Research Article

Existence Results for a Coupled System of Nonlinear Fractional Differential Equation with Four-Point Boundary Conditions

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This paper studies a coupled system of nonlinear fractional differential equation with four-point boundary conditions. Applying the Schauder fixed-point theorem, an existence result is proved for the following system: $D^\alpha u(t) = f(t, v(t), D^m v(t))$, $t \in (0, 1)$, $D^\beta v(t) = f(t, u(t), D^n u(t))$, $t \in (0, 1)$, $u(0) = \gamma u(\zeta)$, $u(1) = \delta u(\eta)$, $v(0) = \gamma v(\zeta)$, $v(1) = \delta v(\eta)$, where $\alpha, \beta, m, n, \zeta, \eta, \gamma, \delta$ satisfy certain conditions.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymerrheology, and so forth involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [1–8] and the references therein.

On the other hand, the study of coupled systems involving fractional differential equations is also important as such systems occur in various problems of applied nature, for instance, see [9, 10]. Recently, in [11], the existence of nontrivial solutions was investigated for a coupled system of nonlinear fractional differential equations with two-point boundary conditions by using Schauder's fixed-point theorem. Reference [12] established the existence of a positive solution to a singular coupled system of fractional order. The existence of nontrivial solutions for a coupled system of nonlinear fractional differential equations with

three-point boundary conditions was investigated in [13] by using Schauder's fixed point theorem.

In this paper, we consider a four-point boundary value problem for a coupled system of nonlinear fractional differential equation given by

$$\begin{aligned} D^\alpha u(t) &= f(t, v(t), D^m v(t)), \quad t \in (0, 1), \\ D^\beta v(t) &= f(t, u(t), D^n u(t)), \quad t \in (0, 1), \\ u(0) &= \gamma u(\zeta), \quad u(1) = \delta u(\eta), \quad v(0) = \gamma v(\zeta), \quad v(1) = \delta v(\eta), \end{aligned} \quad (1.1)$$

where $1 < \alpha, \beta < 2, m, n, \gamma, \delta > 0, 0 < \zeta < \eta < 1, \alpha - n \geq 1, \beta - m \geq 1, \delta \eta^{\alpha-1} < 1, \delta \eta^{\alpha-2} < 1, \gamma \zeta^{\alpha-1} < 1, \gamma \zeta^{\alpha-2} < 1, \delta \eta^{\beta-1} < 1, \delta \eta^{\beta-2} < 1, \gamma \zeta^{\beta-1} < 1, \gamma \zeta^{\beta-2} < 1, D$ is the standard Riemann-Liouville fractional derivative, and $f, g : [0, 1] \times R \times R \rightarrow R$ are given continuous function.

The organization of this paper is as follows. In Section 2, we present some necessary definition and preliminary results that will be used to prove our main results. The proofs of our main results are given in Section 3.

2. Preliminaries

For the convenience of the reader, we present here the necessary definition from fractional calculus theory and preliminary results.

Definition 2.1 (see [5]). The Riemann-Liouville fractional integral of order $q > 0$ of function $f : (0, \infty) \rightarrow R$ is given by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad (2.1)$$

provided that the integral exists.

Definition 2.2 (see [5]). The Riemann-Liouville fractional derivative of order $q > 0$ of function $f : (0, \infty) \rightarrow R$ is given by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{f(s)}{(t-s)^{q-n+1}} ds, \quad (2.2)$$

where $n = [q] + 1$ and $[q]$ denotes the integral part of number q , provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.3 ([5]). Let $n - 1 < \alpha \leq n, D^\alpha u(t)$ exists for $t \in (0, 1)$. Then,

$$I^\alpha D^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n}. \quad (2.3)$$

Remark 2.4. The following properties are useful for our discussion: $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$, $D^\alpha I^\alpha f(t) = f(t)$, $\alpha > 0, \beta > 0, f \in L(0, 1)$; $I^\alpha D^\alpha f(t) = f(t)$, $0 < \alpha < 1, f(t) \in C[0, 1]$ and $D^\alpha f(t) \in C(0, 1) \cap L(0, 1)$; $I^\alpha : C[0, 1] \rightarrow C[0, 1]$, $\alpha > 0$.

For convenience, we introduce the following notation. Let

$$\begin{aligned}\mu_1 &= \gamma \zeta^{\alpha-1}, & \nu_1 &= 1 - \gamma \zeta^{\alpha-2}, & \varpi_1 &= 1 - \delta \eta^{\alpha-1}, & \lambda_1 &= 1 - \delta \eta^{\alpha-2}, \\ \mu_2 &= \gamma \zeta^{\beta-1}, & \nu_2 &= 1 - \gamma \zeta^{\beta-2}, & \varpi_2 &= 1 - \delta \eta^{\beta-1}, & \lambda_2 &= 1 - \delta \eta^{\beta-2}.\end{aligned}\quad (2.4)$$

Let $C(J)$ Denote the space of all continuous functions defined on $J = [0, 1]$. Let $X = \{u(t) : u \in C(J) \text{ and } D^n u \in C(J)\}$ be a Banach space endowed with the norm $\|u\|_X = \max_{t \in J} |u(t)| + \max_{t \in J} |D^n u(t)|$, where $1 < \alpha < 2$, $0 < n \leq \alpha - 1$, see [11] Lemma 3.2, and let $Y = \{v(t) : v \in C(J) \text{ and } D^m v \in C(J)\}$ be a Banach space equipped with the norm $\|v\|_Y = \max_{t \in J} |v(t)| + \max_{t \in J} |D^m v(t)|$, where $1 < \beta < 2$, $0 < m \leq \beta - 1$. Thus, $(X \times Y, \|\cdot\|_{X \times Y})$ is a Banach with the norm defined by $\|(u, v)\|_{X \times Y} = \max\{\|u\|_X, \|v\|_Y\}$ for $(u, v) \in X \times Y$.

