$. If $a\rho$ is an idempotent of $S/\rho$, then an idempotent $e$ can be found in $S$ such that $a\rho e$.

**Remark 2.2.** Since $S$ is an eventually regular semigroup and $\rho$ is a group congruence on $S$, $x\rho$ is an idempotent of $S/\rho$ for all $x \in \langle E_S \rangle$.

**Lemma 2.3.** Let $S$ be a regular semigroup with a unique idempotent, then $S$ is a group.

**Lemma 2.4** (see [5, 6]). Let $S$ be an eventually regular semigroup. Then $W(a) \neq \emptyset$ and $aa', a'a \in E_S$ for all $a \in S$, $a' \in W(a)$.

**Lemma 2.5.** Let $H$ be a subsemigroup of an eventually regular semigroup $S$ and $ab \in H$ for $a, b \in S$. If $H$ is weak self-conjugate, closed, and full, then $axb \in H$ for $x \in \langle E_S \rangle$.

**Proof.** Suppose that there exist $a, b \in S$ such that $ab \in H$ and $x \in \langle E_S \rangle$. Since $H$ is full and weak self-conjugate, we obtain $b'a'axb \in H$, $ab' a' \in H$ for $a' \in W(a)$, $b' \in W(b)$. It follows from $ab \in H$ that $(ab)b'a'axb \in H$. Since $H$ is closed, we claim $axb \in H$. \qed

3. Main Results

We begin the section with the main result of this paper.

**Theorem 3.1.** Let $S$ be an eventually regular semigroup and $H = \langle E_S \rangle\omega$. Then the following statements are true.

1. If $H$ is a weak self-conjugate, closed subsemigroup, then $\sigma_H$ is the minimum group congruence on $S$ and $\ker \sigma_H = H$. 

(2) If the relation $\sigma$ is a group congruence on $S$ and $\ker \sigma = H$, then $\sigma$ is the minimum group congruence on $S$ and $H$ is weak self-conjugate, closed, and full subsemigroup with $\sigma = \sigma_{\ker \sigma}$.

The following lemma plays an important role in the proof of Theorem 3.1.

**Lemma 3.2.** Let $S$ be an eventually regular semigroup and $a, b \in S$. If the subsemigroup $H$ of $S$ is weak self-conjugate, closed, and full, then the following statements are equivalent:

1. $a \sigma_H b$;
2. $ab' \in H$ for $a' \in W(a), b' \in W(b)$;
3. $b'a \in H$ for $b' \in W(b)$.

**Proof.** (1) $\Rightarrow$ (2) Suppose $a \sigma_H b$ for $a, b \in S$, then there exists $a'' \in W(a)$ such that $ab'' \in H$, and so $ab''b'b' \in H$ for $b' \in W(b)$. For any $a' \in W(a)$, $a'a \in E_S$, it follows from Lemma 2.4 that

$$ab''b'(a'a)b' = ab''ba'(a'b') \in H.$$  \hfill (3.1)

Since $H$ is weak self-conjugate, closed, and full, we deduce $ab''ba' \in H$, so that $ab' \in H$. In a similar way, we prove $ba' \in H$ for $a' \in W(a)$.

(2) $\Rightarrow$ (3) Using the statement (2), we conclude that there exists $b' \in W(b)$ such that $ab' \in H$. Since $H$ is weak self-conjugate, we obtain $a'ab'a \in H$ and $a'a \in E_S \subseteq H$, so that $b'\a \in H$.

(3) $\Rightarrow$ (1) For $a, b \in S$, there exists $b' \in W(b)$ such that $b'a \in H$. From the weak self-conjugate of $H$, we deduce $bb'ab' \in H$ and $bb' \in H$. And since $H$ is closed, we have $ab' \in H$, which leads to $a \sigma_H b$.

We now give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** (1) To show that $\sigma_H$ is an equivalence, let $H = \langle E_S \rangle \omega$ be a weak self-conjugate, closed subsemigroup. It is obvious that $H$ is full and $\langle E_S \rangle \subseteq H$. For $a \in S$, there exists $a' \in W(a)$ such that $a' \in E_S \subseteq H$, so that $a \sigma_H a$, and so $\sigma_H$ is reflexive. To prove the symmetry, suppose $a \sigma_H b$ for $a, b \in S$, then there exists $b' \in W(b), a' \in W(a)$ such that $ab' \in H$. Since $H$ is weak self-conjugate, full, we obtain $ab'ba' \in H$, so that $b \sigma_H b$, and so $\sigma_H$ is symmetric. To prove the transitivity, let $a \sigma_H b, b \sigma_H c$ for $a, b, c \in S$. Then there exist $b' \in W(b), c' \in W(c)$ such that $ab' \in H, bc' \in H$, hence $ab'bc' \in H$. And there exists $a' \in W(a)$ such that $a' \in E_S$, and it follows from Lemma 2.4 that $ab'(a'ac') = (ab'ba')ac' \in H$. Since $H$ is weak self-conjugate and full, we deduce $ab'ba' \in H, ac' \in H$, and so $a \sigma_H c$, which says that $\sigma_H$ is transitivity. Therefore $\sigma_H$ is an equivalence, as required.