Lemma 2.5. *Let $y \in C(J)$ be a given function and $1 < \alpha < 2$. Then, the unique solution of*

$$D^\alpha u(t) = y(t), \quad t \in (0, 1), \quad (2.5)$$

$$u(0) = \gamma u(\zeta), \quad u(1) = \delta u(\eta), \quad (2.6)$$

is given by

$$u(t) = \int_0^1 K_1(t, s) y(s) ds, \quad (2.7)$$

where $K_1(t, s)$ is Green's function given by

$$K_1(t, s) = \frac{1}{\Gamma(\alpha)(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \begin{cases} K_{11}(t, s), & 0 \leq t \leq \zeta, \\ K_{12}(t, s), & \zeta < t \leq \eta, \\ K_{13}(t, s), & \eta < t \leq 1, \end{cases} \quad (2.8)$$

$$K_{11}(t, s) = \begin{cases} \left[(t-s)^{\alpha-1} (\varpi_1 \nu_1 + \mu_1 \lambda_1) + \gamma t^{\alpha-2} (\varpi_1 - \lambda_1 t) (\zeta - s)^{\alpha-1} \right. \\ \quad \left. + \delta t^{\alpha-2} (\nu_1 t + \mu_1) (\eta - s)^{\alpha-1} - t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1} \right], & 0 \leq s \leq t, \\ \left[\gamma t^{\alpha-2} (\varpi_1 - \lambda_1 t) (\zeta - s)^{\alpha-1} + \delta t^{\alpha-2} (\nu_1 t + \mu_1) (\eta - s)^{\alpha-1} \right. \\ \quad \left. - t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1} \right], & t < s \leq \zeta, \\ \left[\delta t^{\alpha-2} (\nu_1 t + \mu_1) (\eta - s)^{\alpha-1} - t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1} \right] & \zeta < s \leq \eta, \\ -t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1}, & \eta < s \leq 1, \end{cases}$$

$$\begin{aligned}
K_{12}(t, s) &= \begin{cases} \left[(t-s)^{\alpha-1} (\varpi_1 \nu_1 + \mu_1 \lambda_1) + \gamma t^{\alpha-2} (\varpi_1 - \lambda_1 t) (\zeta - s)^{\alpha-1} \right. \\ \quad \left. + \delta t^{\alpha-2} (\nu_1 t + \mu_1) (\eta - s)^{\alpha-1} - t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1} \right], & 0 \leq s \leq \zeta, \\ \left[(t-s)^{\alpha-1} (\varpi_1 \nu_1 + \mu_1 \lambda_1) + \delta t^{\alpha-2} (\nu_1 t + \mu_1) (\eta - s)^{\alpha-1} \right. \\ \quad \left. - t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1} \right], & \zeta < s \leq t, \\ \left[\delta t^{\alpha-2} (\nu_1 t + \mu_1) (\eta - s)^{\alpha-1} - t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1} \right], & t < s \leq \eta, \\ -t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1}, & \eta < s \leq 1, \end{cases} \\
K_{13}(t, s) &= \begin{cases} \left[(t-s)^{\alpha-1} (\varpi_1 \nu_1 + \mu_1 \lambda_1) + \gamma t^{\alpha-2} (\varpi_1 - \lambda_1 t) (\zeta - s)^{\alpha-1} \right. \\ \quad \left. + \delta t^{\alpha-2} (\nu_1 t + \mu_1) (\eta - s)^{\alpha-1} - t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1} \right], & 0 \leq s \leq \zeta, \\ \left[(t-s)^{\alpha-1} (\varpi_1 \nu_1 + \mu_1 \lambda_1) + \delta t^{\alpha-2} (\nu_1 t + \mu_1) (\eta - s)^{\alpha-1} \right. \\ \quad \left. - t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1} \right], & \zeta < s \leq \eta, \\ \left[(t-s)^{\alpha-1} (\varpi_1 \nu_1 + \mu_1 \lambda_1) - t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1} \right], & \eta < s \leq t, \\ -t^{\alpha-2} (\nu_1 t + \mu_1) (1-s)^{\alpha-1}, & t < s \leq 1. \end{cases}
\end{aligned} \tag{2.9}$$

Proof. For $C_1, C_2 \in R$, the general solution of (2.5) can be written as

$$\begin{aligned}
u(t) &= I^\alpha y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}.
\end{aligned} \tag{2.10}$$