We now prove that $\sigma_H$ is a congruence. Suppose $a \sigma_H b$ for $a, b, c \in S$. Then there exists $(cb)' \in W(cb)$, and so $b(cb)' \in W(c), (cb)'c \in W(b)$. Put $c' = b(cb)\', b' = (cb)'c$. Then $b'c' \in W(cb), (cb)' = b'c'$. It follows from Lemma 3.2 that $ab'' \in H$ for $b'' \in W(b)$, and so $b' = (cb)'c \in W(b)$, so that $ab' \in H$. Since $H$ is weak self-conjugate and $(ca)(cb)' = cab' \in H$, we conclude $ca(cb)' = cab'c' \in H$, so that $ca \sigma_H cb$. Therefore $\sigma_H$ is left compatible. On the other hand, a similar argument will show that $\sigma_H$ satisfies right compatible. Thus $\sigma_H$ is a congruence on $S$.

We now turn to show $\sigma_H$ is a group congruence on $S$. For any $e, f \in E_S$, there exists $f \in W(f) \subseteq \langle E_S \rangle$ such that $ef \in \langle E_S \rangle \subseteq H$, so that $e \sigma_H f$. It follows from Lemma 2.1 that
$S/\sigma_H$ has a unique idempotent. For any $a \in S$, there exists $m \in N$ such that $a^m$ is regular element. Furthermore, there exists $(a^m)' \in W(a^m)$ such that
\[ a^m(a^m)' \in E_S, \quad a^m(a^m)'a(a^m)' = a^m(a^m)' \in E_S, \tag{3.2} \]
and so $a(a^m)' \in (E_S)\omega = H$, which leads to $a\sigma_H a^m$. Therefore, we conclude that $S/\sigma_H$ is a regular semigroup. It follows from Lemma 3.2 that there exists $\rho$ and let $e$ and so $b\rho$ be the identity of $S$. Furthermore, there exists $a\rho$ such that $a\sigmaH \subseteq \rho$. Thus $\sigma_H$ is the minimum group congruence on $S$.

We then show that $\sigma_H$ is the minimum group congruence on $S$. Let $a\sigma_H b$ for $a, b \in S$, and let $\rho$ be any group congruence on $S$ with $e\rho$ as the unique idempotent of $S/\rho$. It follows from Lemma 3.2 that there exists $b' \in W(b)$ such that $ab' \in H$, and so there exists $t \in (E_S)$ such that $tab' \in (E_S)$. Notice that
\[ (tab')\rho = ep = (ap)b'\rho, \quad (a'd')\rho = ep = (ap)a'd'\rho, \tag{3.3} \]
for $a' \in W(a)$, so that $b'\rho$ and $a'd'\rho$ are the group inverse of $ap$. In view of the uniqueness of group inverses, we have $a'\rho = b'\rho$. Since $a'\rho$ is the group inverse of $ap$ and $b'\rho$ is the group inverse of $b\rho$, we claim $ap = bp$, which leads to $\sigma_H \subseteq \rho$. Thus $\sigma_H$ is the minimum group congruence on $S$.

We finally prove $\ker \sigma_H = H$. For any $a \in \ker \sigma_H$, it follows from Lemma 2.1 that there exists $e \in E_S$ such that $a\sigma_H e$. We, by Lemma 3.2, deduce that there exists $e' \in W(e)$ such that $e'\in H$, $ee' \in E_S \subseteq H$. Since $H$ is closed, we have $e' \in H$, $a \in H$, and so $\ker \sigma_H \subseteq H$. To show $\ker \sigma_H \supseteq H$, let $a \in H$. Since there exists $t \in (E_S)$, $(ta)' \in W(ta)$ such that
\[ ta \in (E_S), \quad (ta)\sigma_H \subseteq E \left( \frac{S}{\sigma_H} \right), \quad (ta)(ta)' \in E_S, \tag{3.4} \]
and so $a(ta)' \in H$, so that $a\sigma_H (ta)$. Therefore $a\sigma_H \subseteq E(S/\sigma_H)$, and so $a \in \ker \sigma_H$. Thus $\ker \sigma_H = H$, as required.