By the boundary condition,

$$\begin{aligned}
u(0) &= \gamma u(\zeta), \\
C_2 &= \frac{1}{\nu_1} \left(\gamma \int_0^\zeta \frac{(\zeta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \mu_1 C_1 \right).
\end{aligned} \tag{2.11}$$

By the boundary condition,

$$\begin{aligned}
u(1) &= \delta u(\eta), \\
C_1 &= \frac{1}{\varpi_1} \left(\delta \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds - \lambda_1 C_2 \right).
\end{aligned} \tag{2.12}$$

Substituting (2.11) into (2.12), we get

$$\begin{aligned} C_1 &= \frac{\delta v_1}{(\varpi_1 v_1 + \mu_1 \lambda_1)} \int_0^\eta \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &\quad - \frac{v_1}{(\varpi_1 v_1 + \mu_1 \lambda_1)} \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &\quad - \frac{\gamma \lambda_1}{(\varpi_1 v_1 + \mu_1 \lambda_1)} \int_0^\zeta \frac{(\zeta - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds. \end{aligned} \quad (2.13)$$

Substituting (2.13) into (2.11), we get

$$\begin{aligned} C_2 &= \frac{\gamma \varpi_1}{(\varpi_1 v_1 + \mu_1 \lambda_1)} \int_0^\zeta \frac{(\zeta - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &\quad + \frac{\delta \mu_1}{(\varpi_1 v_1 + \mu_1 \lambda_1)} \int_0^\eta \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &\quad - \frac{\mu_1}{(\varpi_1 v_1 + \mu_1 \lambda_1)} \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds. \end{aligned} \quad (2.14)$$

Thus, the unique solution of (2.5) and (2.6) is

$$\begin{aligned} u(t) &= \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &\quad + \frac{\gamma t^{\alpha-2} (\varpi_1 - \lambda_1 t)}{(\varpi_1 v_1 + \mu_1 \lambda_1)} \int_0^\zeta \frac{(\zeta - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &\quad + \frac{\delta t^{\alpha-2} (v_1 t + \mu_1)}{(\varpi_1 v_1 + \mu_1 \lambda_1)} \int_0^\eta \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &\quad - \frac{t^{\alpha-2} (v_1 t + \mu_1)}{(\varpi_1 v_1 + \mu_1 \lambda_1)} \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &= \int_0^1 K_1(t, s) y(s) ds, \end{aligned} \quad (2.15)$$

where $K_1(t, s)$ is given by (2.8). □

Similarly, the general solution of

$$\begin{aligned} D^\beta v(t) &= y(t), \quad t \in (0, 1), \\ v(0) &= \gamma v(\zeta), \quad v(1) = \delta v(\eta), \end{aligned} \quad (2.16)$$

is

$$v(t) = \int_0^1 K_2(t, s)y(s)ds, \quad (2.17)$$

where $K_2(t, s)$, can be obtained from $K_1(t, s)$ by replacing α with β . Let (K_1, K_2) denote Green's function for the boundary value problem (1.1).

Consider the coupled system of integral equation:

$$\begin{aligned} u(t) &= \int_0^1 K_1(t, s)f(s, v(s), D^p v(s))ds, \\ v(t) &= \int_0^1 K_2(t, s)g(s, u(s), D^q u(s))ds. \end{aligned} \quad (2.18)$$

3. Main Results

Lemma 3.1. Assume that $f, g : J \times R \times R \rightarrow R$ are continuous functions. Then, $(u, v) \in X \times Y$ is a solution of (1.1) if and only if $(u, v) \in X \times Y$ is a solution of (2.18).

Proof. The proof is immediate from Lemma 2.5, so we omit it. $\square$

Let us define an operator $F : X \times Y \rightarrow X \times Y$ as

$$F(u, v)(t) = (F_1 v(t), F_2 u(t)), \quad (3.1)$$

where

$$F_1 v(t) = \int_0^1 K_1(t, s)f(s, v(s), D^p v(s))ds, \quad F_2 u(t) = \int_0^1 K_2(t, s)g(s, u(s), D^q u(s))ds. \quad (3.2)$$

In view of the continuity of K_1, K_2, f, g , it follows that F is continuous. Moreover, by Lemma 3.1, the fixed-point of the operator F coincides with the solution of (1.1).

For the forthcoming analysis, we introduce the growth condition on f and g as

(A₁) there exists a nonnegative function $a(t) \in L(0, 1)$ such that

$$|f(t, x, y)| \leq a(t) + \epsilon_1 |x|^{\rho_1} + \epsilon_2 |y|^{\rho_2}, \quad \epsilon_1, \epsilon_2 > 0, \quad 0 < \rho_1, \rho_2 < 1, \quad (3.3)$$

(A₂) there exist a nonnegative function $b(t) \in L(0, 1)$ such that

$$|g(t, x, y)| \leq b(t) + \delta_1 |x|^{\sigma_1} + \delta_2 |y|^{\sigma_2}, \quad \delta_1, \delta_2 > 0, \quad 0 < \sigma_1, \sigma_2 < 1. \quad (3.4)$$

Let us set the following notations for convenience:

$$\begin{aligned}
A &= \frac{1}{\alpha \Gamma(\alpha + 1) \lceil (\alpha - q + 1)(\varpi_1 v_1 + \mu_1 \lambda_1) \rceil} \\
&\quad \times \left[\gamma \zeta^\alpha \Gamma(\alpha + 1)(\alpha - q)(\varpi_1 - \lambda_1) + \delta \eta^\alpha \Gamma(\alpha + 1)(\alpha - q)(v_1 + \mu_1) \right. \\
&\quad \left. + \alpha \Gamma(\alpha + 1)(v_1(1 + \varpi_1) + \mu_1(1 + \lambda_1)) \right. \\
&\quad \left. + \alpha \lceil (\alpha - q + 1) \left((1 - \delta \eta^\alpha)(v_1 + \mu_1) - \gamma \zeta^\alpha (\varpi_1 - \lambda_1) \right) \rceil \right], \\
B &= \frac{1}{\beta \Gamma(\beta + 1) \lceil (\beta - p + 1)(\varpi_2 v_2 + \mu_2 \lambda_2) \rceil} \\
&\quad \times \left[\gamma \zeta^\beta \Gamma(\beta + 1)(\beta - p)(\varpi_2 - \lambda_2) + \delta \eta^\beta \Gamma(\beta + 1)(\beta - p)(v_2 + \mu_2) \right. \\
&\quad \left. + \beta \Gamma(\beta + 1)(v_2(1 + \varpi_2) + \mu_2(1 + \lambda_2)) \right. \\
&\quad \left. + \beta \lceil (\beta - p + 1) \left((1 - \delta \eta^\beta)(v_2 + \mu_2) - \gamma \zeta^\beta (\varpi_2 - \lambda_2) \right) \rceil \right], \\
m &= \max_{t \in J} \int_0^1 |a(s)k_1(t, s)| ds + \frac{1}{\lceil (\alpha - q)(\varpi_1 v_1 + \mu_1 \lambda_1) \rceil} \\
&\quad \times \left[\gamma (\varpi_1 - \lambda_1) \int_0^\zeta (\zeta - s)^{\alpha-1} a(s) ds + \delta (v_1 + \mu_1) \int_0^\eta (\eta - s)^{\alpha-1} a(s) ds \right. \\
&\quad \left. + (v_1(1 + \varpi_1) + \mu_1(1 + \lambda_1)) \int_0^1 (1 - s)^{\alpha-q-1} a(s) ds \right], \\
n &= \max_{t \in J} \int_0^1 |b(s)k_1(t, s)| ds + \frac{1}{\lceil (\beta - p)(\varpi_2 v_2 + \mu_2 \lambda_2) \rceil} \\
&\quad \times \left[\gamma (\varpi_2 - \lambda_2) \int_0^\zeta (\zeta - s)^{\beta-1} b(s) ds + \delta (v_2 + \mu_2) \int_0^\eta (\eta - s)^{\beta-1} b(s) ds \right. \\
&\quad \left. + (v_2(1 + \varpi_2) + \mu_2(1 + \lambda_2)) \int_0^1 (1 - s)^{\beta-p-1} b(s) ds \right].
\end{aligned} \tag{3.5}$$

Define a ball W in the Banach space $X \times Y$ as

$$W = \{(u(t), v(t)) \mid (u(t), v(t)) \in X \times Y, \|(u(t), v(t))\|_{X \times Y} \leq R, t \in J\}, \tag{3.6}$$

where $R \geq \max\{(3A\epsilon_1)^{1/(1-\rho_1)}, (3A\epsilon_2)^{1/(1-\rho_2)}, (3B\delta_1)^{1/(1-\sigma_1)}, (3B\delta_2)^{1/(1-\sigma_2)}, 3m, 3n\}$.

Theorem 3.2. Assume that the assumptions (A_1) and (A_2) hold. Then, there exists a solution for the four-point boundary value problem (1.1).

Proof. As a first step, we prove that $F : W \rightarrow W$:

$$\begin{aligned}
|F_1 v(t)| &= \left| \int_0^1 k_1(t, s) f(s, v(s), D^p v(s)) ds \right| \\
&\leq \int_0^1 |a(s) k_1(t, s)| ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \int_0^1 |k_1(t, s)| ds \\
&= \int_0^1 |a(s) k_1(t, s)| ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \\
&\quad \times \left(- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds - \frac{\gamma t^{\alpha-2} (\varpi_1 - \lambda_1 t)}{(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \int_0^\zeta \frac{(\zeta-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right. \\
&\quad \left. - \frac{\delta t^{\alpha-2} (\nu_1 t + \mu_1)}{(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{t^{\alpha-2} (\nu_1 t + \mu_1)}{(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right) \\
&= \int_0^1 |a(s) k_1(t, s)| ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \\
&\quad \times \left(- \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha-2} (\nu_1 t + \mu_1) - \gamma \zeta^\alpha t^{\alpha-2} (\varpi_1 - \lambda_1 t) - \delta \eta^\alpha t^{\alpha-2} (\nu_1 t + \mu_1)}{(\varpi_1 \nu_1 + \mu_1 \lambda_1) \Gamma(\alpha+1)} \right) \\
&\leq \int_0^1 |a(s) k_1(t, s)| ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \\
&\quad \times \left(\frac{(1 - \delta \eta^\alpha) (\nu_1 + \mu_1) - \gamma \zeta^\alpha (\varpi_1 - \lambda_1)}{(\varpi_1 \nu_1 + \mu_1 \lambda_1) \Gamma(\alpha+1)} \right), \\
|D^q F_1 v(t)| &= \left| D^q I^\alpha f(t, v(t), D^p v(t)) + \frac{1}{(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \right. \\
&\quad \times \left[\gamma (\varpi_1 D^q t^{\alpha-2} - \lambda_1 D^q t^{\alpha-1}) I^\alpha f(\zeta, v(\zeta), D^p v(\zeta)) \right. \\
&\quad \left. + \delta (\nu_1 D^q t^{\alpha-1} + \mu_1 D^q t^{\alpha-2}) I^\alpha f(\eta, v(\eta), D^p v(\eta)) \right. \\
&\quad \left. - (\nu_1 D^q t^{\alpha-1} + \mu_1 D^q t^{\alpha-2}) I^\alpha f(1, v(1), D^p v(1)) \right] \Big| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} f(s, v(s), D^p v(s)) ds + \frac{1}{[(\alpha-q)(\varpi_1 \nu_1 + \mu_1 \lambda_1)]} \\
&\quad \times \left[\gamma (\varpi_1 - \lambda_1) \int_0^\zeta (\zeta-s)^{\alpha-1} f(s, v(s), D^p v(s)) ds \right. \\
&\quad \left. + \delta (\nu_1 + \mu_1) \int_0^\eta (\eta-s)^{\alpha-1} f(s, v(s), D^p v(s)) ds \right. \\
&\quad \left. + (\nu_1 + \mu_1) \int_0^1 (1-s)^{\alpha-1} f(s, v(s), D^p v(s)) ds \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha - q)} \left[\int_0^t (t-s)^{\alpha-q-1} a(s) ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \int_0^t (t-s)^{\alpha-q-1} ds \right] \\
&\quad + \frac{1}{\Gamma(\alpha - q)(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \\
&\quad \times \left[\gamma(\varpi_1 - \lambda_1) \int_0^\zeta (\zeta - s)^{\alpha-1} a(s) ds + \gamma(\varpi_1 - \lambda_1)(\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \right. \\
&\quad \times \int_0^\zeta (\zeta - s)^{\alpha-1} ds \\
&\quad + \delta(\nu_1 + \mu_1) \int_0^\eta (\eta - s)^{\alpha-1} a(s) ds + \delta(\nu_1 + \mu_1)(\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \\
&\quad \times \int_0^\eta (\eta - s)^{\alpha-1} ds \\
&\quad + (\nu_1 + \mu_1) \int_0^1 (1-s)^{\alpha-1} a(s) ds + (\nu_1 + \mu_1)(\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \\
&\quad \times \left. \int_0^1 (1-s)^{\alpha-1} ds \right] \\
&\leq \frac{1}{\Gamma(\alpha - q)(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \\
&\quad \times \left[\gamma(\varpi_1 - \lambda_1) \int_0^\zeta (\zeta - s)^{\alpha-1} a(s) ds + \gamma(\varpi_1 - \lambda_1)(\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \right. \\
&\quad \times \int_0^\zeta (\zeta - s)^{\alpha-1} ds \\
&\quad + \delta(\nu_1 + \mu_1) \int_0^\eta (\eta - s)^{\alpha-1} a(s) ds + \delta(\nu_1 + \mu_1)(\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \\
&\quad \times \int_0^\eta (\eta - s)^{\alpha-1} ds \\
&\quad + (\nu_1(1 + \varpi_1) + \mu_1(1 + \lambda_1)) \left(\int_0^1 (1-s)^{\alpha-1} a(s) ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \right. \\
&\quad \times \left. \int_0^1 (1-s)^{\alpha-1} ds \right) \Big] \\
&= \frac{1}{\Gamma(\alpha - q)(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \\
&\quad \times \left[\gamma(\varpi_1 - \lambda_1) \int_0^\zeta (\zeta - s)^{\alpha-1} a(s) ds + \delta(\nu_1 + \mu_1) \int_0^\eta (\eta - s)^{\alpha-1} a(s) ds \right. \\
&\quad \times \left. + (\nu_1(1 + \varpi_1) + \mu_1(1 + \lambda_1)) \int_0^1 (1-s)^{\alpha-q-1} a(s) ds \right]
\end{aligned}$$

$$\begin{aligned}
& + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \\
& \times \left[\frac{\gamma \zeta^\alpha (\alpha - q) (\varpi_1 - \lambda_1) + \delta \eta^\alpha (\alpha - q) (v_1 + \mu_1) + \alpha (v_1 (1 + \varpi_1) + \mu_1 (1 + \lambda_1))}{\alpha [(\alpha - q + 1) (\varpi_1 v_1 + \mu_1 \lambda_1)]} \right].
\end{aligned} \tag{3.7}$$

Thus,

$$\begin{aligned}
\|F_1 v(t)\|_x &= \max_{t \in J} |F_1 v(t)| + \max_{t \in J} |D^q F_1 v(t)| \leq m + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) A \\
&\leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R.
\end{aligned} \tag{3.8}$$

Similarly, it can be shown that $\|F_2 v(t)\|_y \leq n + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) B \leq R$. Hence, we conclude that $\|F_1 v(t)\|_{x \times y} \leq R$.