(2) Let $\sigma$ be a group congruence on $S$ and $e\sigma$ the identity of $S/\sigma$. Suppose $aob$ for $a, b \in S$, then there exist $a' \in W(a)$, $b' \in W(b)$ such that $a'\sigma$ is the group inverse of $a\sigma$ and $b'\sigma$ is the group inverse of $b\sigma$. By the uniqueness of group inverses, we have $a'\sigma b'$ and $a'\sigma b'\sigma b'\sigma e$, so that $b\sigma e \in \ker \sigma = H$, and so there exists $t \in (E_S)$ such that $tba' \in (E_S)$. Suppose that $\rho$ is any group congruence on $S$, then
\[ (tba')\rho = (b\sigma e)(a'\sigma) = (b\sigma a')(e\sigma), \tag{3.5} \]
and so $bp$ is the group inverse of $a'd'\rho$. On the other hand, $ap$ is the group inverse of $a'd'\rho$. By the uniqueness of group inverses, we have $apb$, so that $\sigma \subseteq \rho$. Therefore $\sigma$ is the minimum group congruence on $S$.

We now prove $H$ is weak self-conjugate, closed, and full. It is obvious that $\ker \sigma = H$ is a full subsemigroup. For any $a \in S$, $a' \in W(a)$, $x \in \ker \sigma$, then
\[ (axa')\sigma = (a\sigma)x(a'\sigma) = (a\sigma)(a'\sigma) = e\sigma, \tag{3.6} \]
which leads to $axa' \in \ker \sigma$. A similar argument shows that $a'xa \in \ker \sigma$. Therefore $H$ is weak self-conjugate. For $x \in H \omega = (\ker \sigma)\omega$, then there exists $t \in \ker \sigma$ such that $tx \in \ker \sigma$. Hence
\[ e\sigma = (tx)\sigma = t\sigma(x\sigma) = e\sigma(x\sigma) = x\sigma, \]
and so $x \in \ker \sigma$, so that $(\ker \sigma)\omega \subseteq \ker \sigma$. On the other hand, it is obvious that $(\ker \sigma)\omega \supseteq \ker \sigma$. Thus $(\ker \sigma)\omega = \ker \sigma$, and so $H$ is weak self-conjugate, closed, and full subsemigroup of $S$, as required.

We finally prove $\sigma = \sigma_{\ker \sigma}$. To show $\sigma \subseteq \sigma_{\ker \sigma}$, let $a\sigma b$ for $a, b \in S$. Then there exists $b' \in W(b)$ such that $ab'\sigma bb'\sigma e$, and so $ab' \in \ker \sigma$, $a\sigma_{\ker \sigma} b$, which yields to $\sigma \subseteq \sigma_{\ker \sigma}$. We now turn to proving that the converse holds. Let $a\sigma_{\ker \sigma} b$ for $a, b \in S$. Then there exists $b' \in W(b)$ such that $ab' \in \ker \sigma = H$, and so there exists $t \in \langle E_S \rangle$ such that $tab' \in \langle E_S \rangle$. Put $\rho$ is any group congruence on $S$. Notice
\[ (tab')\rho = (tp) a\rho (b'\rho) = e\rho, \]
so that $b\rho$ and $a\rho$ are the group inverse of $b'\rho$. By the uniqueness of group inverses, we claim that $a\rho b$. Since $\sigma$ is the minimum group congruence on $S$, $\sigma$ is the intersection of all group congruence on $S$. Hence $a\sigma b$, so that $\sigma \supseteq \sigma_{\ker \sigma}$, and so $\sigma = \sigma_{\ker \sigma}$. The proof is then completed. 

As a specialization of Theorem 3.1, the following corollary is immediate.

**Corollary 3.3.** Let $S$ be an eventually regular semigroup. Then the following statements are true.

1. If $H$ is a weak self-conjugate, closed, and full subsemigroup, then $\sigma_H$ is a group congruence on $S$ and $\ker \sigma_H = H$.
2. If the relation $\sigma$ is a group congruence on $S$, then $\ker \sigma$ is a weak self-conjugate, closed, and full subsemigroup with $\sigma = \sigma_{\ker \sigma}$.

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**References**

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