Since $F_1 v(t)$, $F_2 v(t)$, $D^q F_1 v(t)$, $D^q F_2 v(t)$ are continuous on J , therefore, $F : W \rightarrow W$.

Now, we show that F is a completely continuous operator. For that we fix

$$M = \max_{t \in J} |f(t, v(t), D^p v(t))|, \quad N = \max_{t \in J} |g(t, u(t), D^p v(t))|. \tag{3.9}$$

For $(u, v) \in W$, $t, \tau \in J(t < \tau)$, we have

$$\begin{aligned}
& |F_1 v(t) - F_1 v(\tau)| \\
&= \left| \int_0^1 (k_1(t, s) - k_1(\tau, s)) f(s, v(s), D^p v(s)) ds \right| \\
&\leq M \left[\int_0^t |(k_1(t, s) - k_1(\tau, s))| ds + \int_t^\tau |(k_1(t, s) - k_1(\tau, s))| ds + \int_\tau^\zeta |(k_1(t, s) - k_1(\tau, s))| ds \right. \\
&\quad \left. + \int_\zeta^\eta |(k_1(t, s) - k_1(\tau, s))| ds + \int_\eta^1 |(k_1(t, s) - k_1(\tau, s))| ds \right] \\
&= \frac{M}{\Gamma(\alpha) (\varpi_1 v_1 + \mu_1 \lambda_1)} \\
&\quad \times \left[\int_0^t \left\{ ((\tau - s)^{\alpha-1} - (t - s)^{\alpha-1}) [\varpi_1 v_1 + \mu_1 \lambda_1] \right. \right. \\
&\quad \left. \left. + (\tau^{\alpha-1} - t^{\alpha-1}) [v_1 (1 - s)^{\alpha-1} + \gamma \lambda_1 (\zeta - s)^{\alpha-1} - \delta v_1 (\eta - s)^{\alpha-1}] \right. \right. \\
&\quad \left. \left. + (\tau^{\alpha-2} - t^{\alpha-2}) [\mu_1 (1 - s)^{\alpha-1} - \gamma \varpi_1 (\zeta - s)^{\alpha-1} - \delta \mu_1 (\eta - s)^{\alpha-1}] \right\} ds \right]
\end{aligned}$$

$$\begin{aligned}
& + \int_t^\tau \left\{ (\tau - s)^{\alpha-1} [\varpi_1 \nu_1 + \mu_1 \lambda_1] + \left(\tau^{\alpha-1} - t^{\alpha-1} \right) \left[\nu_1 (1-s)^{\alpha-1} + \gamma \lambda_1 (\zeta - s)^{\alpha-1} - \delta \nu_1 (\eta - s)^{\alpha-1} \right] \right. \\
& \quad \left. + \left(\tau^{\alpha-2} - t^{\alpha-2} \right) \left[\mu_1 (1-s)^{\alpha-1} - \gamma \varpi_1 (\zeta - s)^{\alpha-1} - \delta \mu_1 (\eta - s)^{\alpha-1} \right] \right\} ds \\
& + \int_\tau^\zeta \left\{ \left(\tau^{\alpha-1} - t^{\alpha-1} \right) \left[\nu_1 (1-s)^{\alpha-1} + \gamma \lambda_1 (\zeta - s)^{\alpha-1} - \delta \nu_1 (\eta - s)^{\alpha-1} \right] \right. \\
& \quad \left. + \left(\tau^{\alpha-2} - t^{\alpha-2} \right) \left[\mu_1 (1-s)^{\alpha-1} - \gamma \varpi_1 (\zeta - s)^{\alpha-1} - \delta \mu_1 (\eta - s)^{\alpha-1} \right] \right\} ds \\
& + \int_\zeta^\eta \left\{ \left(\tau^{\alpha-1} - t^{\alpha-1} \right) \left[\nu_1 (1-s)^{\alpha-1} - \delta \nu_1 (\eta - s)^{\alpha-1} \right] \right. \\
& \quad \left. + \left(\tau^{\alpha-2} - t^{\alpha-2} \right) \left[\mu_1 (1-s)^{\alpha-1} - \delta \mu_1 (\eta - s)^{\alpha-1} \right] \right\} ds \\
& + \int_\eta^1 \left\{ \left(\tau^{\alpha-1} - t^{\alpha-1} \right) \left[\nu_1 (1-s)^{\alpha-1} \right] + \left(\tau^{\alpha-2} - t^{\alpha-2} \right) \left[\mu_1 (1-s)^{\alpha-1} \right] \right\} ds \Bigg] \\
& = \frac{M}{\Gamma(\alpha)(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \\
& \times \left[\int_0^\tau (\tau - s)^{\alpha-1} [\varpi_1 \nu_1 + \mu_1 \lambda_1] ds - \int_0^t (t - s)^{\alpha-1} [\varpi_1 \nu_1 + \mu_1 \lambda_1] ds \right. \\
& \quad + \int_0^\zeta \left[\left(\tau^{\alpha-1} - t^{\alpha-1} \right) \left[\gamma \lambda_1 (\zeta - s)^{\alpha-1} \right] - \left(\tau^{\alpha-2} - t^{\alpha-2} \right) \left[\gamma \varpi_1 (\zeta - s)^{\alpha-1} \right] \right] ds \\
& \quad + \int_0^\eta \left[\left(\tau^{\alpha-1} - t^{\alpha-1} \right) \left[\nu_1 (1-s)^{\alpha-1} - \delta \nu_1 (\eta - s)^{\alpha-1} \right] \right. \\
& \quad \left. + \left(\tau^{\alpha-2} - t^{\alpha-2} \right) \left[\mu_1 (1-s)^{\alpha-1} - \delta \mu_1 (\eta - s)^{\alpha-1} \right] \right] ds \\
& \quad \left. + \int_\eta^1 \left[\left(\tau^{\alpha-1} - t^{\alpha-1} \right) \left[\nu_1 (1-s)^{\alpha-1} \right] + \left(\tau^{\alpha-2} - t^{\alpha-2} \right) \left[\mu_1 (1-s)^{\alpha-1} \right] \right] ds \right] \\
& = \frac{M}{\Gamma(\alpha+1)(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \\
& \times \left[(\tau^\alpha - t^\alpha) [\varpi_1 \nu_1 + \mu_1 \lambda_1] + \left(\tau^{\alpha-1} - t^{\alpha-1} \right) \left[\gamma \zeta^\alpha (\lambda_1 - \varpi_1) + (\nu_1 + \mu_1) (1 - \delta \eta^\alpha) \right] \right], \\
& |D^q F_1 v(t) - D^q F_1 v(\tau)| \\
& = \left| D^q I^\alpha f(t, v(t), D^p v(t)) + \frac{1}{(\varpi_1 \nu_1 + \mu_1 \lambda_1)} \right. \\
& \quad \times \left[\gamma \left(\varpi_1 D^q t^{\alpha-2} - \lambda_1 D^q t^{\alpha-1} \right) I^\alpha f(\zeta, v(\zeta), D^p v(\zeta)) \right. \\
& \quad \left. + \delta \left(\nu_1 D^q t^{\alpha-1} + \mu_1 D^q t^{\alpha-2} \right) I^\alpha f(\eta, v(\eta), D^p v(\eta)) \right] \Bigg|
\end{aligned}$$

$$\begin{aligned}
& -\left(v_1 D^q t^{\alpha-1} + \mu_1 D^q t^{\alpha-2}\right) I^\alpha f(1, v(1), D^p v(1)) \Big] \\
& - D^q I^\alpha f(\tau, v(\tau), D^p v(\tau)) - \frac{1}{(\varpi_1 v_1 + \mu_1 \lambda_1)} \\
& \times \left[\gamma \left(\varpi_1 D^q \tau^{\alpha-2} - \lambda_1 D^q \tau^{\alpha-1} \right) I^\alpha f(\zeta, v(\zeta), D^p v(\zeta)) \right. \\
& \quad \left. + \delta \left(v_1 D^q \tau^{\alpha-1} + \mu_1 D^q \tau^{\alpha-2} \right) I^\alpha f(\eta, v(\eta), D^p v(\eta)) \right. \\
& \quad \left. - \left(v_1 D^q \tau^{\alpha-1} + \mu_1 D^q \tau^{\alpha-2} \right) I^\alpha f(1, v(1), D^p v(1)) \right] \Big| \\
& \leq \frac{1}{\Gamma(\alpha-q)} \left[\left| \int_0^t (t-s)^{\alpha-q-1} f(s, v(s), D^p v(s)) ds - \int_0^\tau (\tau-s)^{\alpha-q-1} f(s, v(s), D^p v(s)) ds \right| \right. \\
& \quad + \frac{M}{\Gamma(\alpha-q)(\varpi_1 v_1 + \mu_1 \lambda_1)} \\
& \quad \times \left| \left(t^{\alpha-q-1} - \tau^{\alpha-q-1} \right) \left\{ -\gamma \lambda_1 \int_0^\zeta (\zeta-s)^{\alpha-1} ds + \delta v_1 \int_0^\eta (\eta-s)^{\alpha-1} ds - v_1 \int_0^1 (1-s)^{\alpha-1} ds \right\} \right| \\
& \quad + \frac{M}{\alpha \Gamma(\alpha-q-1)(\varpi_1 v_1 + \mu_1 \lambda_1)} \\
& \quad \times \left| \left(t^{\alpha-q-2} - \tau^{\alpha-q-2} \right) \left\{ \gamma \varpi_1 \int_0^\zeta (\zeta-s)^{\alpha-1} ds + \delta \mu_1 \int_0^\eta (\eta-s)^{\alpha-1} ds - \mu_1 \int_0^1 (1-s)^{\alpha-1} ds \right\} \right| \\
& \leq \frac{1}{\Gamma(\alpha-q)} \left[\left| \int_0^t (t-s)^{\alpha-q-1} f(s, v(s), D^p v(s)) ds - \int_0^\tau (t-s)^{\alpha-q-1} f(s, v(s), D^p v(s)) ds \right| \right. \\
& \quad \left. + \left| \int_0^\tau (t-s)^{\alpha-q-1} f(s, v(s), D^p v(s)) ds - \int_0^\tau (\tau-s)^{\alpha-q-1} f(s, v(s), D^p v(s)) ds \right| \right] \\
& \quad + \frac{M(\gamma \zeta^\alpha \lambda_1 + v_1(1-\delta \eta^\alpha))}{\alpha \Gamma(\alpha-q)(\varpi_1 v_1 + \mu_1 \lambda_1)} \left(t^{\alpha-q-1} - \tau^{\alpha-q-1} \right) \\
& \quad + \frac{M(\gamma \zeta^\alpha \varpi_1 + \mu_1(1-\delta \eta^\alpha))}{\alpha^2 \Gamma(\alpha-q-1)(\varpi_1 v_1 + \mu_1 \lambda_1)} \left(t^{\alpha-q-2} - \tau^{\alpha-q-2} \right) \\
& \leq \frac{M}{\Gamma(\alpha-q)} \left[\int_0^\tau \left((\tau-s)^{\alpha-q-1} - (t-s)^{\alpha-q-1} \right) ds + \int_t^\tau (t-s)^{\alpha-q-1} ds \right] \\
& \quad + \frac{M(\gamma \zeta^\alpha \lambda_1 + v_1(1-\delta \eta^\alpha))}{\alpha \Gamma(\alpha-q)(\varpi_1 v_1 + \mu_1 \lambda_1)} \left(t^{\alpha-q-1} - \tau^{\alpha-q-1} \right) \\
& \quad + \frac{M(\gamma \zeta^\alpha \varpi_1 + \mu_1(1-\delta \eta^\alpha))}{\alpha^2 \Gamma(\alpha-q-1)(\varpi_1 v_1 + \mu_1 \lambda_1)} \left(t^{\alpha-q-2} - \tau^{\alpha-q-2} \right) \\
& = \frac{M}{\Gamma(\alpha-q+1)} \left(\tau^{\alpha-q} - t^{\alpha-q} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{M(\gamma \zeta^{\alpha} \lambda_1 + \nu_1(1 - \delta \eta^{\alpha}))}{\alpha[(\alpha - q)(\varpi_1 \nu_1 + \mu_1 \lambda_1)]} (t^{\alpha-q-1} - \tau^{\alpha-q-1}) \\
& + \frac{M(\gamma \zeta^{\alpha} \varpi_1 + \mu_1(1 - \delta \eta^{\alpha}))}{\alpha^2[(\alpha - q - 1)(\varpi_1 \nu_1 + \mu_1 \lambda_1)]} (t^{\alpha-q-2} - \tau^{\alpha-q-2}).
\end{aligned} \tag{3.10}$$

Analogously, it can be proved that

$$\begin{aligned}
& |F_1 u(t) - F_1 u(\tau)| \\
& \leq \frac{N}{\Gamma(\beta + 1)(\varpi_2 \nu_2 + \mu_2 \lambda_2)} \\
& \quad \times \left[(\tau^{\beta} - t^{\beta}) [\varpi_2 \nu_2 + \mu_2 \lambda_2] + (\tau^{\beta-1} - t^{\beta-1}) [\gamma \zeta^{\beta} (\lambda_2 - \varpi_2) + (\nu_2 + \mu_2) (1 - \delta \eta^{\beta})] \right] \\
& |D^p F_1 u(t) - D^p F_1 u(\tau)| \\
& \leq \frac{N}{\Gamma(\beta - p + 1)} (\tau^{\beta-p} - t^{\beta-p}) \\
& \quad + \frac{N(\gamma \zeta^{\beta} \lambda_2 + \nu_2(1 - \delta \eta^{\beta}))}{\beta[(\beta - p)(\varpi_2 \nu_2 + \mu_2 \lambda_2)]} (t^{\beta-p-1} - \tau^{\beta-p-1}) \\
& \quad + \frac{N(\gamma \zeta^{\beta} \varpi_2 + \mu_2(1 - \delta \eta^{\beta}))}{\beta^2[(\beta - p - 1)(\varpi_2 \nu_2 + \mu_2 \lambda_2)]} (t^{\beta-p-2} - \tau^{\beta-p-2}).
\end{aligned} \tag{3.11}$$

Since the functions t^{α} , $t^{\alpha-1}$, t^{β} , $t^{\beta-1}$, $t^{\alpha-q}$, $t^{\alpha-q-1}$, $t^{\beta-p}$, $t^{\beta-p-1}$ are uniformly continuous on J , therefore, it follows from the above estimates that FW is an equicontinuous set. Also, it is uniformly bounded as $FW \subset W$. Thus, we conclude that F is a completely continuous operator. Hence, by Schauder's fixed-point theorem, there exists a solution for the four-point boundary value problem (1.1). $\square$

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